Application of an Energy Theorem to Derive a Scaling Law for Structural Behaviors

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Abstract

A general procedure for deriving a scaling law for structural behaviors from a principle of conservation of energy was proposed. This method provides clear scaling results of a full-sized structure, since the whole members are scaled simultaneously rather than considered separately. The concept of this method is that a total potential energy of a similar scaled model has to be proportionate to that of a full-size structure and satisfy the principle of conservation of energy. An applicability of this procedure was demonstrated by deriving scaling factors for static deflection of a beam and plate under various load cases and support conditions. Moreover, the procedure was applied to a dynamic problem, i.e. the scaling factor for frequency response of a plate. All of the derived scaling factors were validated by comparing the scaling solutions of the prototype's behaviors using similitude theory to those computed from closed-form solutions. The derived scaling factors for all cases considered predict the exact results of the prototype's behaviors.

Keywords: energy theorem, similitude, beam, thin plate, deflection, natural frequency

1. Introduction

In order to predict the physical behaviors of a full-sized structure or prototype from its corresponding scaled model, the similitude conditions between the prototype and its scaled model must be satisfied. Together with several methods used in the past, derivation of a scaling law by utilizing a governing differential equation, is the most systematic and rigorous method [1]. This technique has been successfully applied to several problems, for example, buckling of a composite plate [2, 3], free vibration response of a composite plate [3, 4] as well as free and forced vibration response of an elastic supported rectangular plate under a moving point load [5].

In the method of governing equations, after substituting the scaled variables into the governing equation of the prototype, the equation is then compared with an equation of the model. The similitude conditions are obtained by equating the coefficients that appear in both equations. The concept insight of this process can be stated as: the sufficient and necessary condition of similitude between two systems is that the mathematical model of the one be related by a one-to-one transformation to that of the other [1]. Scaling factors obtained by this process are called "explicit scaling factors" [5]. However, the structural behaviors depend on boundary conditions, i.e. methods of restraint. To achieve a complete similitude between model and prototype behaviors, it is necessary to properly scale the boundary conditions also. The scaling factors for boundary conditions are called "implicit scaling factors" [5]. In the method of governing equations, an implicit scaling factor is separately derived by other methods such as dimensional analysis. Therefore, it is worth developing a method that can unify the deriving process of both types of scaling factors.

In this study, the scaling laws are derived by considering the principle of conservation of energy, since the principle considers the whole system, i.e. body, loads, and boundary conditions. The similitude conditions can not be sufficiently derived by scaling each one of the strain energy, external work, and kinetic energy, but they must be derived by scaling all types of energy simultaneously. This process leads to a scaled energy equation which has to obey the principle of conservation of energy for the model. The last requirement introduces the condition of complete similitude between both systems. This idea has been successfully applied to 2-D truss and beam bending problems [6]. The process of deriving a scaling factor by this method is systematic and has no requirement to specify the boundary conditions similar to the method of using a governing equation. This implies that both methods have the same order of generality. However, when the structure is constructed from several members, e.g. truss, frame, etc., the proposed method directly gives the scaling factor for the structural behavior. The method of governing equations derives the scaling factor from the governing equation of a member, then employs a scaling factor to the whole structure. The final derived a scaling factors from both methods are the same. The energy approach is more straightforward in the scaled behaviors.

In the present work, the applicability of the proposed method is further explored by applying it to several structural problems.

2. General procedure

This section presents a general procedure to derive a similitude condition for behaviors of structural members employing an energy principle. The principle of conservation of energy states that if there is no energy loss in the form of heat and chemical reactions, the strain energy U stored within the body is equal to a summation of the work produced by external loads W, and kinetic energy T:

$$U(X_i) = W(Y_j) + T(Z_k) \qquad (1)$$

where X_i, Y_j, and Z_k are the lists of physical variables, i.e. dimensions, material properties, for each type of energy.

Let the physical variables of a model and a prototype for each type of energy be defined similarly to those that appeared in Eq.(1) with an addition of the subscripts "m" and "p", i.e. X_{mi} and X_{pi}, Y_{mj}, and Y_{pj} as well as Z_{mk} and Z_{pk}, respectively. Furthermore, all variables of both systems are related to each other by the equations:

$$X_{pi} = C_i X_{mi} \qquad (2a)$$

$$Y_{pj} = C_j Y_{mj} \qquad (2b)$$

and

$$Z_{pk} = C_k Y_{mk} \qquad (2c)$$

where C_i, C_j and C_k are scaling factors of prototype variables.

Substitute Eq.(2) into (1) yields

$$U(C_i X_{mi}) - W(C_j Y_{mj}) - T(C_k Z_{mk}) = 0 \qquad (3)$$

In order to achieve a complete similitude condition, it is necessary that the equation must be rewritten in the following form:

$$\begin{aligned} &\varphi(C_i)U(X_{mi}) - \chi(C_j)W(Y_{mj}) \\ &-\phi(C_k)T(Z_{mk}) = 0 \end{aligned} \qquad (4)$$

where $\varphi(C_i)$, $\chi(C_j)$ and $\phi(C_k)$ are functional relationships among the scaling factors.

The complete similitude requirements are achieved if and only if the condition of conservation of energy of the model is satisfied, i.e. $U(X_{mi}) - W(Y_{mj}) - T(Z_{mk}) = 0$. Therefore, all of the functional terms in Eq.(4) have to be equal:

$$\varphi(C_i) = \chi(C_j) = \phi(C_k). \qquad (5)$$

3. Application

This section presents an application of a general procedure to determine scaling factors for structural behaviors of a beam and thin plate. These behaviors are static deflection and natural frequency. In section 3.1, the procedure is used in derivation of a scaling factor for the deflection of a beam having different types of support under a concentrated load. The objective is to present an invariant scaling factor for the boundary conditions. In section 3.2, the procedure is used in the derivation of a scaling factor for deflection of a plate subjected to concentrated load and uniformly distributed load. The objective is to present the dependency of the scaling factor to types of load. Finally, in section 3.3, the procedure is used in the derivation of a scaling factor for the frequency response of a plate.

The derived scaling factors are verified by comparing the behavior of a prototype, which is computed from closed-form solutions, with that from a similitude theory.

3.1 Scaling factor for beam deflection

3.1.1 Beam with rigid or free support subjected to a concentrated load

In this case, the reaction forces at the supports produce no work and there is no kinetic energy. Applying the conservation of an energy principle to the prototype which is subjected to a concentrated load Q_p yields:

$$\int_0^{L_p} \frac{E_p I_p}{2} \left(\frac{d^2 w_p}{dx_p^2} \right)^2 dx_p - \frac{1}{2} Q_p \Delta_p = 0 , \quad (6)$$

where E is Young' s modulus, I is area moment of inertia, L is beam length, x is linear distance along the beam length, w is static deflection, and Δ is deflection at the loading point in the direction of the force Q. The subscript " p " is added to emphasize that these variables belong to a prototype.

The relationship among prototype and model variables are:

$$Q_p = C_Q Q_m , E_p = C_E E_m , I_p = C_I I_m$$
$$L_p = C_L L_m , \; x_p = C_x x_m , \; w_p = C_w w_m$$
$$\text{and } \Delta_p = C_w \Delta_m \quad (7)$$

where C_Q, C_E, C_I, C_L and C_w are scaling factors for load, Young's modulus, area moment of inertia, length, and deflection, respectively.

Substitute Eq.(7) into (6) yields:

$$\int_0^{C_I L_m} C_E C_I \frac{E_m I_m}{2} \frac{C_w^2}{C_x^4} \left(\frac{d^2 w_m}{dx_m^2} \right)^2 d(C_x x_m)$$
$$- \frac{1}{2} C_Q C_w Q_m \Delta_m = 0 \quad (8)$$

For geometric similarity, C_x becomes equal to C_L. The complete similitude requirement obtained by Eq.(5) is:

$$\frac{C_E C_I C_w^2}{C_L^3} = C_Q C_w \quad (9)$$

The scaling factor for static deflection derived from the above condition is:

$$C_w = \frac{C_Q C_L^3}{C_E C_I} \quad (10)$$

To verify the above scaling factor, consider a cantilever beam having a length of L and subjected to a single concentrated load Q at a distance a as shown in Fig. 1. The solution for the deflection at the tip (point A) δ_A, obtained by elementary beam theory is:

$$\delta_A = \frac{Qa^2}{EI}(3L + a). \quad (11)$$

The values of prototype variables are:

$E_p = 207 GPa$, $I_p = 10^{-6} m^4$, $L_p = 2\, m$
$a_p = 1\, m$, and $Q_p = 500\, N$

The values of model variables are:

$E_m = 70\, GPa$, $I_m = 2 \times 10^{-9} m^4$, $L_m = 0.4\, m$
$a_m = 0.2\, m$, and $Q_m = 15\, N$

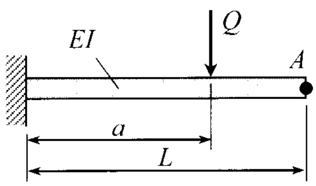

Fig. 1 Cantilever beam subjected to a single concentrated load

The deflection at point A of the prototype beam computed by Eq.(11) is 2.01 mm downward. The deflection at point A of the model computed by Eq.(11) is 0.714 mm downward. The scaling factor for deflection is 2.818. Therefore, the deflection of a prototype beam computed by similitude theory is $2.818 \times 0.714 = 2.01$ mm. It can be concluded that under a complete similitude condition, similitude theory predicts the exact result.

3.1.2 Beam with linear spring support subjected to a concentrated load

In this case, reaction forces at spring locations produce work which is a function of deflection and stiffness of the spring. Consider a beam that is supported by a single linear elastic spring, the principle of conservation of energy for the prototype which is subjected to a concentrated load Q_p can be written as:

$$-\frac{1}{2}k_p\delta_p^{\,2} + \int_0^{L_p}\frac{E_pI_p}{2}\left(\frac{d^2w_p}{dx_p^{\,2}}\right)^2 dx_p - \frac{1}{2}Q_p\Delta_p = 0, \tag{12}$$

where k and δ are spring stiffness and deflection of the beam at the spring support, respectively.

Let the scaling factor for spring stiffness be C_k and is defined as k_p/k_m. It should be noted that the scaling factor for deflection at any point on the beam is equal; therefore, it is not necessary to define a scaling factor for deflection of the spring. Substitute Eq.(7) together with $C_k \equiv k_p/k_m$ into Eq.(12) yields:

$$\left[-\left(C_kC_w^{\,2}\right)\frac{1}{2}k_m\delta_m^{\,2} + \frac{C_EC_IC_w^{\,2}}{C_L^{\,3}}\int_0^{L_m}E_mI_m\left(\frac{d^2w}{dx_m^{\,2}}\right)^2 dx_m - C_QC_w\left(\frac{1}{2}Q_m\Delta_m\right)\right] = 0. \tag{13}$$

The complete similitude requirement obtained by Eq.(5) is:

$$C_kC_w^{\,2} = \frac{C_EC_IC_w^{\,2}}{C_L^{\,3}} = C_QC_w \tag{14}$$

Equation (14) provides the scaling factor for static deflection as:

$$C_w = \frac{C_QC_L^{\,3}}{C_EC_I} \tag{15a}$$

and the scaling factor for spring stiffness as:

$$C_k = \frac{C_EC_I}{C_L^{\,3}} \tag{15b}$$

To verify the above scaling factor, consider a beam with built-in end and spring support (Fig.2). Dimensions of beam as well as section properties and load are equal to those in the previous section. The stiffness of the prototype spring k_p is $5000\ N/m$. The spring stiffness of the model which is required so that both systems are similar, and similitude theory is applicable can be determined from Eq.(15b). The result is $k_m = 422.7\ N/m$. From the theoretical solution, deflection at point A, δ_A can be determined from:

$$\delta_A = \frac{\frac{5}{16}Q}{k\left(1+3\frac{EI}{kL^3}\right)} \tag{16}$$

For the prototype beam, the computed deflection is 1.89 mm. For the model beam the computed deflection and scaling factor are 0.67 mm and 2.818 mm, respectively. Using the model' s result and similitude theory, the prototype result is predicted as $2.818 \times 0.67 = 1.89$ mm. Therefore, under a complete similitude condition, similitude theory predicts the exact result.

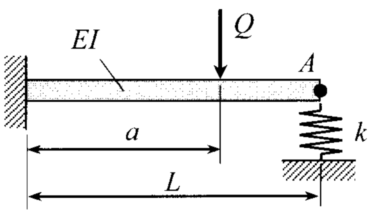

Fig. 2 Beam with built-in end and spring support

3.2 Scaling factor for deflection of a thin plate
3.2.1 Plate subjected to uniformly distributed load

For a thin plate of uniform thickness h subjected to a uniformly distributed load q, the conservation of an energy equation for a prototype is [7] :

$$\frac{1}{2}\iint_{A_p}\frac{E_p h_p^3}{12\left(1-v_p^2\right)}\left\{\left(\frac{\partial^2 w_p}{\partial x_p^{\,2}}+\frac{\partial^2 w_p}{\partial y_p^{\,2}}\right)^2-2\left(1-v_p\right)\left[\frac{\partial^2 w_p}{\partial x_p^{\,2}}\frac{\partial^2 w_p}{\partial y_p^{\,2}}-\left(\frac{\partial^2 w_p}{\partial x_p\partial y_p}\right)^2\right]\right\} -\frac{1}{2}\iint_{A_p} q_p w_p dx_p dy_p = 0 \quad (17)$$

where A is the area of the plate surface and v is Poisson' s ratio.

Let the variables of the prototype be related to those of the model through the scaling factors as follows:

$$x_p = C_x x_m,\; y_p = C_y y_m,\; h_p = C_h h_m$$
$$q_p = C_q q_m,\; v_p = C_v v_m$$
$$E_p = C_E E_m,\text{ and } w_p = C_w w_m \quad (18)$$

where C_h, C_v are scaling factors for plate thickness and Poisson's ratio, respectively.

Substitute Eq.(18) into (17) yields:

$$\frac{1}{2}\iint_{A_m}\frac{C_E C_h^{\,3} E_m h_m^{\,3}}{1-C_v^{\,2}v_m^{\,2}}\left\{\left(\frac{C_w}{C_x^{\,2}}\frac{\partial^2 w_m}{\partial x_m^{\,2}}+\frac{C_w}{C_y^{\,2}}\frac{\partial^2 w_m}{\partial y_m^{\,2}}\right)^2 + 2\left(1-C_v v_m\right)\left[\frac{C_w^{\,2}}{C_x^{\,2}C_y^{\,2}}\frac{\partial^2 w_m}{\partial x_m^{\,2}}\frac{\partial^2 w_m}{\partial y_m^{\,2}} - \frac{C_w^{\,2}}{C_x^{\,2}C_y^{\,2}}\left(\frac{\partial^2 w_m}{\partial x_m\partial y_m}\right)^2\right]\right\}C_x C_y dx_m dy_m - \left(C_w C_q C_x C_y\right)\iint_{A_m} w_m q_m dx_m dy_m = 0 \quad (19)$$

For geometric similarity,

$$C_x = C_y \equiv C_L \quad (20)$$

where C_L is the geometric scaling factor.

However, to satisfy the condition in the form of Eq.(4), it is necessary that:

$$C_v = 1 \quad (21)$$

The complete similitude requirement obtained from Eq.(19) is

$$\frac{C_E C_h^{\,3} C_w^{\,2}}{C_L^{\,2}} = C_w C_q C_L^{\,2} \quad (22)$$

The scaling factor for deflection is:

$$C_w = \frac{C_q C_L^{\,4}}{C_E C_h^{\,3}} \quad (23)$$

To verify the above scaling factor, consider a regular pentagon plate of a thickness of h which is simply supported at all edges and subjected to a uniformly distributed load q (Fig. 3). A general solution for a deflection, w is [8]:

$$w \approx \frac{12\left(1-v^2\right)qa^4}{Eh^3}\left[\beta^4/64 + 0.02489\left(1-1.831\beta^2\right) + 0.007126\beta^5\left(1-0.3591\beta^2\right)\cos(\pi\theta) + 0.0001263\beta^{10}\left(1+0.2669\beta^2\right)\cos(2\pi\theta) + 0.00001097\beta^{15}\left(1-0.2525\beta^2\right)\cos(3\pi\theta)\right] \quad (24)$$

where $\beta = r/a$, r is the radial distance from the center of the plate and a is the radius of a circle that circumscribes the pentagon.

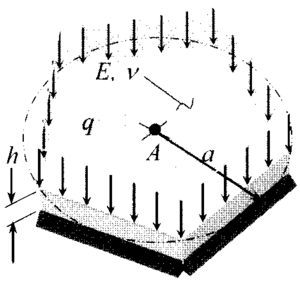

Fig. 3 Pentagon plate which is simply supported at all edges subjected to a uniformly distributed load.

The values of prototype variables are:

$E_p = 207GPa$, $v_p = 0.3$, $h_p = 20\,mm$

$a_p = 1.5\,m$, and $q_p = 10^4\ N/m^2$

The values of model variables are:

$E_m = 70GPa$, $\nu_m = 0.3$, $h_m = 1mm$
$a_m = 0.3\,m$, and $q_m = 10^2\ N/m^2$

The maximum deflection of a prototype plate computed by Eq.(24) is 8.31 mm downward. The maximum deflection of a model computed by the same equation is 3.15 mm downward. The scaling factor for deflection computed by Eq.(23) is 2.642. Then the deflection of prototype beam determined from similitude theory is 2.642×3.15 =8.31 mm. Therefore, for a complete similitude between model and prototype, the derived scaling factor is exact solution.

3.2.2 Plate subjected to a concentrated load

In the case of a plate subjected to a concentrated load Q, the conservation of an energy equation for a prototype is:

$$\frac{1}{2}\iint_{A_p}\frac{E_p h_p^{\ 3}}{12\left(1-\nu_p^{\ 2}\right)}\left\{\left(\frac{\partial^2 w_p}{\partial x_p^{\ 2}}+\frac{\partial^2 w_p}{\partial y_p^{\ 2}}\right)^2 - 2\left(1-\nu_p\right)\left[\frac{\partial^2 w_p}{\partial x_p^{\ 2}}\frac{\partial^2 w_p}{\partial y_p^{\ 2}}-\left(\frac{\partial^2 w_p}{\partial x_p \partial y_p}\right)^2\right]\right\}dx_p dy_p - \frac{1}{2}Q_p\Delta_p = 0 \tag{25}$$

Substitute Eq.(18), except C_q which is replaced by C_Q defined as $C_Q = Q_p/Q_m$, into Eq.(25) yields:

$$\left(C_E C_h^{\ 3}\frac{C_w^{\ 2}}{C_L^{\ 2}}\right)\frac{1}{2}\iint_{A_m}\frac{E_m h_m^{\ 3}}{12\left(1-\nu_m^{\ 2}\right)}\left\{\left(\frac{\partial^2 w_m}{\partial x_m^{\ 2}}+\frac{\partial^2 w_m}{\partial y_m^{\ 2}}\right)^2 + 2\left(1-\nu_m\right)\left[\frac{\partial^2 w_m}{\partial x_m^{\ 2}}\frac{\partial^2 w_m}{\partial y_m^{\ 2}}-\left(\frac{\partial^2 w_m}{\partial x_m \partial y_m}\right)^2\right]\right\}dx_m dy_m - \left(C_w C_Q\right)\frac{1}{2}Q_m\Delta_m \ = 0 \tag{26}$$

Complete similitude requirement obtained from Eq.(26) is

$$\frac{C_E C_h^{\ 3} C_w^{\ 2}}{C_L^{\ 2}} = C_w C_Q \tag{27}$$

Thus, a scaling factor for deflection is:

$$C_w = \frac{C_Q C_L^{\ 2}}{C_E C_h^{\ 3}} \tag{28}$$

To verify the above scaling factor, consider a regular pentagon plate with free edges and corner supports subjected to a concentrated load Q at the middle of the plate (Fig. 4). A general solution for a deflection, w is [8]:

$$w \approx \frac{12\left(1-\nu^2\right)Qa^2}{\pi E h^3}\left\{\frac{\beta^2\ln\beta}{8} + \frac{\left(0.2109+0.04566\nu\right)\left(1-\beta^2\right)}{1+\nu} + \frac{1}{1-\nu}\left[0.02299(1-0.3088\nu)\left(1-\beta^5\cos(\pi\theta)\right) - 5.696\times10^{-5}(1-3.169\nu)\left(1-\beta^{10}\cos(2\pi\theta)\right) - 1.34\times10^{-6}(1-1.4376\nu)\left(1-\beta^{15}\cos(3\pi\theta)\right)\right]\right\} \tag{29}$$

where $\beta = r/a$, r is the radial distance from the center of the plate and a is the radius of a circle that circumscribes a pentagon.

The values of prototype variables are:

$E_p = 207GPa$, $h_p = 20mm$, $a_p = 1.5\,m$, $Q_p = 10^4\,N$, and $\nu_p = 0.3$.

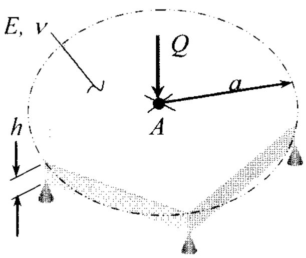

Fig. 4 Pentagon plate with free edges and corner supports subjected to a concentrated load Q.

The values of model variables are:

$E_m = 70GPa$, $h_m = 1mm$, $a_m = 0.3\,m$

$Q_m = 5N$, and $\nu_m = 0.3$

The maximum deflection of a prototype plate computed by Eq. (29) is 9.57 mm downward. The maximum deflection of a model computed by the same equation is 4.53 mm downward. The scaling factor for deflection is 2.642. Then the deflection of a prototype beam determined from similitude theory is $2.114 \times 4.53 = 9.57$ mm. Therefore, for a complete similitude between the model and prototype, the derived scaling factor is the exact solution.

3.3 Free vibration of a thin plate

The conservation of an energy equation for prototype plate is expressed by the following equation [9]:

$$\frac{1}{2}\iint_{A_p}\frac{E_p h_p^{\ 3}}{12\left(1-\nu_p^{\ 2}\right)}\left\{\left(\frac{\partial^2 w_p}{\partial x_p^{\ 2}}+\frac{\partial^2 w_p}{\partial y_p^{\ 2}}\right)^2+\right.$$

$$\left.2\left(1-\nu_p\right)\left[\left(\frac{\partial^2 w_p}{\partial x_p \partial y_p}\right)^2-\frac{\partial^2 w_p}{\partial x_p^{\ 2}}\frac{\partial^2 w_p}{\partial y_p^{\ 2}}\right]\right\}dx_p dy_p$$

$$-\frac{1}{2}\iint_{A_p} h_p \rho_p \left(\frac{\partial w_p}{\partial t_p}\right)^2 dx_p dy_p = 0 \qquad (30)$$

where ρ is density of material and t is time.

Substitute Eq.(18) except C_q, together with C_ρ and C_t defined as ρ_p/ρ_m and t_p/t_m, respectively, into Eq.(30) yields:

$$\left(\frac{C_E C_h^{\ 3} C_w^{\ 2}}{C_L^{\ 2}}\right)\frac{1}{2}\iint_{A_m}\frac{E_m h_m^{\ 3}}{12\left(1-\nu_m^{\ 2}\right)}\left\{\left(\frac{\partial^2 w_m}{\partial x_m^{\ 2}}+\frac{\partial^2 w_m}{\partial y_m^{\ 2}}\right)^2\right.$$

$$\left.+2\left(1-\nu_m\right)\left[\left(\frac{\partial^2 w_m}{\partial x_m \partial y_m}\right)^2-\frac{\partial^2 w_m}{\partial x_m^{\ 2}}\frac{\partial^2 w_m}{\partial y_m^{\ 2}}\right]\right\}dx_m dy_m$$

$$-\left(\frac{C_h C_\rho C_w^{\ 2} C_L^{\ 2}}{C_t^2}\right)\frac{1}{2}\iint_{A_m} h_m \rho_m \left(\frac{\partial w_m}{\partial t_m}\right)^2 dx_m dy_m = 0 \qquad (31)$$

The complete similitude requirement conditions for this problem is:

$$\frac{C_E C_h^{\ 3} C_w^{\ 2}}{C_L^{\ 2}} = \frac{C_h C_\rho C_w^{\ 2} C_L^{\ 2}}{C_t^{\ 2}}$$

Solve for the scaling factor for time yields:

$$C_t = \sqrt{\frac{C_\rho C_L^{\ 4}}{C_E C_h^{\ 2}}} \qquad (32)$$

Scaling of time is equivalent to scaling of the vibration period. Thus the scaling factor for frequency response is an inverse of C_t:

$$C_\omega = \sqrt{\frac{C_E C_h^{\ 2}}{C_\rho C_L^{\ 4}}}\,. \qquad (33)$$

To verify the above scaling factor, consider a rectangular plate with all edges simply supported (Fig. 5). A general solution for natural frequency of the fundamental mode is [9]:

$$\omega = \pi^2\sqrt{\frac{Eh^2}{12\rho\left(1-\nu^2\right)}}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \qquad (34)$$

The values of prototype variables are:

$E_p = 207GPa$, $\rho_p = 7000\frac{kg}{m^3}$, $\nu_p = 0.3$

$a_p = 1m$, $b_p = 2m$, and $h_p = 5mm$

The values of model variables are:

$E_m = 70GPa$, $\rho_m = 2000\frac{kg}{m^3}$, $\nu_m = 0.3$

$a_m = 0.2m$, $b_m = 0.4m$, and $h_m = 1mm$

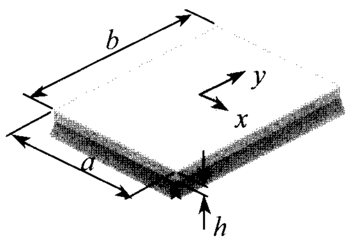

Fig. 5 Rectangular plate with all edges simply supported.

The natural frequency of a prototype plate computed by Eq. (34) is 101.51 rad/sec. The natural frequency of a model computed by the same equation is 552.17 rad/sec. The scaling factor for frequency is 0.184. Therefore, the natural frequency of a prototype beam computed by similitude theory is $0.184 \times 552.17 = 101.51$ rad/sec. Therefore, under a complete similitude condition, similitude theory predicts the exact result.

5. Discussion

As already mentioned, the proposed method is not restricted to specific boundary conditions; therefore, a scaling factor derived by this method is independent of the boundary conditions. This has already been shown in Eqs.(10) and (15a), where the scaling factor for deflection of a cantilever beam with free end and that with elastic support, are the same. Furthermore, this method can derive an implicit scaling factor, i.e. scaling factor for spring stiffness together with a scaling factor for structural behaviors. So this approach guarantees that no similitude requirement is missed.

Although the proposed method is based on the principle of conservation of energy which can be solved for a deflection of a single-load system, it is applicable to the case of a structure subjected to multiple applied loads of the same type or multiple elastic supports. This is possible because, the scaling factor is derived without solving the conservation of energy equation. The complete similitude requirement in this case requires that all applied loads and all elastic supports have to be scaled with the same proportion.

6. Conclusions

A general procedure for deriving a scaling law by an energy theorem was described. The applicability of the method was demonstrated by deriving a scaling law for both static and dynamic behaviors of a beam and plate. All of the derived scaling factors are verified with problems having an exact solution. The results showed that the derived scaling factors can help exactly predict the prototype's behavior, when the complete similitude requirements are fulfilled. The scaling factors depend on the structure type, loading types and behaviors of interest.

7. References

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