The $P(G, V)$ –Theorem

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Abstract

Let G be a group, p a prime number and V a faithful $F_p[G]$-module, where F_p is a field of p elements. Call $P(G, V)$ the set of all non-trivial elementary abelian p-subgroups A of G such that $|A|\,|C_V(A)| \geq |B|\,|C_V(B)|$ for any subgroup B of A. In this note we prove that if $A \in P(G, V)$, then A normalizes every component of G. This is another version of Thomson factorizations.

Keywords: prime number, p elements, p-subgroups, Thomson factorizations

1. Introduction

This section provides necessary information which will be used in Sections 2 and 3. Sections 2 and 3 give important information that we need for the proof of the following result:

Theorem 1.1 ($P(G, V)$ - Theorem) Let $A \in P(G, V)$. Then A normalizes every component of G.

By Definitions 1.2 and 1.4 below, in the case that $p = 2$, Theorem 1.1 has been well-known. The first published proof, due to Aschbacher, appeared in [4] in 1982, under the assumption that G is a K-group if all simple sections of G are in K, the collection of known finite simple groups. In 1982, Timmesfeld made an improvement on the "Thompson Replacement Theorem" and used it to give a fairly short proof of Theorem 1.1, for $p = 2$, removing the assumption that G is a K-group. Timmesfeld's proof cannot really be described as elementary, since it relies on the theory of groups generated by odd transpositions in order to reduce to the case when every component K of G is isomorphic to $SL(2, 2^n)$. Also, Andrew Chermak found a proof of Theorem 1.1, which handles the case that p is odd. A key step in Chermak's proof is provided by the Timmesfeld Replacement Theorem, which will be used to obtain Theorem 1.1 as a corollary to Theorem 3.4 in Section 3.

All groups and fields in this paper are finite. For a prime p, F_{p^n} denotes a field with p^n elements.

Definition 1.2 Let G be a group, p a prime, and V a faithful $F_p[G]$-module. Define $P(G, V)$ to consist of the nontrivial elementary abelian p-subgroups A of G such that:

$$|A|\,|C_V(A)| \geq |B|\,|C_V(B)|$$

for every subgroup B of A.

Notice that if B is a nontrivial subgroup of $A \in P(G, V)$ then this inequality is an equality. Then, B is in $P(G,V)$. If A is in $P(G, V)$, then $|A| \geq |V/C_V(A)|$. To see this, just take $B = 1$ in the inequality defining membership in $P(G, V)$.

Definition 1.3 Let G be a group, p a prime, and V a faithful $F_p[G]$-module. Define $P^*(G,V)$ to be the set of the minimal members of $P(G, V)$ under the partial order $B <^* A$ if $B \leq A$ and

$$|A|\,|C_V(A)| = |B|\,|C_V(B)|.$$

Note that if $A \in P^*(G, V)$, $B \leq A$, and $B \in P(G, V)$, then either $A = B$ or $B = 1$.

Definition 1.4 A group K is *quasisimple* if $K = [K,K]$ and $K/Z(K)$ is simple. A quasisimple subnormal subgroup of a group G is said to be a *component* of G.

Definition 1.5 For any p-group S, let $A(S)$ be the set of elementary abelian subgroups of maximal order of S. Define:

$$J(S) = \{A \mid A \in A(S)\}.$$

$J(S)$ is called the *Thompson subgroup* of S.

Since any automorphism of S clearly permutes the elements of $A(S)$ among themselves, it leaves $J(S)$ invariant and so $J(S)$ is a characteristic subgroup of S. Some basic properties of Thompson subgroups are listed in the following lemma.

Lemma 1.6 (i) If A is in $A(S)$, then $A = C_S(A)$. In particular, $Z(S) \subseteq A$.

(ii) If A is in $A(S)$ and $A \le R \le S$, then $A(R) \subseteq A(S)$ and $J(R) \subseteq J(S)$.

(iii) Let S be a Sylow p-subgroup of H. Then $J(S) = J(R)$ for each p-subgroup R of H containing $J(S)$.

Proof. (i) Indeed, as A is abelian, so is $\langle A, x\rangle$ for any x in $C_S(A)$. The maximality of A forces $x \in A$ and (i) follows.

(ii) If $A \le R \le S$ with $A \in A(S)$, then obviously $A \in A(R)$. Thus the elements of $A(R)$ are of the same order as those of $A(S)$ and so $A(R) \subseteq A(S)$, whence $J(R) \subseteq J(S)$.

(iii) From (ii), we have $J(S) = J(R)$, if $J(S) \subseteq R$.

Theorem 1.7 (Thompson) Let $A \in A(S)$ and suppose that $M = [x, A]$ is abelian for any element x of S. Then $MC_A(M) \in A(S)$.

Proof. Set $C = C_A(M)$. Since M is abelian, clearly MC is an abelian group. Hence we need only show that $|MC| \ge |A|$, for then $MC \in A(S)$ by definition of $A(S)$. Since $C_S(A) = A$ and M is abelian, we have $C \cap M \subseteq A \cap M = C_M(A)$. Hence:

(1) $|MC| = |M||C| / |C \cap M|$

$$\ge |M||C_A(M)| / |C_M(A)|.$$

Hence the desired conclusion $|MC| \ge |A|$ will follow from (1) provided we can prove that:

(2) $$|M/C_M(A)| \ge |A/C_A(M)|.$$

To establish (2), it will clearly suffice to show that if $u, v \in A$ lie in distinct cosets of $C_A(M)$, then the elements $[x, u]$ and $[x, v]$ of M lie in distinct cosets of $C_M(A)$. Suppose that $[x, u] \equiv [x, v] \pmod{C_M(A)}$. Then $y = [x, u]^{-1}[x, v] \in C_M(A)$. But $y = (x^{-1}x^u)^{-1}(x^{-1}x^v) = (x^u)^{-1}x^v$ and y centralizes A, so we have $y = y^{u^{-1}} = [x, vu^{-1}]$. Since y centralizes A, $[x, vu^{-1}, a] = 1$ for every a in A. But then $[x, a, vu^{-1}] = 1$, as A and $[x, A]$ are abelian. Thus vu^{-1}centralizes $[x, a]$ for all a in A and we conclude that $vu^{-1} \in C_A(M)$, contrary to the fact that u and v lie in distinct cosets of $C_A(M)$ in A.

Lemma 1.8 (Frattini argument) Suppose N is a normal subgroup of a finite group G and p is a prime. If T is a Sylow p-subgroup of N, then

$$G = N_G(T)N = NN_G(T).$$

Proof. Let $g \in G$. Since T^g is a Sylow p-subgroup of N, $T^g = T^n$ for some $n \in N$. Now $gn^{-1} \in N_G(T)$. Thus $g \in N_G(T)N = NN_G(T)$.

Lemma 1.9 (Three-Subgroup Lemma) Let X, Y, Z be subgroups of a group G with $[X, Y, Z] = [Y, Z, X] = 1$. Then $[Z, X, Y] = 1$.

Proof. See 8.7 in [1].

Definition 1.10 The *generalized Fitting subgroup* of G is $F^*(G) = F(G)E(G)$, where $F(G)$, the *Fitting subgroup* of G, is the largest nilpotent normal subgroup of G, and $E(G)$ is the subgroup of G generated by the set of components of G.

Lemma 1.11 $C_G(F^*(G)) \le F^*(G)$.

Proof. See 31.13 in [1].

We wish to say something about the motivation for the $P(G, V)$. Suppose now that S is a Sylow p-subgroup of a group H such that the generalized Fitting subgroup $F^*(H)$ is equal to $O_p(H)$. In this case, we set $V = \Omega_l(Z(O_p(H)))$. By Lemma 1.11, we obtain $C_H(O_p(H)) \le O_p(H)$, and so $Z(S) \subseteq Z(O_p(H))$. Thus, $\Omega_l(Z(S)) \subseteq \Omega_l(Z(O_p(H))) = V$. If now $J(S)$ centralizes V, then $J(S) \le C_H(V) \le H$. Putting $R = S \cap C_H(V)$, we see that R is a Sylow p-subgroup of $C_H(V)$ containing $J(S)$. Then, the Frattini argument gives $H = C_H(V)N_H(R)$. It follows from Lemma 1.6 (iii) that

$$H = C_H(V)\, N_H(J(S)).$$

So, as $\Omega_l(Z(S)) \subseteq V$, $H = C_H(\Omega_l(Z(S)))N_H(J(S))$, and so called Thompson Factorization is achieved.

Alternatively, if $[V, J(S)] \ne 1$, then there exists an element A^* of $A(S)$ with $A^*C_H(V) > C_H(V)$. Put $G = H/C_H(V)$ and let $A =$

$A^*C_H(V)/C_H(V)$ denote the image of A^* in G. For any subgroup B of A, let B^* denote the preimage of B in A^*. We claim that $A \in P(G, V)$. Note that $C_{A^*}(V) \leq B^* \leq A^*$ and that $A_0 = A^*C_V(A^*)$ is an elementary abelian p-group of S. So, as $A^* \in A(S)$, $|A_0| \leq |A^*|$. Hence $A^*=A_0$ since $A^* \leq A_0$. Thus $A^*=A^*C_V(A^*)$, and then we have $C_V(A^*)=A^*\cap V$. Put $B_0=B^*C_V(B^*)$. Then B_0 is elementary abelian, and by the maximality of A^* we have $|A^*| \geq |B^*C_V(B^*)|$ for each $B^* \leq A^*$. As $C_{A^*}(V) = C_{B^*}(V)$, we have:

$$\begin{aligned}|A^*| &= |A|\,|C_{A^*}(V)|\\ &\geq |B^*C_V(B^*)|\\ &= |B|\,|C_{B^*}(V)|\,|C_V(B^*)|/|B^*\cap C_V(B^*)|\\ &= |B|\,|C_{A^*}(V)|\,|C_V(B^*)|\,/\,|B^*\cap C_V(B^*)|.\end{aligned}$$

Then,

$$|A| \geq |B|\,|C_V(B^*|/|B^*\cap C_V(B^*)|.$$

Because $B^*\cap C_V(B^*) \leq A^*\cap V$, we have:

$$|A| \geq |B|\,|C_V(B^*)|\,/\,|A^*\cap V|.$$

So, as $C_V(A^*) = A^*\cap V$, it follows that:

$$|A| \geq |B|\,|C_V(B^*)|\,/\,|C_V(A^*)|.$$

So that $|A|\,|C_V(A^*)| \geq |B|\,|C_V(B^*)|$. But $C_V(A^*) = C_V(A)$ and $C_V(B^*) = C_V(B)$, so we now get:

$$|A|\,|C_V(A)| \geq |B|\,|C_V(B)|.$$

Hence $A \in P(G, V)$, as claimed.

Thus $P(G, V)$ provides a means by which to measure the possible failure of Thompson factorization.

2. The Timmesfeld Replacement Theorem

In this section, G is a group, V is a faithful $F_p[G]$-module, and $A \in P(G, V)$.

Definition 2.1 Let $S(G)$ denote the set of all subgroups of G. Define

$$m = m(G, V) = max\{|A|\,|C_V(A)| : A \in S(G)\}.$$
$$M = M(G, V) = \{A \in S(G) : |A|\,|C_V(A)| = m\}.$$

Lemma 2.2 Let $A,B \in M(G, V)$. Then $A \cap B \in M(G, V)$.

Proof. Let A and B be in M. Then, as $C_V(\langle A, B\rangle) = C_V(AB)$, we have:

$$\begin{aligned}|A|\,|C_V(A)| &\geq |\langle A, B\rangle|\,|C_V(\langle A, B\rangle)|\\ &\geq |AB|\,|C_V(AB)|.\end{aligned}$$

Then also:

$$|A|\,|C_V(A)| \geq |A|\,|B|\,|C_V(AB)|\,/\,|A\cap B|,$$

and so $|A\cap B|\,|C_V(A)| \geq |B|\,|C_V(AB)|$. Thus,

$$\begin{aligned}&|A\cap B|\,|C_V(A\cap B)|\\ &\quad\geq |B|\,|C_V(AB)|\,|C_V(A\cap B)|\,/\,|C_V(A)|\end{aligned}$$

Now $C_V(A)C_V(B) \leq C_V(A\cap B)$ and so we have:

$$\begin{aligned}&|C_V(A)|\,|C_V(B)|\,/\,|C_V(A)\cap C_V(B)|\\ &\quad= |C_V(A)C_V(B)| \leq |C_V(A\cap B)|.\end{aligned}$$

Since $C_V(A) \cap C_V(B) = C_V(AB)$, it follows that:

$$|C_V(B)| \leq |C_V(A\cap B)|\,|C_V(AB)|\,/\,|C_V(A)|,$$

and hence:

$$\begin{aligned}&|B|\,|C_V(B)|\\ &\quad\leq |B|\,|C_V(A\cap B)|\,|C_V(AB)|\,/\,|C_V(A)|\\ &\quad\leq |A\cap B|\,|C_V(A\cap B)|.\end{aligned}$$

Since $B \in M$, it now follows that $A\cap B \in M$.

Theorem 2.3 (Thompson Replacement) Let $x \in V$, $W = [x, A]$ and $A_x = C_A(W)$. Then :

(i) $A_x \in P(G, V)$.

(ii) If $A \in P^*(G, V)$ then $[V, A, A] = 0$.

Proof. (i) Let $c \in C_A(W)$. Notice that if $[W, c] = [x, A, c] = 0$, then $[x, c, A] = 0$ by the Three Subgroups Lemma, so $[x, c] \in C_W(A)$. Indeed for $a, b \in A$, if $aC_A(W) = bC_A(W)$, then $a = bc$ for some $c \in C_A(W)$. Hence $[x, a] + C_W(A) = [x, bc]+C_W(A) = [x, c] + [x, b]^C + C_W(A)$. As $[x, b]^C = [x, b]$ and $[x, c] \in C_W(A)$, we have $[x, a] + C_W(A) = [x, b] + C_W(A)$. It follows that there is a well-defined mapping

$$\phi : A/C_A(W) \rightarrow W/C_W(A)$$

given by $\phi(aC_A(W)) = [x, a] + C_W(A)$ for any $a \in A$. We claim that ϕ is injective. For $a, b \in A$, suppose that $\phi(a\,C_A(W)) = \phi(b\,C_A(W))$. Then we have $[x, b] \in [x, a] + C_W(A)$, so $[x, b] - [x, a] \in C_W(A)$. Conjugating by a^{-1}, we obtain $[x, ba^{-1}] \in C_W(A)$. The Three Subgroups Lemma gives $[x, A, ba^{-1}] = [W, ba^{-1}] = 0$, so $b \in aC_A(W)$, and hence $aC_A(W) = bC_A(W)$. Thus ϕ is injective, as claimed. We now have that $|A/C_A(W)| \leq |W/C_W(A)|$. Then :

$$|A|\,|C_W(A)| \leq |W|\,|C_A(W)|.$$

Then also :

$$\begin{aligned}&|A|\,|C_V(A)|\\ &\quad= |A|\,|C_W(A)|\,|C_V(A) : C_W(A)|\\ &\quad\leq |W|\,|C_A(W)|\,|C_V(A) : C_W(A)|\\ &\quad= |W|\,|C_A(W)|\,|C_V(A) : C_V(A)\cap W|\\ &\quad= |C_A(W)|\,|W + C_V(A)|.\end{aligned}$$

But $A \in P(G, V)$, so the above inequalities are equalities. Thus, we have $C_V(A_x) = [x, A] + C_V(A)$, and $|A_x|\,|C_V(A_x)| = |A|\,|C_V(A)|$. Therefore $A_x \in P(G, V)$ or $A_x = 1$, and in the latter case $V = C_V(A_x) = [x,$

$A] + C_V(A) = W + C_V(A)$. But $[V/C_V(A), A] \neq V$ and so $W + C_V(A) \neq V$, a contradiction. Thus $A_x \in P(G, V)$.

(ii) If $A \in P^*(G, V)$ then either $A = A_x$ or $A_x = 1$. But $A_x \neq 1$, and so $A = A_x$ centralizes $[x,A]$ for each $x \in V$, and therefore $[V,A,A] = 0$.

Theorem 2.4 (Timmesfeld Replacement) Let $B = C_A([V, A])$. Then $C_V(B) = [V, A] + C_V(A)$, and

$$|B|\,|C_V(B)| = |A|\,|C_V(A)|.$$

In particular, we then have $B \in P(G, V)$.

Proof. Put $U = [V, A] + C_V(A)$, put $m = m(A, V)$, $M = M(A, V)$, $n = m(A, U)$, and $N = M(A, U)$. Then $m = |A|\,|C_V(A)|$, by the definition of $P(G, V)$. Further, for any subgroup B of A, we have:

$$|B|\,|C_U(B)| \leq |B|\,|C_V(B)| \leq |A|\,|C_V(A)| = |A|\,|C_U(A)|.$$

This shows that $m = n$. If now $B \in N$ we can conclude that $B \in M$, and so $N \subseteq M$. For any B in N we have $C_U(B) = C_V(B)$. By Theorem 2.3, we have $A_x \in M$ for every $x \in V$, and $C_V(A_x) = [x, A] + C_V(A) \subseteq [V, A] + C_V(A) = U$. Hence $A_x \in N$ for all x. Setting $B = C_A([V, A])$, we have $B = \cap\{A_x\}_{x \in V}$, and then $B \in N$, by Lemma 2.2. Therefore $C_U(B) = C_V(B)$, so $C_V(B) \subseteq U$. Since $B \subseteq A_x$ for all $x \in V$, we have $C_V(A_x) \subseteq C_V(B)$ for all $x \in V$, and so $\cup\{C_V(A_x)\}_{x \in V} \subseteq C_V(B)$. But $U = \cup\{[x, A] + C_V(A)\}_{x \in V}$, that is, $U = \cup\{C_V(A_x)\}_{x \in V}$. Hence $C_V(B) = U$, that is, $C_V(B) = [V, A] + C_V(A)$. Then also $|B|\,|C_V(B)| = |A|\,|C_V(A)|$, therefore $B = 1$ or $B \in P(G, V)$. If now $B = 1$, then $V = C_V(B)$, and so $V = U$, a contradiction. Hence $B \in P(G, V)$.

3. Theorem 3.4

This section is devoted to supplying important information, which will be used in the proof of the $P(G, V)$-Theorem.

Lemma 3.1 Let p be a prime, V a vector space over F_p and G a subgroup of $GL(V)$. Let W be a G-invariant subspace of V and $A \in P(G, V)$. Then

(i) $AC_G(W)/C_G(W) \in P(G/C_G(W), W)$.

(ii) If $A \in P^*(G, V)$ then $AC_G(W)/C_G(W) \in P^*(G/C_G(W), W)$. Moreover if $|A : C_A(W)| = |W : C_W(A)|$, then $V = W + C_V(A)$.

Proof. (i) Let $C_A(W) \leq B \leq A$, and put $X = C_V(B)$ and $Y = C_V(A)$. Then :

$$\begin{aligned}|B|\,|W\cap X|\,|X : W\cap X| &= |B|\,|X| \\ &\leq |A|\,|Y| \\ &= |A|\,|W\cap Y|\,|Y : W\cap Y|.\end{aligned}$$

If now $|A|\,|W\cap Y| < |B|\,|W\cap X|$ we conclude that $|X : W\cap X| < |Y : W\cap Y|$, and so

$$|X + W : W| < |Y + W : W|,$$

contrary to $Y \subseteq X$. Thus, we have $|A|\,|W\cap Y| \geq |B|\,|W\cap X|$. But $W \cap Y = C_W(A)$ and $W \cap X = C_W(B)$, hence $|A|\,|C_W(A)| \geq |B|\,|C_W(B)|$ for every subgroup B of A. Notice that $C_A(W) = C_B(W)$, so

$$\begin{aligned}|A|\,|C_W(A)|\,/\,|C_G(W)\cap A| &= |A|\,|C_W(A)|\,/\,|C_A(W)| \\ &\geq |B|\,|C_W(B)|\,/\,|C_B(W)| \\ &= |B|\,|C_W(B)|\,/\,|C_G(W)\cap B|.\end{aligned}$$

Then, we have: $|AC_G(W)/C_G(W)|\,|C_W(A)| \geq |BC_G(W)/C_G(W)|\,|C_W(B)|$, as required.

(ii) See 3.2 in [2].

The results which are not utterly elementary that we require for the proof of Theorem 3.4, are included in the following list.

(1) Let V be an irreducible $F_p[G]$-module, and let F be the field $End_G(V)$ of G-endomorphisms of V. Suppose that $G = H \times K$ is a direct product, U is an irreducible $F[H]$-submodule, and W an irreducible $F[K]$-submodule of V. Then for any fixed choice of bases for U and W, over F, there is a canonical identification of V with the tensor product $U \otimes_F W$. This is an essential result of Theorem 3.7.1 in [5].

(2) Any quasisimple subgroup of $SL(2, 2^m)$ is conjugate to a natural $SL(2, 2^n)$-subgroup (consisting of all matrices in $SL(2, 2^m)$ written over F_{2^n}), for some divisor n of m.

(3) Concerning automorphisms of $SL(2, 2^n)$, any non-inner automorphism of $SL(2, 2^n)$ of order 2 is conjugate to a field automorphism.

To begin the proof of Theorem 3.4, we state the following hypothesis.

Hypothesis 3.2 Let G be a finite group, p a prime, V a faithful $F_p[G]$-module, and let K be a component of G. Let A be a non-cyclic elementary abelian p-subgroup of G, and let B be a maximal subgroup of A. Assume that

(i) $[V, A, B] = 0$;

(ii) $C_A(K) = 1$ and

(iii) $N_A(K) \subseteq B$.

Put $L = \langle K^A \rangle$. Without loss of generality, we may assume that $G = LA$. Let Λ denote the set of components of G and write $\Lambda = \{K_1, K_2, \ldots, K_t\}$. We then have $L = K_1K_2\ldots K_t$, where $[K_i, K_j] = 1$ for all i and j with $i \neq j$. So $t = |A : N_A(K)|$.

For any $K_i, K_j \in \Lambda$, we have $K_i = K^{a_i}$ and $K_j = K^{a_j}$ for some $a_i, a_j \in A$. Then we have $a = a_i^{-1}a_j \in A$ with $K_i^a = K_j$. Thus A acts transitively on Λ by conjugation.

Lemma 3.3 Let G, V, K, A, and B satisfy Hypothesis 3.2. Suppose $a \in A\text{-}B$, and X is a component of $C_G(a)$. Then the following conditions hold:

(i) X^A is the set of all components of $C_G(a)$.

(ii) $L = \langle X^G \rangle$.

(iii) $X \subseteq [B, X] \subseteq \langle B^X \rangle$.

Proof (i). We have $N_A(K) \subseteq B$ by assumption, so $a \notin N_A(K)$. The transitivity of A on Λ implies that a fixes no component of G. Without loss of generality, we may assume that indexing has been chosen so that the orbits for $\langle a \rangle$ on Λ are

$$\{K_1, \ldots, K_p\}, \ldots, \{K_{(s-1)p+1}, \ldots, K_t\},$$

where $t = sp$. Put:

$$Y_i = K_{(i-1)p+1} \ldots K_{ip},$$

$$X_i = \{ x x^a x^{a^2} \ldots x^{a^{p-1}} \mid x \in K_{(i-1)p+1} \}$$

for $1 \le i \le s$. Then $X_i \cong K_{(i-1)p+1}$, for $1 \le i \le s$, and then also $X_1, \ldots, X_s$ are the components of $C_G(a)$. Since A permutes the $\langle a \rangle$-orbits on Λ transitively, we obtain part (i) of the lemma.

(ii). Since $\langle X^G \rangle$ is a normal, perfect subgroup of G contained in L, it is a product of components of L. But evidently no proper sub-products of components of G contain each X_i, so we obtain $L = \langle X^G \rangle$.

(iii). Take $X = X_1$ and $K = K_1$. Since $C_A(K) = 1$, we have $[B, K] \neq 1$, and since $X \subseteq Y_1 = \langle K^{\langle a \rangle} \rangle$, it follows that $[B, X] \neq 1$. Note that X is a component of $C_G(a)$ and $B \subseteq C_G(a)$, it follows that $[B, X] \subseteq \langle B^X \rangle$. In particular, we have $X \subseteq [B, X]$, and this yields (iii).

Theorem 3.4 Let G, V, K, A, and B satisfy Hypothesis 3.2. Then $p = 2$, $B = N_A(K) \subseteq C_G(K)K$, and $K/Z(K)$ is isomorphic to $SL(2, 2^n)$ for some integer n, $n > 1$.

Proof. Let M be a non-trivial irreducible G-invariant section of V, and let N be a non-trivial irreducible L-submodule of M. Since N is L-invariant, so $\langle N^A \rangle$ is L-invariant. As $\langle N^A \rangle$ is A-invariant, it follows that $\langle N^A \rangle$ is G-invariant. Hence $M = \langle N^A \rangle$. Since $[M, L] \neq 0$, we also have $[N, L] \neq 0$. For any a in $A\text{-}B$, it follows from Lemma 3.3 (ii) that there exists a component X of $C_G(a)$ with $[N, X] \neq 0$. On the other hand, we have $[V, B, a] = 0$ by hypothesis, so $[N, \langle B^X \rangle, a] = 0$. From Lemma 3.3 (iii), $X \subseteq \langle B^X \rangle$, so $[N, X, a] = 0$. Thus, we have shown that:

$$0 \neq [N, X] \subseteq C_N(a) \subseteq N \cap N^a$$

for any a in $A\text{-}B$. Since $N \cap N^a$ is a L-submodule and N is irreducible for L, we conclude that $N = N \cap N^a$. Since $\langle A\text{-}B \rangle = A$ we can conclude that N is A-invariant. Thus, $M = \langle N^A \rangle = N$, and any non-trivial irreducible G-invariant section of V is in fact irreducible for L. Let F be the field $End_L(N)$ of L-endomorphisms of N, and U is an irreducible K-submodule of N over F, and let A_0 be a complement in A to $N_A(K)$, that is, $A_0 \le A$ such that $A = N_A(K)A_0$ and $N_A(K) \cap A_0 = 1$. Notice that, for any a in A_0, U^a is an irreducible F-submodule of N for K^a. It then follows that, as an $F[L]$-module, N can be written as the tensor product over F of the subspaces U^a, for $a \in A_0$. Using A_0 to identify each K_i with K, we may write:

$$N = U \otimes U \otimes \ldots \otimes U$$

where L is represented on N in the ordinary component-wise way, and where A_0 acts by regularly permuting the tensor factors. Now, for each pair of elements u and v of U, define elements $x_{u,v}$, $y_{u,v}$, and $c_{u,v}$ of N as follows:

$$x_{u,v} = v \otimes u \otimes u \otimes \ldots \otimes u,$$

$$y_{u,v} = u \otimes v \otimes u \otimes \ldots \otimes u, \text{ and}$$

$$c_{u,v} = y_{u,v} - x_{u,v} = (u \otimes v - v \otimes u) \otimes u \otimes \ldots \otimes u.$$

By hypothesis, $N_A(K) \subseteq B$, so $A = N_A(K)A_0 \subseteq BA_0$. But $BA_0 \subseteq A$ it then follows that $A =$

$N_A(K)A_0 = BA_0 = A_0B$, and we, without loss of generality, may assume that there exists $a \in A_0 - B$ with $K_1^a = K_2$. Notice that then $y_{u,v} = x^a_{u,v}$, and that $[x_{u,v}, a] = -x_{u,v} + x^a_{u,v} = -x_{u,v} + y_{u,v} = c_{u,v}$. By hypothesis we have $[V, A, B] = 0$ and $[x_{u,v}, a] \in [V, A]$, we then get B centralizes $c_{u,v}$. Suppose that $t \neq 2$. There then exists a component X of $C_G(a)$ such that X is not contained in K_1K_2. But $X \subseteq \langle B^X \rangle$, and it follows that X centralizes $c_{u,v}$ for all choices of u and v. By our construction, X acts non-trivially on some tensor factors of N after the first two. So, we have a contradiction, and thus $t = 2$. It follows that:

$$|A : N_A(K)| = |A_0| = 2,$$

and either $B \cap A_0 = 1$ or $B \cap A_0 = A_0$. If now $B \cap A_0 = A_0$, then $|A| = |B|$, and it contradicts $|A : B| = p$. Thus $B \cap A_0 = 1$, then also $B = N_A(K)$ and $p = 2$.

It will be convenient to work with the semidirect product $G = LA$, where $L = K \times K$ and A operates on L by the formulas: $(g,h)^a = (h,g)$ and $(g,h)^b = (g^b, h^b)$, for $(g,h) \in L$ and for $a \in A_0$, $b \in N_A(K)$. Put $Y = C_L(a)$ where $a \in A_0 - B$. Then $Y = \{ (g,g) \mid g \in K \} \cong K$ and $Y \subseteq \langle B^Y \rangle$. By hypothesis, we have that $[V, a, B] = 0$. Hence $[N, a, Y] = 0$. Observe that:

$$[N, a] = \langle u \otimes v + v \otimes u \mid u,v \in U \rangle = U \wedge U$$

and thus Y centralizes $U \wedge U$, where U is a non-trivial irreducible Y - module. If we identify $U \otimes U$ with $Hom_F(U^*, U)$, we then have $U \wedge U \subseteq Hom_{F[Y]}(U^*, U)$, and thus $dim_F(U \wedge U) \leq 1$, by Schur's Lemma. Then $dim_F U \leq 2$, and so $K/C_K(U)$ is isomorphic to a quasisimple subgroup of $SL(2, F)$. We conclude that $K/C_K(U)$ is a natural $SL(2, 2^n)$ - subgroup of $SL(2, F)$, for some n with $n > 1$. If we write $F = F_{2^m}$, then $m = rn$ for some integer r, and U is a direct sum of r copies of a natural module for K over F_{2^n}. Then also N is a direct sum of r submodules for L. But N was chosen to be irreducible as a module for L, we conclude that $r = 1$, also $F = F_{2^n}$, and $K/C_K(U) \cong SL(2, 2^n)$.

We now show that $B \subseteq C_G(K)K$. Suppose that it is not true, there exists $b \in B$ such that b induces an outer automorphism of K by $b: x \to (x)b$, $x \in K$. Then b also induces an outer automorphism on $K/Z(K)$ by $\pi: Z(K)x \to Z(K)(x)b$, $Z(K)x \in K/Z(K)$. The structure of $Aut(K/Z(K))$ is well known, and one may conclude that b is conjugate to a field automorphism of $K/Z(K)$. Since $Z(K) = C_K(N)$, b also induces a field automorphism on N, where N is isomorphic to a direct sum of two copies of a natural $SL(2, 2^n)$-module for K. But then b centralizes no F-subspaces of N, which is contrary to $[N, A, b] = 0$. Thus $B \subseteq C_G(K)K$, as required.

Theorem 2.4 and Theorem 3.4 are technical results necessary to the proof of the $P(G, V)$-Theorem. An important step in the proof of the $P(G, V)$-Theorem begins by assuming that the theorem is false. Then let $A \in P(G, V)$, let K be a component of G such that A does not normalize K, and put $B = C_A([V, A])$, $L = \langle K^A \rangle$ and $G = LA$. Then we must show that Hypothesis 3.2 holds. Toward that we use Theorem 2.4 and Theorem 3.4 to show that V has an irreducible K-submodule Z of W, where W is an irreducible G-submodule, and finally we show that Z is L-invariant which contradicts the irreducibility of W. This proof is thereby complete.

References

[1] Aschbacher, M., Finite Group Theory, Cambridge University Press, Cambridge, 1986.

[2] Aschbacher, M., On the Failure of the Thompson Factorization in 2-Constrained Group, Proc. London Math. Soc., Vol. 43, pp. 425-449, 1981.

[3] Aschbacher, M., A Factorization Theorem for 2-Constrained Group, Proc. London Math. Soc., Vol. 43, pp. 450-477, 1981.

[4] Aschbacher, M., GF(2)-Representations of Finite Groups, Amer. J. Math., Vol. 104, pp. 683-771, 1982.

[5] Chermak, A., An Elementary Proof of the P(G,V)-Theorem, Department of Mathematics, Kansas State University, Manhattan, USA.

[6] Glauberman, G., Weakly Closed Element of Sylow Subgroups, Math. Z., Vol. 107, pp. 1-20, 1968.

[7] Gorenstein, D., Finite Groups, Harper and Row, New York, 1968.