Lie Algebraic Methods of Light Optics for Lens System Design used in OTIS Architecture

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Abstract

The use of Lie algebraic Methods offers the calculation of high order aberrations and also new insight into the origin and possible correction of aberrations. In this paper, it will be shown using Lie algebraic methods, how to describe the transposition of the OTIS, characterise its optical elements and simplify the calculation of their aberrations. A preliminary numerical study of its optical elements indicates that, by simultaneously removing almost all third-order (and higher-order) aberrations, it is possible to design a lens system, a Fourier transform lens, having an error function approaching zero.

Keywords: Lie algebraic method, Lie Optics, Optical interconnection.

1. Introduction

The propagation of a ray in an optical system can be described in many cases by a Hamiltonian. The Hamiltonian formulation of geometrical optics describes light rays as points (**q**, **p**) in an optical *phase space*. The position coordinate of a ray $\mathbf{q}=(q_x,q_y)$ determines the intersection with the z = 0 plane. The momentum coordinate **p** $=(p_x,p_y)$ is the projection, on the plane, of the ray direction vector. Fermat's principle leads to an optical Lagrangian [1,2,3] from which the canonical momentum **p** is shown to be a vector in the z = constant plane, along the projection of the ray on the plane, of magnitude $p=n\sin\theta$, where n is the refractive index of the medium at (**q**, z), and θ is the angle between the ray and the z axis. The initial and final values of the system are represented by w^i and w^f, respectively. The set of vectors w is the optical phase space. The initial conditions determine the final conditions and can be expressed in terms of a functional relationship that is denoted formally as the optical transfer map or optical symplectic map. The optical symplectic map can be written as a product of Lie transformations. The Lie transformation is a linear operator acting on functions of phase space, and is formally defined by the exponential series.

In thin lens (axis-symmetry) system as shown in Figure 1, the *object* phase space$(\mathbf{q}^o,\mathbf{p}^o)$ in geometrical optics is transformed canonically to the corresponding *image* space $(\mathbf{q}^i,\mathbf{p}^i)$ through its representation matrix which is the composite of the operators of paraxial free propagation ($\mathcal{F}_1$ and $\mathcal{F}_2$) in a homogeneous medium of refractive index n after the object and before the screen, by distances z_1 and z_2, and the action of the refraction surfaces ($\mathcal{R}_t$). The concatenation of these operators produces the Lie map of the system $\mathcal{M}=\mathcal{F}_1\mathcal{R}_t\mathcal{F}_2$. These operators are Lie (exponential) operators, and they are given by

$$\mathcal{F}_1 := \exp\left[\frac{iz_1}{\lambda\!\!\!^{-}}\left(-n+\frac{1}{2n}\sum_j \mathrm{P}_j\mathrm{P}_j\right)\right]=\mathcal{G}\left\{0,-\frac{z_1 n}{\lambda\!\!\!^{-}{}^2},\begin{pmatrix}1 & -z_1/n1\\ 0 & 1\end{pmatrix}\right\}$$

$$\mathcal{R}_t := \exp\left[-\frac{i(n-n')}{\lambda\!\!\!^{-}}+\sum_{j,k}\mathrm{Q}_j B_{jk}\mathrm{Q}_k\right]=\mathcal{G}\left\{0,0,\begin{pmatrix}1 & 0\\ B1 & 1\end{pmatrix}\right\}$$

$$\mathcal{F}_2 := \exp\left[\frac{iz_2}{\lambda\!\!\!^{-}}\left(-n+\frac{1}{2n}\sum_j \mathrm{P}_j\mathrm{P}_j\right)\right]=\mathcal{G}\left\{0,-\frac{z_2 n}{\lambda\!\!\!^{-}{}^2},\begin{pmatrix}1 & -z_2/n1\\ 0 & 1\end{pmatrix}\right\} \tag{1}$$

where $\lambda\!\!\!^{-}=\lambda/2\pi$ and n' is the refractive index of thin lens. Let $z_1=z_2=f$ (focal length of thin lens), n (air) = 1, and $B=-\frac{1}{f}$. The transformation $\mathcal{M}$ of the thin lens system is given by

$$
\begin{aligned}
\mathcal{M} &= \mathcal{G}\left\{\mathbf{0},-\frac{f}{\lambda^2},\begin{pmatrix}\mathbf{1} & f\mathbf{1}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}\right\}\mathcal{G}\left\{\mathbf{0},0,\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ -1/f\,\mathbf{1} & \mathbf{1}\end{pmatrix}\right\}\mathcal{G}\left\{\mathbf{0},-\frac{f}{\lambda^2},\begin{pmatrix}\mathbf{1} & f\mathbf{1}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}\right\}\\
&= \mathcal{G}\left\{\mathbf{0},-\frac{2f}{\lambda^2},\begin{pmatrix}\mathbf{0} & f\mathbf{1}\\ -1/f\,\mathbf{1} & \mathbf{0}\end{pmatrix}\right\}\\
&= \exp\{(-f/2):\mathbf{p}^2:\}\exp\{(1-n')/(2r):\mathbf{q}^2:\}\exp\{(-f/2):\mathbf{p}^2:\}
\end{aligned} \tag{2}
$$

where r is the radius of thin lens. The Lie operators $:h:$ is defined as:

$$
:h: := \sum_{i=1}^{2}\left(\frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}-\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i}\right) := \{h,\bullet\} \tag{3}
$$

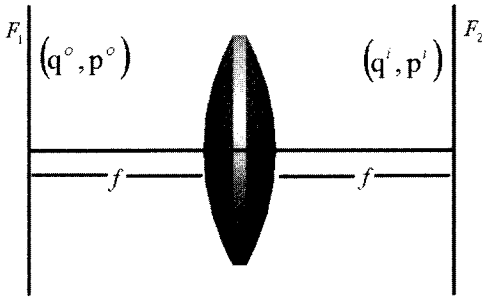


Figure 1. Thin lens system.

The matrix $\begin{pmatrix}\mathbf{0} & f\mathbf{1}\\ -1/f\,\mathbf{1} & \mathbf{0}\end{pmatrix}$ from equation (2) is corresponding to the transformation matrix in geometrical optics [5], i.e.:

$$
\begin{pmatrix}q_2\\ p_2\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q_1\\ p_1\end{pmatrix}=M\begin{pmatrix}q_1\\ p_1\end{pmatrix} \tag{4}
$$

where M are 2×2 matrices of determinant 1. These kinds of matrices are called (linear) canonical transformations or symplectic transformations. Any refracting lens system can be considered as the composite of several systems of two basic types:

(a) Afer translation by a distance t, the formula is:

$$
\begin{pmatrix}q_2\\ p_2\end{pmatrix}=\begin{pmatrix}1 & T\\ 0 & 1\end{pmatrix}\begin{pmatrix}q_1\\ p_1\end{pmatrix}
$$

$$
\det\begin{pmatrix}1 & T\\ 0 & 1\end{pmatrix}=1,\ T={t}/{n} \tag{5}
$$

where n is the refractive index of the medium.

(b) Refraction at the boundary surface between two regions of refractive indices n_1 and n_2, the formula is:

$$
\begin{pmatrix}q_2\\ p_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ -\rho & 1\end{pmatrix}\begin{pmatrix}q_1\\ p_1\end{pmatrix}
$$

$$
\det\begin{pmatrix}1 & 0\\ -\rho & 1\end{pmatrix}=1,\ \rho=\frac{n_2-n_1}{R} \tag{6}
$$

where R is the radius of the refracting surface.

From these two results, we can calculate the matrix of the thin lens (a double convex lens) between a reference planes F_1 located a distance f (a focal length of the thin lens) to the left of the lens and a reference plane F_2 located a distance f to the right (Figure 1), then we have:

$$
\begin{pmatrix}q_2\\ p_2\end{pmatrix}=\begin{pmatrix}0 & f\\ -{1}/{f} & 0\end{pmatrix}\begin{pmatrix}q_1\\ p_1\end{pmatrix} \tag{7}
$$

where $\frac{1}{f}=(n_2-n_1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$, R_1 is the radius of the left refracting surface and R_2 is the radius of the right refracting surface. From equation (7), the elements A and D of the matrix are zeros. This means there are two cases we will consider as: 1) $D = 0$. This means that all rays entering the input plane from the same point emerge at the output plane making the same angle with the axis, no matter at what angle the rays enter the system. In another words, the position of the ray is transformed into an angle. 2) $A = 0$. This means that all rays entering the system at the same angle will pass through the same point in the output plane. Therefore, the angle is transformed into the position of the ray [6]. With two cases it shows that the lens system transforms the coordinate system (q,p) by rotating the axes by 90^o :

$$
(q,p)\rightarrow(-p,q) \tag{8}
$$

In this paper, we used Lie algebraic methods to describe the system called the Optical Transpose Interconnection System (OTIS). The OTIS system, which is described in Ref. 4, has three stages of lenses, an array of microlenses in stage I, a macrolens in stage II, and an array of microlenses in stage III, as shown in Figure 2.

2. Fundamental Theory of the OTIS

The transpose interconnection of the system is a one-to-one mapping between $N \times M$ input to $N \times M$ output beamlets. The input and output beamlets are arranged as an $\sqrt{N} \times \sqrt{M}$ array of $\sqrt{N} \times \sqrt{M}$ sub-array. Each $\sqrt{N} \times \sqrt{M}$ sub-array of input and output beamlets are at the front and back focal planes of each of the lenses of stage I and III which are arranged as an $\sqrt{N} \times \sqrt{M}$ array, respectively. Each input and output beamlets has a coordinate (n,m) where $n,m = 1,\ldots,\sqrt{N} \times \sqrt{M}$. The input beamlet with the coordinate (n,m) is mapped to the output beamlet with the coordinate (m,n), called the transposition of the input. We can write as the equation:

$$(n,m) \rightarrow (m,n) \tag{9}$$

The sources of input beamlets are telecentric sources, i.e. the chief ray of each source is parallel to the optical axis of the corresponding lens in stage I. The microlenses in stage I are positioned one focal length away from the sources. The macrolens in stage II and microlenses array in stage III are positioned as shown in Figure 2(a) so that the images of the input beamlets can be Fourier transformed at the output. Figure 2(b) shows the diagram of the system for N and M = 16. The system composites of 256 input and output beamlets are arranged as an 4×4 array of 4×4 sub-array, an 4×4 array of lenses in stage I, a macrolens in stage II, and an 4×4 array of lenses in stage III. The input beamlet with coordinate (1,1), for example, on the microlens n = 1 in stage I is mapped to the output beamlet with coordinate (1,1) on the microlens n = 1 in stage III, similarly the input beamlet with coordinate (1,2) on the microlens n = 1 in stage I is mapped to the output beamlet with coordinate (2,1) on the microlens n = 2 in stage III, and so on. Hence, the transposition of the input coordinates to the output coordinates can be done by three lenses. Therefore, the optical transpose interconnection system composite of three stages of lenses to transform the input beamlets with coordinates $\begin{pmatrix} q \\ p \end{pmatrix}_{nm}$ into the coordinates of the output beamlets $\begin{pmatrix} q \\ p \end{pmatrix}_{mn}$. Equation (10) below is the equation of the system:

$$\begin{pmatrix} q \\ p \end{pmatrix}_{nm} = \begin{pmatrix} 0 & -\dfrac{f_1 f_3}{f_2} \\ \dfrac{f_2}{f_1 f_3} & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}_{mn} \tag{10}$$

or

$$\begin{pmatrix} q \\ p \end{pmatrix}_{nm} = \begin{pmatrix} -\dfrac{f_1 f_3}{f_2} p \\ \dfrac{f_2}{f_1 f_3} q \end{pmatrix}_{mn} \tag{11}$$

where f_1, f_2, and f_3 are the focal lengths of lenses in stage I, II, and III, respectively, and the effective focal length of the system is $\frac{f_1 f_3}{f_2}$. By Gaussian beam analysis and linear optical design, the optical transpose interconnection system [8] has been designed and simulated as shown in Figure 3.

3. Aberrations in the OTIS

Let $\mathcal{M}$ be any symplectic map having a Taylor series expansion of the form,

$$\begin{aligned} z_a^f &= K_a + \sum_b R_{ab} z_b^i + \sum_{bc} T_{abc} z_b^i z_c^i \\ &+ \sum_{bcd} U_{abcd} z_b^i z_c^i z_d^i + \ldots \end{aligned} \tag{12}$$

An incoming initial ray is specified by the phase-space coordinates z^i, and the outgoing final ray has the corresponding phase-space coordinates z^f. From equation (12), it can be shown that $\mathcal{M}$ can be written uniquely in the factor product form,

$$\begin{aligned} \mathcal{M} = &\exp(:f_1:)\exp(:f_2:)\exp(:f_3:)\exp(:f_4:)\ldots \\ &\exp(:f_m:) \end{aligned} \tag{13}$$

where each function f_m is a homogeneous polynomial of degree m in the variables z^i. By definition, knowledge of $\mathcal{M}$ is equivalent to knowledge of the relation between the initial conditions z^i and the final conditions z^f. According to equation (13), knowledge of $\mathcal{M}$ amounts to determining certain homogeneous polynomials $f_1, f_2, f_3 f_4$, etc. In light optics, the factorization theorem indicates that the effect of any collection of elements can be characterized by a set of homogeneous polynomials. It can be shown that the polynomial f_1 reproduces the constant terms K_a in the Taylor expansion, and the polynomial f_2 reproduces the linear matrix (paraxial optics) terms R_{ab}. And the polynomials f_3, f_4, etc. describe departures from paraxial optics and reproduce the nonlinear terms T_{abc}, U_{abcd}, etc. Moreover, unlike the terms K_a, R_{ab}, T_{abc}, etc., the polynomials f_m are all independent. Much of geometrical light optics is concerned with systems composed of lenses having axial symmetry and constant index of refraction. For such systems the transfer map $\mathcal{M}$ can be written in the form

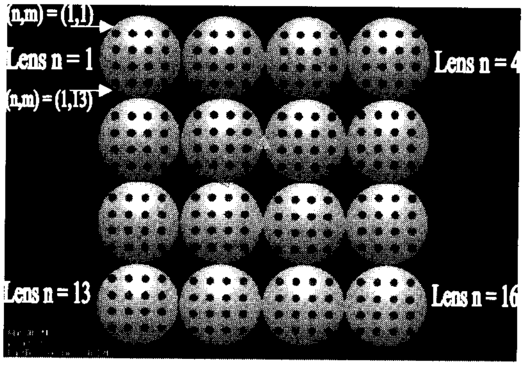
$$\mathcal{M} = \exp(:f_2:)\exp(:f_4:)\exp(:f_6:)\ldots \quad (14)$$

Only f_m in equation (13) with even m occurs due to symmetry, and it can depend only on the variables $\mathbf{p}^2, \mathbf{q}^2$, and $\mathbf{p}\cdot\mathbf{q}$. From equation (14), the results of the equation (2) is one part of an expression of the form

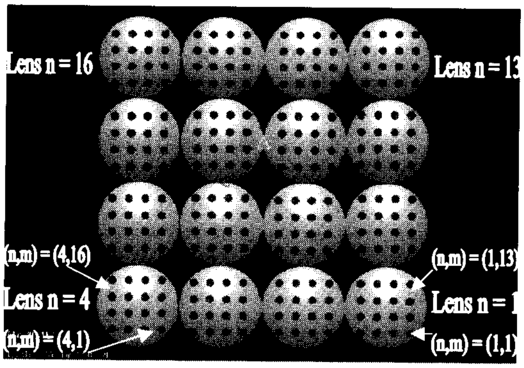
$$\mathcal{M} = \mathcal{M}_G \exp(:h_4^*:)\ldots \quad (15)$$

The quantity $\mathcal{M}_G$ denotes the Gaussian (paraxial) portion of the map ($\exp(:f_2:)$) and is given by the relation of the equation (2). The last term, $\exp(:h_4^*:)$, describes third order departure (aberrations) from Gaussian optics. In general, h_4^* has an expression of the form

$$h_4^* = A^*(\mathbf{p}^2)^2 + B^*\mathbf{p}^2(\mathbf{p}\cdot\mathbf{q}) + C^*(\mathbf{p}\cdot\mathbf{q})^2 + D^*\mathbf{p}^2\mathbf{q}^2 + E^*(\mathbf{p}\cdot\mathbf{q})\mathbf{q}^2 + F^*(\mathbf{q}^2)^2 \quad (16)$$



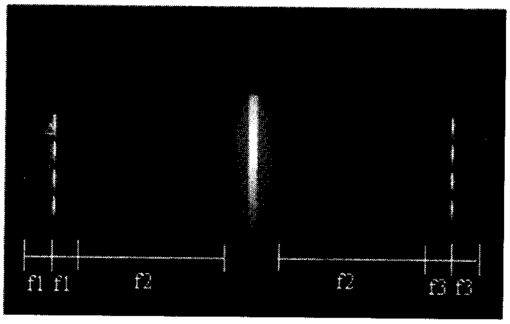
Coordinates of lenses and inputs in stage I



Coordinates of lenses and outputs in stage III

(b) Front View of inputs, outputs, and mesolenses in stage I and III.

Figure 2. Diagram of the OTIS with $N = M = 16$ [7].



(a) Side View of layout of the OTIS

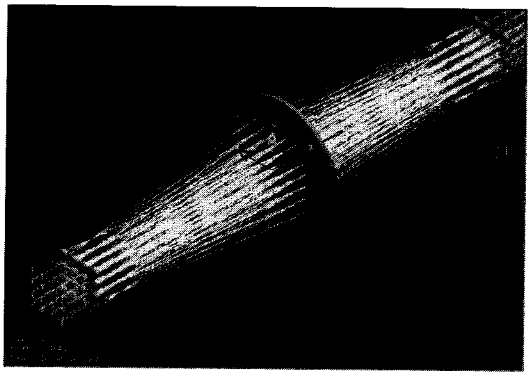


Figure 3. Modelling of the OTIS [7].

The Seidel third order aberrations are classified [9,10,11] and given in Table 1 below

Coefficient	Term	Seidel Aberration
A^*	$(\mathbf{p}^2)^2$	Spherical aberration
B^*	$\mathbf{p}^2(\mathbf{p}\cdot\mathbf{q})$	Coma
C^*	$(\mathbf{p}\cdot\mathbf{q})^2$	Astigmatism
D^*	$\mathbf{p}^2\mathbf{q}^2$	Curvature of Field
E^*	$(\mathbf{p}\cdot\mathbf{q})\mathbf{q}^2$	Distortion
F^*	$(\mathbf{q}^2)^2$	Pocus

Table 1. The Seidel aberrations.

The determination of the function h_4^* and its coefficients can be carried out using the Campbell-Baker-Hausdorff formula [8]. Each lens of the system can be expressed as a linear part and third-order terms, which are nonlinear (cubic) transformations of phase space, i.e., aberrations. There are also inhomogeneous terms in the system because the translations of the optical axis of the microlenses in stage I and III. To examine the effect on phase space of each of the aberration coefficients $(A^*, B^*, C^*, D^*, E^*, F^*)$ and translation, the cubic monomials, which are produced out of $\mathbf{p}$ and $\mathbf{q}$ through the action of the operators of the system, are considered. These monomials are $\mathbf{p}^2\mathbf{p}$, $\mathbf{p}^2\mathbf{q}$, $\mathbf{p}\cdot\mathbf{q}\mathbf{p}$, $\mathbf{p}\cdot\mathbf{q}\mathbf{q}$, $\mathbf{q}^2\mathbf{p}$, $\mathbf{q}^2\mathbf{q}$. To consider the effect on phase space of the aberration coefficients and translation at a time, whilst leaving everything else perfect. Therefore, the classification of the effect on phase space of the aberration coefficients is as follows:

Translation:

$$\begin{pmatrix}\mathbf{p}'\\ \mathbf{q}'\end{pmatrix} = \mathcal{M}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_G\\ \mathbf{q}_G\end{pmatrix} \tag{17}$$

This translation produces change in the ray direction and the position.

<u>Microlenses in stage I</u>

$\mathbf{p}_{\mathrm{I}}' = \mathbf{p}_{G^{\mathrm{I}}} = [1-(n'-1)f_1]\mathbf{p} - [(n'-1)/r_1]\mathbf{q} + \tilde{n}\Delta_1$

$\mathbf{q}_{\mathrm{I}}' = \mathbf{q}_{G^{\mathrm{I}}} = [(1+r_1)f_1 - (n'-1)f_1^2]\mathbf{p} + [(r_1-(n'-1)f_1)/r_1]\mathbf{q} + \tilde{n}\Delta_1$

<u>Macrolens in stage II</u>

$\mathbf{p}_{\mathrm{II}}' = \mathbf{p}_{G^{\mathrm{II}}} = [1-(n'-1)f_2]\mathbf{p}_{\mathrm{I}}' - [(n'-1)/r_2]\mathbf{q}_{\mathrm{I}}' - \tilde{n}\Delta_1$

$\mathbf{q}_{\mathrm{II}}' = \mathbf{q}_{G^{\mathrm{II}}} = [(1+r_2)f_2 - (n'-1)f_2^2]\mathbf{p}_{\mathrm{I}}' + [(r_2-(n'-1)f_2)/r_2]\mathbf{q}_{\mathrm{I}}' - \tilde{n}\Delta_1$

<u>Microlenses in stage III</u>

$\mathbf{p}_{\mathrm{III}}' = \mathbf{p}_{G^{\mathrm{III}}} = [1-(n'-1)f_3]\mathbf{p}_{\mathrm{II}}' - [(n'-1)/r_3]\mathbf{q}_{\mathrm{II}}' + \tilde{m}\Delta_3$

$\mathbf{q}_{\mathrm{III}}' = \mathbf{q}_{G^{\mathrm{III}}} = [(1+r_3)f_3 - (n'-1)f_3^2]\mathbf{p}_{\mathrm{II}}' + [(r_3-(n'-1)f_3)/r_3]\mathbf{q}_{\mathrm{II}}' + \tilde{m}\Delta_3$

Spherical aberration (A^*):

$$\mathcal{M}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_G\\ \mathbf{q}_G - 4A^*p^2\mathbf{p}_G\end{pmatrix} \tag{18}$$

This aberration unfocuses $\mathbf{q}_G$, the ray position with translation, but produces no change in the ray direction with translation.

Coma (B^*):

$$\mathcal{M}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_G + B^*p^2\mathbf{p}_G\\ \mathbf{q}_G - B^*[2p\cdot q\mathbf{p}_G + p^2\mathbf{q}_G]\end{pmatrix} \tag{19}$$

Here rays on a cone ($|\mathbf{p}_G|$ constant) issuing from an object point $\mathbf{q}_G$ fall on a circle with center at $\mathbf{q}_C = \mathbf{q}_G(1-B^*p^2)$ and radius $p = B^*p^2|\mathbf{q}_G|$. This gives rise to the familiar 60° "comet" image of points. The two meridional rays on the cone fall on the same point in the direction of $\mathbf{q}_G$, and the two sagittal rays (perpendicular to the former) on the diametrical opposed point, also in the direction of $\mathbf{q}_G$. Momentum space $\mathbf{p}_G$ is also distorted, i.e., $\mathbf{p}_G \to \mathbf{p}_G(1+B^*p^2)$ but not unfocused because $\mathbf{p}'$ is independent of $\mathbf{q}_G$.

Astigmatism (C^*):

$$\mathcal{M}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_G + 2C^*p\cdot q\mathbf{p}_G\\ \mathbf{q}_G - 2C^*p\cdot q\mathbf{q}_G\end{pmatrix} \tag{20}$$

Due to the common $p\cdot q$ coefficient, sagittal rays are unaffected but meridional ones are. Rays on a cone map onto straight segments, centered and directed along $\mathbf{q}_G$, of half-length $2C^*q^2|\mathbf{p}_G|$.

Curvature of field (D^*):

$$\mathcal{M}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_G + 2D^*p^2\mathbf{q}_G\\ \mathbf{q}_G - 2D^*q^2\mathbf{p}_G\end{pmatrix} \tag{21}$$

Positions are unfocused as well as momenta of the phase space origin. Rays on a cone map onto a circle with center $\mathbf{q}_G$ and radius $2D^{*}q^2|\mathbf{p}_G|$. They fall into focus in $\mathbf{q}_G$ on a paraboloid $z = 2D^{*}q^2$.

Distortion (E^{*}):

$$\mathcal{M}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_G + E^{*}[2p\cdot q\mathbf{q}_G + q^2\mathbf{p}_G]\\ \mathbf{q}_G - E^{*}q^2\mathbf{q}_G\end{pmatrix} \quad (22)$$

This is the Fourier conjugate of coma. Position two-space is distorted ($E^{*} > 0$ it is of the "barrel" type), but it remains in focus, while in momentum space the comatic phenomenon appears.

Pocus (F^{*}):

$$\mathcal{M}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} = \begin{pmatrix}\mathbf{p}_G + 4F^{*}q^2\mathbf{q}_G\\ \mathbf{q}_G\end{pmatrix} \quad (23)$$

This is an $\mathbf{p}_G$-unfocusing aberration, Fourier conjugate to spherical aberration. Position space is neither distorted nor unfocused, so this has not been usually counted among the Seidel aberrations of the system. For $F^{*} > 0$, ray slopes increase cubically with $|\mathbf{q}_G|$ in the direction of $\mathbf{q}_G$ and thus depth of field decreases away from the origin.

The aberration coefficients in equation (16) are given by

$$A^{*} = -f/4 + [(n'-1)(2n'+1)/4n'r]f^2 - [(n'-1)(3n'-1)/4r^3]f^3 + [(n'-1)(4n'^2-7n'+4)/8r^3]f^4 - [(n'-1)^4/8r^4]f^5 \quad (24)$$

$$B^{*} = -[(n'^2-1)/2n'r]f + [3n'(n'-1)/2r^2]f^2 - [(n'-1)(3n'^2-5n'+3)/2r^3]f^3 + [(n'-1)^4/2r^4]f^4 \quad (25)$$

$$C^{*} = -[(n'^2-1)/2r^2]f + [(n'-1)(2n'^2-3n'+2)/2r^3]f^2 - [(n'-1)^4/2r^4]f^3 \quad (26)$$

$$D^{*} = (n'-1)/4n'r - [(n'^2-1)/4r^2]f + [(n-1)(2n'^2-3n'+2)/4r^3]f^2 - [(n'-1)^4/4r^4]f^3 \quad (27)$$

$$E^{*} = (n'-1)/2r^2 - [(n'-1)(n'^2-n'+1)/2r^3]f + [(n'-1)^4/2r^4]f^2 \quad (28)$$

$$F^{*} = n'(n'-1)/8r^3 \quad (29)$$

where f_1, f_2, and f_3 are the focal lengths of the lenses in stage I, II, and III, respectively. r_1, r_2, and r_3 are the radii of the lenses in stage I, II, and III, respectively. n' is the refractive index of the lens. $\tilde{n} = 1,\ldots,\sqrt{N}$ and $\tilde{m} = 1,\ldots,\sqrt{M}$. Δ_1 and Δ_3 are the pitches of the lenses in stage I and III, respectively.

4. Discussions

1. Equations (17)-(23) show that spherical aberration and coma are the dominant imperfections on phase space. It also shows that the macrolens in stage II is the most critical component in the system because the focal length and the radius of the macrolens are large.

2. The diameters of microlenses and a macrolens can be scaled down to $1.8\lambda\sqrt{N}$ and $19.96\lambda\sqrt{N}$ [8], respectively. The focal lengths of microlenses and a macrolens are also scaled down to $1.8\lambda\sqrt{N}$ and $9.98\lambda\sqrt{N}$, respectively. This implies that the radii of the microlenses and the macrolens can be also scaled down to the magnitude of wavelength. This means that the effects on phase space of each aberration coefficient can be scaled down to the magnitude of the propagatory wavelength. By using the design of optical MEMS with many optical codes such as GLAD™, microlenses and macrolenses for the size of wavelength can be fabricated.

3. The third-order terms in each equation represent aberration with Seidel aberration coefficients $A^{*}, B^{*}, C^{*}, D^{*}, E^{*}$, and F^{*}. These aberration coefficients are derived from the intrinsic aberration coefficients, i.e., A, B, C, D, E, F, plus the transition from the front focal point to the lens and the transition from the lens to the back focal point [8]. From theoretical points of view, therefore, the intrinsic aberrations of a single lens can be corrected by a well-designed compound lens system but there are still imperfections from the transition and misalignments.

4. The translation terms depend on the positions of the microlenses in stage I and III, i.e., the term $\tilde{n}\Delta_1$ and $\tilde{m}\Delta_3$.

5. The macrolens in stage II is a bi-convex single lens, which performs as a Fourier transform lens. From those equations, the aberration coefficients of each lens in stage I, II, and III can be obtained as:

Microlenses in stage I (Unit: mm.)

$A^* = 5.46$	$B^* = 0.162$
$C^* = -0.001$	$D^* = 0.002$
$E^* = -1.97\times10^{-4}$	$F^* = 4.75\times10^{-6}$

Macrolens in stage II

$A^* = 30.375$	$B^* = 0.33$
$C^* = -0.001$	$D^* = 4.48\times10^{-4}$
$E^* = -0.6\times10^{-5}$	$F^* = 2.55\times10^{-8}$

Microlenses in stage III

$A^* = 5.46$	$B^* = 0.162$
$C^* = -0.001$	$D^* = 0.002$
$E^* = -1.97\times10^{-4}$	$F^* = 4.75\times10^{-6}$

From these results, spherical aberration and coma are the dominant imperfections. This shows that the macrolens of the OTIS system is the most critical component with respect to these effects.

5. Conclusions

This paper discusses the use of Lie Methods to characterize the aberration of third order of the macrolens in stage II of the OTIS. This approach, like that of Hamiltonian optics, embodies the restrictions imposed by Fermat's principle. Moreover, it has the feature that the relation between initial and final quantities is always given in explicit form. In particular, it is possible to represent the action of each separate element of a compound optical system, including all departures from Gaussian optics, by a certain operator. These operators can then be concatenated by well-defined rules of Lie algebraic methods to obtain a resultant operator that characterizes the entire system. That is, the use of Lie methods provides an operator extension of the matrix methods of Gaussian optics to the general case. Finally, their use, as is the case with characteristic functions, facilitates the treatment of symmetries in an optical system and the classification of aberrations. This is one of the major advantages of Lie methods to provide additional insight concerning the sources of various aberrations. The expressions for the primary (third order) aberrations of a single lens depend on the surface curvatures of the lens. The dependence of aberrations on the surface curvatures can be expressed in terms of the shape factor of a thin lens. The shape factor of a thin lens is defined as

$$X = \frac{c_1 + c_2}{c_1 - c_2} \qquad (30)$$

where c_1 and c_2 are the surface curvatures of a thin lens. X can also be written in terms of the lens power, K, and refractive index of a thin lens, n', as

$$X = \frac{n'-1}{K}(c_1 + c_2) \qquad (31)$$

The following relations are also easily proven:

$$c_1 = \frac{1}{r_1} = \frac{K(X+1)}{2(n'-1)} \qquad (32)$$
$$c_2 = \frac{1}{r_2} = \frac{K(X-1)}{2(n'-1)}$$

The shape factor is also often called *bending factor*. If we substitute these equations into the equations of aberration coefficients, it yields the expressions of aberrations depending on the bending factor. Therefore, the best way to remove the aberrations is by bending lenses and by using a system of compound lenses. For example, with two thin lenses in contact, it is possible to eliminate both spherical aberration and coma, by judicious choice of the lens shape. The macrolens in stage II is a bi-convex single lens. By using these results the macrolens can be designed using CODE V (lens design software, Optical Research Associates, U.S.A.) as a compound lens system with 6 lenses as shown in Figure 4.

Figure 4 shows a Fourier transform lens [8] which performs the same function (Fourier transform) as the bi-convex single lens in stage II does. This compound lens system can eliminate almost all aberrations as shown in Table 2 below.

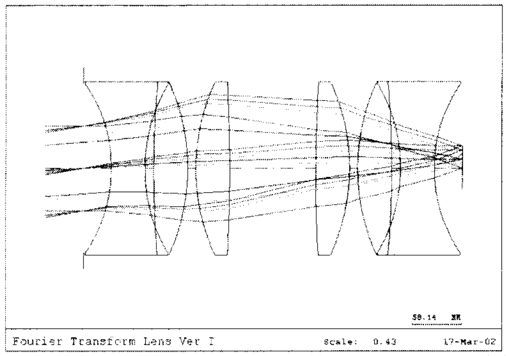


Figure 4. A Fourier Transform Compound Lens.

THO SO..I
Fourier Transform Lens Ver I
Position 1, Wavelength = 632.8 NM

	SA	TCO	TAS	SAS	PTB	DST	AX	LAT	PTZ
STO	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.132109	-0.324151	0.416215	0.239469	0.151095	-0.195859	0.000000	0.000000	0.001315
3	4.357567	6.466562	3.402359	1.269853	0.203600	0.628148	0.000000	0.000000	0.001772
4	-1.276026	-2.823565	-2.135946	-0.747517	-0.053303	-0.551364	0.000000	0.000000	-0.000464
5	-0.084134	0.110863	-0.209266	-0.176803	-0.160572	0.077657	0.000000	0.000000	-0.001397
6	-0.715383	-1.698704	-1.539173	-0.642809	-0.194627	-0.508792	0.000000	0.000000	-0.001693
7	-0.265263	0.403486	-0.206727	-0.070342	-0.002149	0.035665	0.000000	0.000000	-0.000019
8	0.167886	-0.270596	0.143232	0.046311	-0.002149	-0.024881	0.000000	0.000000	-0.000019
9	-2.753462	-2.831718	-1.165360	-0.518205	-0.194627	-0.177644	0.000000	0.000000	-0.001693
10	0.210645	-0.391930	0.082505	-0.079546	-0.160572	0.049335	0.000000	0.000000	-0.001397
11	-1.549839	-1.710493	-0.682570	-0.263059	-0.053303	-0.096776	0.000000	0.000000	-0.000464
12	1.735852	2.856966	1.770987	0.726063	0.203600	0.398332	0.000000	0.000000	0.001772
13	-0.151709	0.151614	0.100589	0.134260	0.151095	-0.044725	0.000000	0.000000	0.001315
SUM	-0.191757	-0.061667	-0.023154	-0.082325	-0.111910	-0.410904	0.000000	0.000000	-0.000974

Table 2. Third order Aberrations of the compound lens

6. References

[1] Kurt Bernardo Wolf, *Symmetry in Lie Optics*, Annals of Physics, Vol. 172, pp. 1-25, 1986.

[2] Tetsundo Sekiguchi and Kurt Bernardo Wolf, *The Hamiltonian formulation of optics*, Am. J. Phys., Vol. 55, pp. 830-835, 1987.

[3] Octavio Castanos, Enrique Lopez-Moreno, and Kurt Bernardo Wolf, *Canonical transforms for paraxial wave optics*, Lie Methods in Optics, Proceedings, Leon, Mexico, edited by J. Sanchez Mondragon and Kurt Bernardo Wolf, Lecture Notes in Physics, Springer-Verlag, 1985.

[4] W. Pijitrojana and T. J. Hall, *Optical Transpose Interconnection System Architecture*, The 26th Electrical Engineering Conference (EECON-26), Bangkok, Thailand, Vol. III, pp. 1599-1604, 2003.

[5] A. Gerrard and J. M. Burch, *Introduction to Matrix Methods in Optics*, Dover Publications, Inc., 1975.

[6] Kurt Bernardo Wolf, *Fourier transform in metaxial geometric optics*, J. Opt. Soc. Am. A, Vol. 8, pp. 1399-1403, 1991.

[7] The pictures in Figure 2 are simulated by LightTools which is an illumination design program based on 3D interactive solid modelling with optical accuracy. The software is developed by Optical Research Associates, www.opticalres.com.

[8] W. Pijitrojana, *Optical Transpose Interconnection System Design in Optical Phase Space*, A Ph.D. thesis submitted to the University of London, 2002.

[9] Alex J. Dragt, Lectures on Nonlinear Orbit Dynamics, University of Maryland, College Park, Maryland, American Institute of Physics, 1982.

[10] Stanly Steinberg, *Lie series, Lie transformations, and their applications*, Lie Methods in Optics, Proceedings, Leon, Mexico, edited by J. Sanchez Mondragon and Kurt Bernardo Wolf, Lecture Notes in Physics, Springer-Verlag, 1985.

[11] Alex J. Dragt, Etienne Forest, and Kurt Bernardo Wolf, *Foundations of a Lie algebraic theory of geometrical optics*, Lie Methods in Optics, Proceedings, Leon, Mexico, edited by J. Sanchez Mondragon and Kurt Bernardo Wolf, Lecture Notes in Physics, Springer-Verlag, 1985.