Optimal Stabilization of A Set of Linear Time-Invariant Systems by Lyapunov Equations

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Abstract

This paper describes the determination of a single controller that will simultaneously stabilize a finite number of linear time-invariant systems in simple steps. To accomplish this, an overall performance index is defined as the summation of quadratic cost functions with constraints on Lyapunov equations for each system. The quadratic cost functions are used to yield both good performance of the systems and suitable magnitude of the controller output while satisfaction of Lyapunov equations guarantee the stability of the systems. The numerical optimization is then applied to minimize the overall performance index in order to reach a sub-optimal control provided that the solution of the simultaneous stabilization problem exists. The simulation results demonstrate that the proposed technique can determine one control which stabilizes a set of linear time-invariant systems and provides good transient behavior of the systems.

1. Introduction

Simultaneous stabilization is an interesting problem in the area of robust control. It involves determining a single controller that will stabilize a collection of systems. For instance, the dynamic model of a system can unexpectedly change when the failure of a certain mode of controller occurs. Another example is linearized models of a non-linear plant at different operating conditions. One may seek a single controller which will stabilize such systems despite the parameter variations.

Research has been done on the simultaneous stabilization problem for a collection of systems described in transfer matrix form [1-3]. For the existence of a stabilizing non-linear state feedback controller, a sufficient condition has been derived by [4]. In 1989, Schmitendorf and Hollot considered a simultaneous stabilization problem with linear state feedback controller by solving a non-smooth optimization [5,6]. The sufficient condition was derived in [7]. In [8], the algorithm was based on minimizing the largest real part of the eigenvalues of the closed loop systems. The definition of this objective function only emphasized the maximal stability margin. Paskota [9] proposed the stability constraints with Hurwitz' s necessary and sufficient conditions.

The aim of this paper is to determine a suitable controller which not only simultaneously stabilizes the finite number of linear time-invariant systems but also provides good transient behavior of the systems. This can easily be achieved by making use of the quadratic cost functions and Lyapunov equations for performance and stability of closed-loop systems respectively.

2. Preliminary Background

Consider a time-invariant linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

with the linear state feedback control:

$$u(t) = -Kx(t) \quad (3)$$

where $x(t) \in R^n$, $u(t) \in R^r$, $y(t) \in R^m$, $A \in R^{n\times n}$, $B \in R^{n\times r}$ $C \in R^{m\times n}$ and $K \in R^{r\times n}$. Assume (A,B) is controllable. According to [10], the quadratic cost function of the system is defined as:

$$J(K) = E\left\{\frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)dt\right\} = \frac{1}{2}tr(PX_0) \quad (4)$$

where E is the expectation, the initial state $x(0)$ is a zero-mean random variable with covariance matrix X_0 and P is the positive definite solution of the matrix Lyapunov equation:

$$PA_c + A_c^T P + Q + K^T RK = 0 \quad (5)$$

where $A_c = (A - BK)$is a stable closed-loop system matrix. Q and R are positive semidefinite and positive definite matrixes respectively.

Note the system dynamics in the equation (1)-(2) are given by its physics. The quadratic cost function in equation (4) is chosen to achieve the desired response of the system. The weighting matrixes Q and R are selected in order to trade-off comformability in dynamic performance and magnitude of the required controls.

3. Problem formulation

Now, consider the collection of m linear time-invariant systems:

$$\dot{x}_k(t) = A_k x_k(t) + B_k u_k(t) \quad (6)$$

and state feedback controls:

$$u_k(t) = -Kx_k(t) \quad (7)$$

where k=1, 2, 3,..., m

The simultaneous stabilization problem is written as follows:

$$\text{minimize} \quad S(K) = \sum_{k=1}^{m} w_k J_k(K) \quad (8)$$

subject to

$$P_k(A_c)_k + (A_c)_k^T P_k + Q_k + K^T R_k K = 0$$

for k =1,2,3,...,m

over $K \in \Omega$

where w is weighting scalar which is used to specify how tight the certain systems should be controlled and Ω is a subset of $R^{r\times n}$ such that if $K \in \Omega$, then the closed-loop matrix $(A_k - B_k K)$ is stable for each k.

The problem above can be solved by the following steps:

1. Find the initial K. This can be accomplished by:
   1.1 algorithm provided by[11] or
   1.2 randomly generating a number of K, checking their feasibility and then discarding those which are not feasible. The feasible K satisfy the constraints in problem (8).
2. Apply the conventional Quasi-Newton method with BFGS formula and line search method [12] to problem (8).
3. If the obtained K satisfies the desired performance, then stop. Otherwise, go to step 1. while appropriately changing Q and R.

4. Example

The linear model of the lateral dynamics of an aircraft [13] has the states and controls as $x = [p \quad r \quad \beta \quad \phi]^T$ and $u = [\delta a \quad \delta r]^T$ respectively and

$$A_0 = \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where p is the roll rate, r is the yaw rate, β is an incremental sideslip angle, ϕ is an increment roll angle, δa and δr are the incremental changes in the aileron and rudder angle respectively. Suppose malfunction manipulations of the inputs δa and δr are regarded as case 1 and 2 respectively. The collection of linear time-invariant systems consists of the following.

$$B_0 = \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 20 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where $A_0 = A_1 = A_2$.

It is first verified that only systems of (A_0, B_0) and (A_0, B_1) are controllable whereas the system of (A_0, B_2) is not controllable. This means the aircraft can be rolled by using only the ailerons, but it cannot be made to turn. Only the nominal system (A_0, B_0) and system (A_0, B_1) in case 1 can be stabilized by using a state feedback control since controllability guarantees that all the states can be manipulated to zero in finite time.

To demonstrate the effectiveness of the proposed technique, case 1 of the malfunction actuator is not considered for a moment. Using the proposed technique and [$Q_0 = I_{4\times4}$ and $R_0 = I_{2\times2}$], the controller gain matrix K_{des} is obtained in Table 1. In Fig. 1, the controller gain matrix K_{des} provides a performance close to K_{lqr} which is obtained from the conventional LQR method[14]. It is found that the controller gain matrix determined from LQR method can be obtained by this technique if an initial controller gain matrix is chosen around LQR solution. However, the controller gain matrix K_{des} destabilizes the system (A_0, B_1) and the dynamics of the states are shown as Fig 2. Therefore, the proposed technique is applied to the systems of (A_0, B_0) and (A_0, B_1) and this yields the single controller gain matrix $K_{s,un}$ which simultaneously stabilizes the systems of (A_0, B_0) and (A_0, B_1). The overall performance index, weighting matrixes and controller gain matrix $K_{s,un}$ are presented as a unmodified control case in Table 2. By implementing a single controller $K_{s,un}$, the performances of systems (A_0, B_0) and (A_0, B_1) are illustrated in Fig. 3 and Fig 4 respectively.

Now, the state response in Fig 4 is required to be faster to reach steady state. The weighting matrix Q_1 is now made larger. For this modified control, the overall performance index, weighting matrixes and controller gain matrix $K_{s,mod}$ are indicated in the second row of Table 2. According to the state response in Fig 5, the modified control does not have much affect on the system (A_0, B_0) with respect to state response in Fig 3. On the other hand, the modified controller [$K_{s,mod}$] drives all the states in the system (A_0, B_1) to steady state faster than the unmodified controller [$K_{s,un}$] does, as is expected.

5. Conclusion

In this paper, control of a set of linear time-invariant systems with a single state feedback controller is considered. To determine the suitable controller gain matrix, the conventional BFGS optimization procedure is implemented to minimize a weighted sum of the quadratic objective functions with constraints on the stability Lyapunov equations provided that the solution exists. It is found that this procedure is straightforward and takes little computation and effort. Therefore, the proposed technique is a practical approach to obtain a sub-optimal stabilizing control and also provides the good transients through the choice of the weighting matrixes in the objective function. The effectiveness of this proposed technique is demonstrated in the example.

Table 1 Controller gain matrix obtained from the LQR method and the proposed method

	Value of objective cost function (J)	Controller gain matrix
LQR method	1.6013	$K_{lqr} = \begin{bmatrix} .6590 & .0770 & -.2612 & .9973 \\ .0802 & -.7184 & -.2743 & .0733 \end{bmatrix}$
Proposed method	1.6306	$K_{des} = \begin{bmatrix} 2.2730 & .0669 & .0371 & 2.3653 \\ 1.2119 & -.7296 & -.0496 & 1.0413 \end{bmatrix}$

Table 2 Numerical values of overall performance index, weighting matrix and controller gain matrix

	Initial S	Sub-optimal S	Weighting matrix	Sub-optimal controller gain matrix
Unmodified control	47.8385	15.0360	$Q_0=Q_1=I_{4\times4}$, $R_0=R_1=I_{2\times2}$	$K_{s,un} = \begin{bmatrix} -.0817 & .1358 & -.4257 & .3160 \\ -.9320 & -.8203 & -.6250 & -.8911 \end{bmatrix}$
Modified control	90.7103	41.3098	$Q_0=I_{4\times4}$, $Q_1=5\times I_{4\times4}$, $R_0=R_1=I_{2\times2}$	$K_{s,mod} = \begin{bmatrix} -.0675 & .1628 & -.1694 & .3707 \\ -1.1487 & -1.2294 & -1.3232 & -1.2975 \end{bmatrix}$

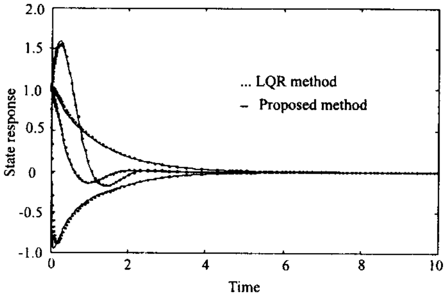

Fig. 1 Performance of comparison of controls obtained from the conventional LQR method and the proposed method.

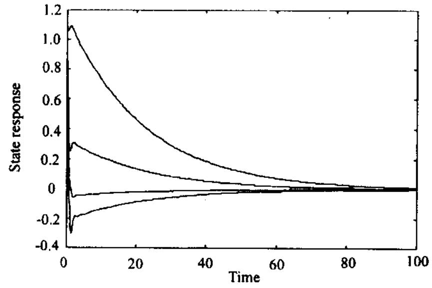

Fig. 4 State response of stable closed loop system $(A_0, B_1, K_{s,un})$

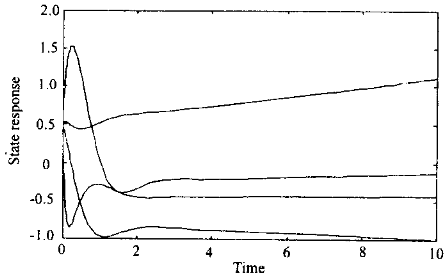

Fig. 2 State response of unstable closed loop system corresponding to (A_0, B_1, K_{uks}) [one mode of controller fails]

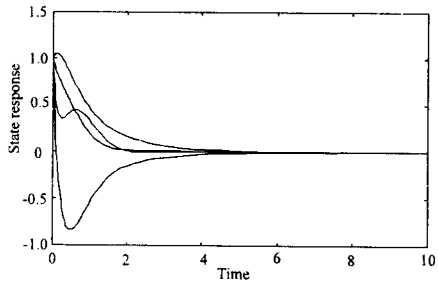

Fig. 5 State response of stable closed loop system $(A_0, B_0, K_{s,mod})$ in case of modified control

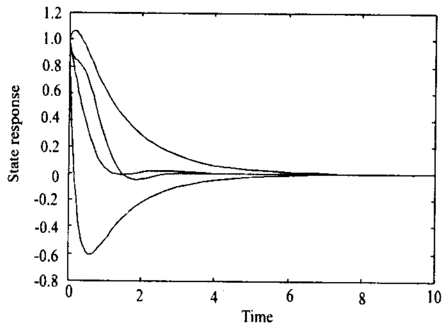

Fig. 3 State response of stable closed loop system $(A_0, B_0, K_{s,un})$

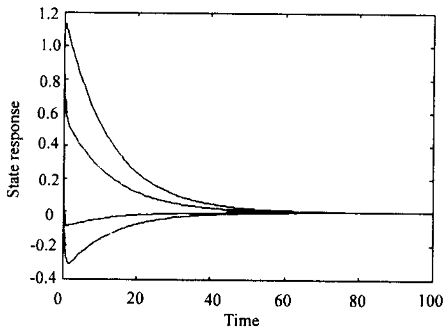

Fig. 6 State response of stable closed loop system $(A_0, B_1, K_{s,mod})$ in case of modified control

6. References

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