A New Approach for Establishing Structural Similitude for Buckling of Symmetric Cross-Ply Laminated Plates Subjected to Combined Loading

Variddhi Ungbhakorn
Professor in the Mechanical Engineering Department
Chulalongkorn University , Bangkok 10330, Thailand.

Abstract

Similitude theory is a powerful tool to establish the sufficient and necessary conditions of similarity between the model and prototype. Recently a few authors [1-4] have published several papers on laminated plates to develop the similarity conditions for plate buckling problems. However, they applied the similitude transformation to the *solutions* of the governing differential equations instead of to the governing differential equations directly. This causes serious limitation on the applicability of the similitude theory because exact or approximate analytical solutions must be available before they can apply the similitude transformation. This research paper is to demonstrate the merits of the similitude theory by applying the similitude transformation to the governing differential equations of symmetric cross-ply laminated plates directly. Then the similitude invariants, hence the scaling laws, for buckling loads of plates subjected to biaxial and shear loads are derived. The validity of this approach is confirmed by reducing the obtained scaling laws to the case of isotropic plates. Numerical examples for the two loading types show exact agreement between results predicting from the similitude invariants and the available analytical solutions. For isotropic cases, the scaling laws also show exact agreement when using the different material property, the Young's modulus, for the model and prototype. Therefore, flexibility in choosing the material of the model to economize the experiment is at our disposal.

1. Introduction

Laminated composites are gaining wider use in mechanical and aerospace applications due to their high specific stiffness and high specific strength. Many mathematical models of these complex components are seldom amenable to rigorous solution. Any new design base on composite materials usually requires extensive experiment evaluation before going to production. This arises the need for efficient experimental design to verify the approximate analytical or numerical solutions and the need for determining the relationships among system parameters by measurement. This is where the similitude method appears as an indispensable tool. Similitude theory can be roughly stated to be a branch of science concerned with sufficient and necessary conditions of similarity among phenomena. As will be seen that the statement of similitude theory is founded on a rigorous mathematical basis

Simitses, et. al [1-4] have published several papers on laminated plates that deal with the establishment of the similarity conditions between the two phenomena, the model and the prototype. Then they use these similarity conditions, or "scaling laws" to design scaled-down models and make use of theoretical calculations of these models to predict the behavior of the prototypes. However, they have applied the similitude theory to the solutions of the governing differential equations (GDE) instead of to the GDE directly. This procedure puts serious limitation on the applicability of the concept of similitude theory because some forms of exact or approximate analytical solutions must be obtained before they can apply the similitude transformation. The main objective of this study is to demonstrate the merits of the similitude theory by applying it to the GDE directly. Then the similitude invariants for the buckling loads of the cross-ply laminated plates subjected to biaxial and shear loads have been established, hence the scaling laws have been derived. To demonstrate the validity, the scaling laws for the laminated plates have been reduced to the cases of isotropic rectangular plates subjected to uniaxial compression and shear loads. The well-known analytical solutions of the latter are then used to compare with the prediction by the scaling laws obtained from the similitude transformation. In this study, in absence of test data of the model, the author has theoretically calculated the buckling loads of the model and prototype from the well-known solutions, then substitute the results of the model into the scaling laws to predict the buckling loads of the prototype.

2. Conditions for Complete Similitude

Considering all variables, geometric and physical, of the prototype and the model denoted by X_{pi} and X_{mi} respectively, where i = 1,2,.... n. The two systems or phenomena are similar if their corresponding characteristics are connected by bi-unique (one-to-one) mappings such that

$$\mathbf{X_p = CX_m}$$

and $$\mathbf{X_m = C^{-1} X_p}$$

Hence, the following theorem can be stated [5]: *The sufficient and necessary condition of similitude between two systems is that the mathematical model of the one be related by a bi-unique transformation to that of the other.*

Mathematical models of similar systems are invariable under similitude transformation Hence, the differential equations of any two similar systems must coincide, ie.

$$L(X_{mi}) = L(X_{pi}) \qquad (1)$$

Let the model and prototype variables be related to each other by the equations:

$$X_{pi} = C_i X_{mi} \qquad (2)$$

Substitute eq. (2) into (1) ,

$$L(X_{mi}) = L(C_i X_{mi})$$

From the above theorem , it is necessary that ,

$$L(X_{mi}) = \varphi(C_i)\, L(X_{mi})$$

where $\varphi(C_i)$ is the functional relationship among the transformation parameters. Therefore it is compulsory that

$$\varphi(C_i) = 1$$

Hence, the condition for the two systems to be similar is that the function linking the transformation parameters equals to unity. The equation $\varphi(C_i) = 1$, is accordingly called the conditional equation or similitude invariant. Now the author shall apply the above theory to the GDE of the symmetric cross-ply laminated plate buckling directly.

3. Buckling of Symmetric Cross-Ply Laminated Rectangular Plates Subjected to Biaxial Loading

For laminates that are symmetric in both geometry and material properties about the middle surface, the general stiffness equations will simplify considerably. The symmetric cross-ply laminates have their major principal material directions alternating at 0^0 and 90^0 to the laminate axes, for example, $(0/90/0)_s$. When thicknesses, locations, and material properties of the laminate are symmetric about the middle surface of the laminate, coupling between bending and extension is eliminated. In this case the following stiffnesses are zero [6].

$$B_{ij} = 0,\ A_{16} = A_{26} = D_{16} = D_{26} = 0.$$

Therefore, symmetric laminates are least difficult to analyze and commonly used as structural components where weight is an important factor.

Consider the symmetric cross-ply laminated plates subjected to the inplane normal and shear loads, $\bar{N}_{xx}$, $\bar{N}_{yy}$, $\bar{N}_{xy}$, the GDE for buckling analysis, after dropping the variational symbol δ, is [6]:

$$A_{11}\, u_{,xx} + (A_{12} + A_{66})\, v_{,xy} + A_{66}\, u_{,yy} = 0 \qquad (3)$$

$$(A_{12} + A_{66})\, u_{,xy} + A_{66}\, v_{,xx} + A_{22}\, v_{,yy} = 0 \tag{4}$$

$$D_{11}\, w_{,xxxx} + 2(D_{12}+2D_{66})\, w_{,xxyy} + D_{22}\, w_{,yyyy} + \bar{N}_{xx}\, w_{,xx} + \bar{N}_{yy}\, w_{,yy} + 2\,\bar{N}_{xy}\, w_{,xy} = 0 \tag{5}$$

3.1 Case I:Biaxial Loading

Consider the case of plates subjected to $\bar{N}_{xx}, \bar{N}_{yy}$ only, and let the in-plane load ratio be such that $\bar{N}_{yy} = P\bar{N}_{xx}$, and since the above equations are decoupled, hence, the required GDE for classical buckling analysis is:

$$D_{11}\, w_{,xxxx} + 2(D_{12}+2D_{66})\, w_{,xxyy} + D_{22}\, w_{,yyyy} + \bar{N}_{xx}(w_{,xx} + P w_{,yy}) = 0 \tag{6}$$

Let the variables of the prototype be related to those of the model through the similitude scaling factors as follows.

$$x_p = C_x x_m, \quad y_p = C_y y_m, \quad w_p = C_w w_m,$$
$$(D_{ij})_p = C_{Dij}(D_{ij})_m, \quad P_p = C_P P_m \quad \text{and}$$
$$(\bar{N}_{xx})_p = C_{\bar{N}xx}(\bar{N}_{xx})_m$$

By applying similitude transformation to eq.(6), the following necessary conditions for the models to behave exactly as the prototype are derived:

$$\frac{C_{D11}}{C_x^4} = \frac{C_{D12}}{C_x^2 C_y^2} = \frac{C_{D66}}{C_x^2 C_y^2} = \frac{C_{D22}}{C_y^4} = \frac{C_{\bar{N}xx}}{C_x^2} = \frac{C_p C_{\bar{N}xx}}{C_y^2} \tag{7}$$

Let $\quad K_{xx} = \dfrac{\bar{N}_{xx} b^2}{E_{22} h^3}$

which yields the following similitude relation :

$$C_{\bar{N}xx} = \frac{C_{Kxx} C_{E22} C_h^3}{C_b^2}$$

Let the model and prototype have complete geometric similarity, therefore from eq.(7) the following conditional equations are obtained:

$C_P = 1$ and

$$C_{Kxx} = \frac{C_{D11}}{C_{E22} C_h^3} = \frac{C_{D12}}{C_{E22} C_h^3} = \frac{C_{D22}}{C_{E22} C_h^3} = \frac{C_{D66}}{C_{E22} C_h^3} \tag{8}$$

Hence, for complete similarity between the prototype and its model it is required that

$$C_{D11} = C_{D12} = C_{D22} = C_{D66} \tag{9}$$

That is, the scaling factors of all laminate flexural stiffnesses must be equal and the load ratio must be the same for both the model and prototype. Let the scaling factors of the flexural stiffnesses be equal to $C_{stiff,}$ then eq.(8) yields the following similitude invariant for the symmetric cross-ply laminated plates subjected to biaxial loading.

$$\frac{C_{Kxx} C_{E22} C_h^3}{C_{stiff}} = 1 \tag{10}$$

From eq.(10), the following *scaling law* can be written:

$$K_{xxp} = K_{xxm}\, C_{stiff} \frac{\left(E_{22} h^3\right)_m}{\left(E_{22} h^3\right)_p} \tag{11}$$

It is seen that C_{Dij} in eq.(9) depend on the material properties, number of plies and stacking sequence but are independent of the ply thickness, therefore two laminated plates with different ply thicknesses but with the same stacking sequence will satisfy the similarity conditions.

3.1.1 Special Case: Isotropic plates

In this case $D_{ij} = D$, hence, eq.(10) is reduced to:

$$\frac{C_{Kxx}\, C_E\, C_h^3}{C_D} = 1,$$

or the scaling law is

$$\left(\frac{K_{xx}\, E h^3}{D}\right)_p = \left(\frac{K_{xx}\, E h^3}{D}\right)_m$$

If both the model and prototype materials have about the same value of ν, then $K_{xxp} = K_{xxm}$ and the predicted $\bar{N}_{xxp}$ can be calculated by the scaling law eq.(12) as follows.

$$\left(\bar{N}_{xx}\right)_p = \left(\bar{N}_{xx}\right)_m \left(\frac{b_m}{b_p}\right)^2 \left(\frac{h_p}{h_m}\right)^3 \left(\frac{E_p}{E_m}\right) \qquad (12)$$

Numerical Examples

Model data : a = 500 mm., b = 250 mm
h = 2.5 mm., E = 67 GN/m^2, $\nu = 0.3$
Prototype data : a = 5,000 mm., b = 2,500 mm.,
h = 25 mm., E = 67 GN/m^2, $\nu = 0.3$

Consider plates with all sides being simply-supported, the exact solution of buckling load from reference [7] is:

$$\bar{N}_{xx} = \frac{4\pi^2 D}{b^2}$$

From which the theoretical results of the model and prototype are :

$\bar{N}_{xxm}$ = 60.5 N/mm. , $\bar{N}_{xxp}$ = 605 N/mm.

In using eq. (12) to calculate the predicted value from the similitude theory also gives $\bar{N}_{xxp}$ = 605 N/mm. .

Now let the materials of the test model differs from that of the prototype such that $E_P = 3E_m = 201$ GN/m^2 and all other parameters remain the same, it can be shown that the following results are obtained:

From theory: $\bar{N}_{xxm}$ = 60.5 N/mm.
$\bar{N}_{xxp}$ = 1,815 N/mm.

From eq (12): N_{xxp} = 1,815 N/mm.

Notice that the similitude invariant eq.(10) is independent of the boundary conditions of the plates. Hence, it applies to all boundary conditions, since the effects of boundary condition will be included in the test data of the models as will be seen in the next case. This is one of the advantages of using the similitude transformation in designing experiments.

3.2 Case II : Shear Load :

Consider the case of plates subjected to $\bar{N}_{xy}$ only , the G DE is:

$$D_{11} w_{,xxxx} + 2(D_{12} + 2D_{66})\, w_{,xxyy} + D_{22} w_{,yyyy} + 2\bar{N}_{xy} w_{,xy} = 0$$

Let $$K_s = \frac{\bar{N}_{xy}\, b^2}{E_{22}\, h^3}$$

Applying the similitude transformation to the above GDE and proceed as the biaxial loading case one can obtain the following similitude invariants:

$$C_{D11} = C_{D12} = C_{D22} = C_{D66} = C_{stiff}$$

$$\frac{C_{Ks}\, C_{E22}\, C_h^3}{C_{stiff}} = 1 \qquad (13)$$

Therefore the *scaling law* for this case is:

$$K_{sp} = K_{sm}\, C_{stiff} \frac{\left(E_{22}h^3\right)_m}{\left(E_{22}h^3\right)_p} \tag{14}$$

3.2.1 Special case: Isotropic Plates

By proceeding similarly as in previous case , the following results are obtained:

$$\left(\frac{K_s\, Eh^3}{D}\right)_p = \left(\frac{K_s\, Eh^3}{D}\right)_m$$

$$\left(\bar{N}_{xy}\right)_p = \left(\bar{N}_{xy}\right)_m \left(\frac{b_m}{b_p}\right)^2 \left(\frac{h_p}{h_m}\right)^3 \left(\frac{E_p}{E_m}\right) \tag{15}$$

Numerical Examples

Consider clamped plates subjected to shear load with the same data for the model and prototype of the previous example. From reference [7], the shear buckling load is:

$$\bar{N}_{xy} \cong \frac{10.2\,\pi^2 D}{b^2}$$

which gives

$\bar{N}_{xym} = 154.2$ N/mm. , $\bar{N}_{xyp} = 1542$ N/mm.

Using eq.(15) to predict $\bar{N}_{xyp}$ yields:

$\bar{N}_{xyp} = 1542$ N/mm.

4. Conclusion and Discussion

The paper has employed a new approach of the similitude transformation to establish the similitude invariants and the scaling laws for the stability of the symmetric cross-ply laminated plates buckling by applying the similitude transformation to the governing differential equations directly. For special cases, the predicted buckling loads of the isotropic prototypes using the known data of the models have shown exact agreement with the available theoretical results. Calculations using different Young's modulus of elasticity for the model and prototype also show exact agreement. This means that, for isotropic cases, the scaling laws allow the flexibility in choosing different materials for the models. Hence, the application of the similitude theory appropriately can help obtain certain answers of the complex unknown phenomena without the need for the solutions of the complicate differential equations.

The conditional requirement in eq.(9) is the same as eq.(16) in reference [2] but their scaling laws eq.(7)-(9) will be the same as eq.(10) in this paper only if their λ_m, λ_n, and λ_R are equal to unity. However, the derivation presented herein is much easier and without the need of employing the closed-form solution of the plates.

In certain situation, complete similarity is difficult to fulfil for the test model, then one might relax certain condition in the similitude invariant to enable the construction of the test model. However, employing the approximate similitude or partial similarity is allowed provided, the complete set of similitude criteria to be taken into consideration is known on the basis of the mathematical model and the error caused by disregarding the criterion can be assessed beforehand. The method presented in this research paper can also be used to verify the accuracy and correctness of other numerical methods such as the finite element and boundary element analyses or certain problems whose analytical solutions are not available but only test model data.

5. Nomenclature

a	plate length
A_{ij}	laminate extensional stiffnesses
b	plate width
B_{ij}	laminate coupling stiffnesses
C	scaling factor matrix
C_i	similitude scaling factors
D	isotropic flexural stiffness, $= Eh^3/12(1-\nu^2)$

D_{ij} laminate flexural stiffnesses

E_{ij} Young's moduli of elasticity

h total laminate thickness

K_{xx}, K_s non-dimensional buckling loads

L(...) differential or algebraic operator

$\bar{N}_{xx}, \bar{N}_{yy}$ inplane applied normal load

$\bar{N}_{xy}$ inplane applied shear load

m model

p prototype

P load ratio

u, v, w middle surface displacements

$\mathbf{X}_m$ vector of model variables

$\mathbf{X}_p$ vector of prototype variables

ν isotropic Poisson's ratio

6. References

[1] Simitses, G.J. and Rezaeepazhand, J., Structural Similitude for Laminated Structures, J., Composites Engineering, Vol.3, Nos. 7-8, pp.751-765, 1993.

[2] Simitses, G.J., and Rezaeepazhand, J., Structural Similitude and Scaling Laws for Buckling of Cross-Ply Laminated Plates, Proceedings of the American Society for Composites Oct 19-21, pp.265-274, 1993 1994.

[3] Rezaeepazhand, J., Simitses, G.J. and Starnes, J.H.,Jr, Design of Scaled Down Models for Stability of Laminated Plates, AIAA Journal , Vol.33 , No.3 , pp. 515-519 , 1995.

[4] Simitses, G.J. Starnes, J.H.,Jr, Rezaeepazhand, J., Structural Similitude and Scaling Laws for Plates and Shells: A Review, Collection of Technical Papers-AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference 1 I Apr 3-6 2000 AIAA, pp. 393-403 0273-4508.

[5] Szucs, Errin, Similitude, and Modelling , Elsevier Scientific Publishing Co., New York , 1980.

[6] Jones, R.M., Mechanics of Composite Materials, Scripta Book Co., Washington, DC, 1975.

[7] Brush, D.O. and Almroth, B.O., Buckling of Bars, Plates, and Shells, McGraw-Hill Book Co., 1975.