Research Article

Performance of EWMA Chart for Trend Autoregressive Model with Exponential White Noise

Wannaporn Suriyakat

Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand
Corresponding author. Email address: suriyakat@psru.ac.th

Received September 28, 2016; Accepted August 22, 2017

Abstract

In this paper, we propose an explicit formula of the performance for Exponentially Weighted Moving Average (EWMA) chart by using the Fredholm integral equation of the second kind and the data are described by trend Exponential Autoregressive order p (EAR(p)) model. The performance of the chart is measured by the Average Run Length (ARL). The solution is compared with numerical approximations and we found that the computational time of the explicit formula takes approximately 1 s while the numerical computations were approximately 10 min.

***Key Words*: ARL; Autoregressive model; EWMA chart; Integral equation**

Introduction

The Exponentially Weighted Moving Average (EWMA) chart was first introduced by Robert (1959), is often used for detecting the shift in the sequence of independent normal distribution data. The chart is used for detection the mean over time as they occur. The autocorrelation has a large impact on the chart developed under the independence assumption. The performance of the chart is measured by the average run length (ARL). ARL is a widely accepted measure of performance of a chart. The ARL_0 denote the average number of observations before an in-control process is taken to signal to be out of control, and the ARL_1 denote the average number of observations taken from out of control. An appropriate scheme provides large ARL_0 for in control processes and small ARL_1 for out of control processes. The EWMA chart is based on the EWMA statistic:

$$Z_t = (1 - \lambda) Z_{t-1} + \lambda Y_{t-1},\ t = 1, 2, 3, \dots ;\ Z_0 = z \quad (1)$$

where Z_t is the EWMA value of a statistic after t data, z is an initial value for Z_t and a smoothing parameter $\lambda(0 < \lambda < 1)$. Let Y_t be an observed value of the trend EAR(p) model defined below:

$$Y_t = a + bt + \phi_1 Y_{t-1} + \phi_2 Y + \dots + \phi_p Y_{t-p} + \zeta_t \quad (2)$$

where ϕ_i be an autoregressive coefficient ($-1 \le \phi_i \le 1$, i = 1, 2, 3, …, p), ζ_t be an exponential white noise, Y_t is the trend EAR (p) model when a is a constant, and b is the trend slope in term of t. Let c be an upper limit on Z_t for determining that the process is an out-of-control and let the first passage time c be defined as the time at the process is an out-of-control process. The process is initially out-of-control at $\tau_c = 1$ by the random data Z_t over the positive value c is

$$\tau_c = \inf (t \ge 0: Z_t \ge c) \quad (3)$$

The ARL of the chart at a given quality value c is defined to be expected value of c. The methods for calculating the performance of EWMA chart are Monte Carlo simulation approach (MC) (Haq et al., 2015; Morais et al., 2015; Riaz & Ahmad, 2016; Rabyk & Schmid, 2016), Markov Chain approach (MCA) (Chang & Wu, 2011; Huang et al., 2012; Saleh et al., 2013; Zhang et al., 2014), Integral Equation approach (IE) (Crowder, 1987; Calzada & Scariano, 2003, 2004), and analytical approach (Areepong, 2009; Mititelu et al., 2010; Suriyakat et al., 2012). The MC can be used when exact analytical formulas are not available. The MCA was first studied by Brook and Evans (1972). The IE for the ARL can be expressed in terms of Fredholm Integral Equation of the second kind by Crowder (1987). The Analytical formulas can be easy to calculate and program. The article is structured as follows. Section 2 presents EWMA chart

ISSN 1905-9159 (Print), ISSN 2586-842x (Online)

and performance. Section 3 presents the uniqueness of the solution for the ARL integral equation. Section 4 show solution for the integral equation of EWMA Chart with the trend EAR (p) model, Section 5 and 6 are the present numerical solution to the ARL integral equation and conclusions, respectively.

EWMA Chart and Performance

The assumption of random variables $\zeta_1, \zeta_2, \zeta_3, \ldots$ is identically independent distribution with probability density function $f(x, \alpha)$ where the parameter $\alpha = \alpha_0$ before the change point time $\theta \le \infty$, at the parameter α changes to α_1. The parameter does not change from α_0 the mean that $\theta \le \infty$. The $f(x,\alpha)$ is an absolute continuous with respect to $f(x, \alpha_0)$. The first passage time for this type of chart for the statistic Z_t defined as in equation (1) is typically

$$\tau_c = \inf(t \ge 0: Z_t \ge c)$$

where c is a constant parameter known as the control limit on the value of Z_t.

Uniqueness of the Solution for the ARL Integral Equation

Mititelu et al. (2010), Busaba et al. (2012), Kanita et al. (2014, 2015) proposed the uniquenes s of the solution for the ARL integral equation. The ARL of EWMA chart at a given level defined as $L(u)$ = ARL = $E\tau_c < \infty$, the solution of the following integral equation for the case where ζ_t are continuous distribution i.i.d. random variables with exponential distribution given by $f(x) = (1/\alpha)\exp(-x/\alpha)$, $x \ge 0$. The EWMA chart can be calculated in term of the Fredholm Integral Equation of the second kind by Crowder (1987) is applied below

$$L(u) = 1 + \int_0^{\frac{c- (1 - \lambda)u - \lambda(a + b + \sum_{j=1}^{p}\phi_j Y_{1-j})}{\lambda}} L\Big((1 - \lambda)u + \lambda(a + b + v \textstyle\sum_{j=1}^{p}\phi_j + x)\Big) f(x)dx \qquad (4)$$

After a change of variable, while the corresponding AR(p) model is υ. Then the following holds

$$L(u) = 1 + \frac{1}{\lambda}\int_0^c L(x) f\Big(\frac{x - (1 - \lambda)u - \lambda(a + b + v \sum_{j=1}^{p}\phi_j)}{\lambda}\Big)dx,\ u \in [0, c] \qquad (5)$$

However, the right-hand side of equation (5) contain only continuous function, it is clear that solution of the integral equation (5) is a continuous function. Recall the complete metric space of all continuous function $(C(T),\|\ \|_1)$ where I are a compact interval and the norm defined as $\|L\| = \sup_{u \in I}|L(u)|$, the operator T is named the contraction it there exist a number $0 \le q \le$ such that $\|T(L_1) - T(L_2)\| \le q\|L_1 - L_2\|$ for all $L_1, L_2 \in C(I)$. In our case, let T be an operator in the class of all continuous function C(I), where I = [0, c] and let T be defined by

$$T(L(u)) = 1 + \frac{1}{\lambda}\int_0^c L(x) f\Big(\frac{x - (1 - \lambda)u - \lambda(a + b + v \sum_{j=1}^{p}\phi_j)}{\lambda}\Big)dx,\ u \in [0, c] \qquad (6)$$

Where $f(\cdot)$ is the probability density function of the distribution. Then equation (6) can be written in operator form as $T(L(u) = L(u)$. To prove the uniqueness of the solution of equation (6) we first prove the following theorem.

Theorem 1 On the metric space $(C(I),\|\ \|_\infty)$ with the norm $\|L\|_\infty = \sup_{u \in I}|L(u)|$ the operator T is a contraction.

Proof To show that T is a contraction we need to prove that for all $u \in I$ and $L_1, L_2 \in C(I)$ we have the inequality $\|T(L_1) - T(L_2)\| \le q\|L_1 - L_2\|$, where $0 \le q \le 1$. From equation (6) we have

$$\begin{aligned}
\|T(L_1) - T(L_2)\|_\infty &\le \sup_{u \in [0,c]}\Big(\frac{1}{\lambda}\int_0^c (L(x_1) - L(x_2)) f\Big(\frac{x - (1 - \lambda)u - \lambda(a + b + v \sum_{j=1}^{p}\phi_j)}{\lambda}\Big)dx\Big) \\
&\le \|L_1 - L_2\|_\infty \sup_{u \in [0,c]}\Big(\frac{1}{\lambda}\int_0^c f\Big(\frac{x - (1 - \lambda)u - \lambda(a + b + v \sum_{j=1}^{p}\phi_j)}{\lambda}\Big)dx\Big) \\
&= \frac{1}{\lambda}\int_0^c f\Big(\frac{x - (1 - \lambda)u - \lambda(a + b + v \sum_{j=1}^{p}\phi_j)}{\lambda}\Big)dx\ \|L_1 - L_2\|_\infty \\
&= q\|L_1 - L_2\|_\infty
\end{aligned}$$

where $0 < q = 1/\lambda\int f(x- (1 - \lambda)u - \lambda(a + b + \upsilon(\phi_1 + \phi_2 + \ldots + \phi_p))/\lambda)dx < 1$.

We have used the triangle inequality for norms and the fact that

$$|L_1(0) - L_2(0)| \le \sup_{u \in [0, c]} | L_1(x) - L_2(x) = \|L_1 - L_2\|$$

So, the unique of the solution is guaranteed by Theorem 1 and the Banach Fixed Point Theorem.

Solution for Integral Equation of EWMA Chart with Trend EAR(p) Model

In Theorem 2, we derive explicit formula for ARL of one-sided EWMA chart with the trend EAR(p) model by the integral equation. Since Suriyakat et al. (2012) presented the explicit formula for ARL of EWMA chart for AR(1) model with exponential white noise. The uniqueness of solution is guaranteed by Theorem 1.

Theorem 2 The solution of equation (5) is

$$L(u) = 1 - \frac{\lambda \exp\left(\frac{(1-\lambda)u}{\alpha\lambda}\right)\left(\exp\left(-\frac{c}{\alpha\lambda}\right) -1\right)}{\lambda \exp\left(-\frac{a+b+\upsilon\sum_{j=1}^{p}\phi_j}{\alpha}\right) + \exp\left(-\frac{c}{\alpha\lambda}\right) -1},\ u \in [0, c]. \tag{7}$$

Proof We have $u \in [0, c]$ that

$$L(u) = 1 + \frac{1}{\lambda}\int_0^c L(x) f\left(\frac{x - (1-\lambda)u - \lambda(a+b+\upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right)dx. \tag{8}$$

The function $f(\cdot)$ can be written as

$$f\left(\frac{x - (1-\lambda)u - \lambda(a+b+\upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right) = \frac{1}{\alpha}\exp\left(\frac{x - (1-\lambda)u - \lambda(a+b+\upsilon\sum_{j=1}^{p}\phi_j)}{\alpha\lambda}\right) \tag{9}$$

The function $L(u)$ can be written as

$$L(u) = 1 + \frac{1}{\alpha\lambda}\int_0^c L(x)\exp\left(-\frac{x - (1-\lambda)u - \lambda(a+b+\upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right)dx \tag{10}$$

Let

$$C(u) = \exp\left(\frac{(1-\lambda)u + \lambda(a+b+\upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right),\ u \in [0, c] \tag{11}$$

Then we can write as

$$L(u) = 1 + \frac{C(u)}{\alpha\lambda}\int_0^c L(x)\exp\left(-\frac{x}{\alpha\lambda}\right)dx \tag{12}$$

Let $d = \int_0^c L(x)\exp\left(-\frac{x}{\alpha\lambda}\right) dx$. we can obtain

$$L(u) = 1 + \frac{C(u)}{\alpha\lambda} d \tag{13}$$

we can express the constant d as

$$d = \int_0^c L(x)\exp\left(-\frac{x}{\alpha\lambda}\right) dx = - \frac{\alpha\lambda(\exp\left(-\frac{c}{\alpha\lambda}\right) -1}{1 + \exp\left(-\frac{c}{\alpha}\right) - 1\frac{\exp(a+b+\upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}} \tag{14}$$

where to substitute d into equation (13) the solution for the integral equation is

$$L(u) = 1 - \frac{\lambda \exp\left(\frac{(1-\lambda)u}{\alpha\lambda}\right)\left(\exp\left(-\frac{c}{\alpha\lambda}\right) - 1\right)}{\lambda \exp\left(a + b + \upsilon\sum_{j=0}^{p}\phi_j\right) + \exp\left(-\frac{c}{\alpha}\right) - 1} \quad (15)$$

Numerical Solution for Integral Equation of EWMA Chart with Trend EAR(p) Model

In this section, we present a numerical method to evaluate the solution $L(u) = E\tau_c$ of the equation (5) for ARL of the trend EAR(p) model. First, we recall equation (5) in the form

$$L(u) = 1 + \frac{1}{\lambda}\int_0^c L(x) f\left(\frac{x - (1-\lambda)u - \lambda(a + b + \upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right) dx. \quad (16)$$

where $f(\cdot)$ is an exponential distribution function. We can approximate $L(u)$ using the Gauss-Legendre quadrature rule as follows:

$$L(a_i) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j - (1-\lambda)a_i - \lambda(a + b + \upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right), \; i = 1, 2, 3, ..., m \quad (17)$$

where $w_j = \frac{c}{m} \geq 0$ and $a_j = \frac{c}{m}(j - 0.5) \geq 0$ for $j = 1, 2, 3, ..., m$. The equation (16) then becomes a system of m linear of equation (17) in the m unknowns $L(a_1), L(a_2), ..., L(a_m)$. For numerical method, it can be written in matrix form as

$$(I_m - R_{m\times m})\, L_m = 1_m \quad (18)$$

where $L_{m\times 1} = (L(a_1)\; L(a_2)\; ...\; L(a_m))'$ and

$$R_{m\times m} = \begin{bmatrix} \frac{1}{\lambda} w_1 f\left(\frac{a_1 - (1-\lambda)a_1 - \lambda(a + b + \upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right) & \cdots & \frac{1}{\lambda} w_m f\left(\frac{a_m - (1-\lambda)a_1 - \lambda(a + b + \upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right) \\ \vdots & \vdots & \vdots \\ \frac{1}{\lambda} w_1 f\left(\frac{a_1 - (1-\lambda)a_m - \lambda(a + b + \upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right) & \cdots & \frac{1}{\lambda} w_m f\left(\frac{a_m - (1-\lambda)a_m - \lambda(a + b + \upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right) \end{bmatrix}$$

and $I_m = \text{diag}(1, 1, 1, ..., 1)$ is the unit matrix of order m. If there exists $(I_m - R_{m\times m})^{-1}$, then the solution of the matrix equation (18) is

$$L_m = (I_m - R_{m\times m})^{-1} 1_m \quad (19)$$

We can approximate the function $L(u)$ in equation (16) as

$$\bar{L}(u) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a) f\left(\frac{a_j - (1-\lambda)a_i - \lambda(a + b + \upsilon\sum_{j=1}^{p}\phi_j)}{\lambda}\right), \quad (20)$$

where $w_j = \frac{c}{m}$ and $a_j = \frac{c}{m}(j - 0.5)$ for $j = 1, 2, 3, ..., m$.

Numerical Results

In this section, we compare the closed form expression given in Theorem 2 for $L(u)$ = ARL for the EWMA chart when data is the trend EAR(p) model with the approximate numerical solution for the ARL = $\bar{L}(u)$ given in equation (20). We define the relative error as a measure of accuracy of the comparisons:

$$\varepsilon_r = \frac{|L(u) - \bar{L}(u)|}{L(u)}$$

Table 1 Upper limits by explicit formula with λ = 0.25

ARL_0	ρ = 0.05	0.1	0.2	0.3	0.4
500	0.2709690	0.2555894	0.2285049	0.2043548	0.1829966
1000	0.2712625	0.2561741	0.2287596	0.2458660	0.1832074

We used equations (15) and (20) to compute the ARL for the second order trend autoregressive model

(AR(2)) with α = 1, the pairs (ϕ_1, ϕ_2) considered was $\rho = \phi_1, \phi_2$ corresponding to these pairs are the values of $\lambda = 0.25$; ρ = 0.05, 0.1, 0.2, 0.3, 0.4; $a = 0$; $b = 0.2$; and initial value is 1, respectively. All results were performed with programs written in the MATLAB software. The upper limits for EWMA chart were chosen by given desired ARL_0 = 500, 1000 are shown in Table 1.

Tables 2 and 3 show a comparison of exact and numerical solutions for the trend EAR(2) model for given ARL_0 = 500. In Table 2, we assume $\lambda = 0.25$, $\rho = 0.05$, $c = 0.2709690$ and the number of division points in the Gauss-Legendre rule $m = 500$. For α = 1 the process is in control whereas for α > 1 the process is out of control. In Table 3, we assume $\lambda = 0.25$, $\rho = 0.2$, $c = 0.2285049$ and the number of division points in the Gauss-Legendre rule $m = 500$. The results obtained from both methods are again in good agreement with less than 0.04 relative errors for the range of the values. The CPU time based on the exact solution take less than 1 second while the numerical integral equation takes approximately 10 minutes.

Table 2 Comparison of ARL_1 values for the trend EAR(2) model from explicit (Exact) formula and numerical approximation (IE) for ARL_0 = 500, c = 0.2709690, λ = 0.25 and m = 500

α	ρ = 0.05		ε_r
	Exact	IE	
1.01	85.7022	85.6153	0.0010
1.03	32.8471	32.7477	0.0030
1.05	20.6157	20.6099	0.0003
1.07	15.1756	15.1596	0.0011
1.10	11.0140	11.0013	0.0012
1.30	4.3976	4.4864	0.0253
1.50	3.0527	3.0058	0.0154

Table 3 Comparison of ARL_1 values for the trend EAR(2) model from explicit (Exact) formula and numerical approximation (IE) for ARL_0 = 500, c = 0.2285049, λ = 0.25 and m = 500

α	ρ = 0.2		ε_r
	Exact	IE	
1.01	77.6699	77.5616	0.0014
1.03	29.4613	29.4127	0.0016
1.05	18.4689	18.2244	0.0132
1.07	13.5995	13.3409	0.0190
1.10	9.8829	9.8329	0.0051
1.30	3.9911	3.8656	0.0314
1.50	2.7980	2.6983	0.0356

Tables 4 and 5 show a comparison of exact and numerical solutions for the trend EAR (2) model for given ARL_0 = 1000. In Table 4, we assume $\lambda = 0.25$, $\rho = 0.05$, $c = 0.2712625$ and the number of division points in the Gauss-Legendre rule $m = 500$. For α = 1 the process is in control whereas for α > 1 the process is out of control. In Table 3, we assume $\lambda = 0.25$, ρ = 0.2, $c = 0.2287596$ and the number of division points in the Gauss-Legendre rule $m = 500$.

Table 4 Comparison of ARL_1 values for EAR(2) model from explicit (Exact) formula and numerical 171 approximation (IE) for ARL_0 = 1000, c = 0.2712625, λ = 0.25 and m = 500

α	**ρ = 0.05**		**ε_r**
	Exact	**IE**	
1.01	93.6261	93.5223	0.0011
1.03	33.9211	33.9904	0.0020
1.05	21.0238	21.0037	0.0010
1.07	15.3906	15.2987	0.0060
1.10	11.1231	11.0926	0.0027
1.30	4.4119	4.4022	0.0022
1.50	3.0586	3.0023	0.0184

Table 5 Comparison of ARL_1 values for the trend EAR(2) model from explicit (Exact) formula and 178 numerical approximation (IE) for ARL_0 = 1000, c = 0.2287596, λ =0.25 and m = 500

α	**ρ = 0.2**		**ε_r**
	Exact	**IE**	
1.01	84.1188	84.0272	0.0011
1.03	30.3208	30.2719	0.0016
1.05	18.7948	18.6904	0.0056
1.07	13.7712	13.6537	0.0085
1.10	9.9701	9.8812	0.0089
1.30	4.0027	4.0001	0.0006
1.50	2.8028	2.7991	0.0013

Conclusions

We have presented that the explicit formulas for ARL_0 and ARL_1 of one-sided EWMA charts for the trend EAR(p) model. We have shown that suggested formulas are very accurate, and are easy to calculate and program. The suggested formulas obviously take the computational times much less than IE approximation. Using the formulas, we have been able to provide tables for the optimal weights, boundaries and approximations for ARL_0 and ARL_1 for one-sided EWMA charts for the trend EAR(p) model.

Acknowledgements

This research was supported by A New Researcher Scholarship of CSTS, MOST. Coordinating Center for Thai Government Science and Technology Scholarship Students (CSTS). National Science and Technology Development Agency NSTDA.

References

Areepong, Y. (2009). *An Integral Equation Approach for Analysis of Control Charts*. Ph.D. Thesis, University of Technology, Sydney, Australia.

Brook, D., & Evans, D. A. (1972). An approach to the probability distribution of CUSUM run length. *Biometrika*, *59*(3), 539-549.

Busaba, J., Sukparungsee, S., Areepong, Y., & Mititelu, G. (2012). Analysis of average run length for CUSUM procedure with negative exponential data. *Chiang Mai Journal of Science*, *39*(2), 200-208.

Calzada, M. E., & Scariano, S. M. (2003). Reconciling the integral equation and Markov chain approaches for computing EWMA average run lengths. *Communications in Statistics-Simulation and Computation*, *32*(2), 591-604.

Calzada, M. E., & Scariano, S. M. (2004). Average run length computations for the three-way chart. *Communications in Statistics-Simulation and Computation*, *33*(2), 505-524.

Chang, Y. M., & Wu, T. L. (2011). On average run lengths of control charts for autocorrelated processes. *Methodology and Computing in Applied Probability*, *13*(2), 419-431.

Crowder, S. V. (1987). A simple method for studying run–length distributions of exponentially weighted moving average charts. *Technometrics*, *29*(4), 401-407.

Haq, A., Brown, J., & Moltchanova, E. (2015). New exponentially weighted moving average control charts for monitoring process mean and process dispersion. *Quality and Reliability Engineering International*, *31*(5), 877-901.

Huang, W., Shu, L., & Jiang, W. (2012). Evaluation of exponentially weighted moving variance control chart subject to linear drifts. *Computational Statistics & Data Analysis*, *56*(12), 4278-4289.

Mititelu, G., Areepong, Y., Sukparungsee, S., & Novikov, A. (2010). Explicit analytical solutions for the average run length of CUSUM and EWMA charts. *East-West Journal of Mathematics*, 253-265.

Morais, M. C., Okhrin, Y., & Schmid, W. (2015). Quality surveillance with EWMA control charts based on exact control limits. *Statistical Papers*, *56*(3), 863-885.

Petcharat, K., Areepong, Y., Sukparungsee, S., & Mititelu, G. (2014). Exact solution for average run length of CUSUM charts for MA(1) Process. *Chiang Mai Journal of Science*, *41*, 1449-1456.

Petcharat, K., Sukparungsee, S., & Areepong, Y. (2015). Exact solution of the average run length for the cumulative sum chart for a moving average process of order q. *ScienceAsia*, *41*, 141-147.

Rabyk, L., & Schmid, W. (2016). EWMA control charts for detecting changes in the mean of a long-memory process. *Metrika*, *79*(3), 267-301.

Riaz, M. & Ahmad, S. (2016). On designing a new Tukey-EWMA control chart for process monitoring. *The International Journal of Advanced Manufacturing Technology*, *82*(1-4), 1-23.

Roberts, S. W. (1959). Control Chart Tests Based on Geometric Moving Averages. *Technometrics*, *20*(1), 239-250.

Saleh, N. A., Mahmoud, M. A., & Abdel-Salam, A. S. G. (2013). The performance of the adaptive exponentially weighted moving average control chart with estimated parameters. *Quality and Reliability Engineering International*, *29*(4), 595-606.

Suriyakat, W., Areepong, Y., Sukparungsee, S., & Mititelu, G. (2012). On EWMA procedure for AR(1) observations with exponential white noise. *International Journal of Pure and Applied Mathematics*, *77*, 73-83.

Zhang, M., Nie, G., He, Z., & Hou, X. (2014). The Poisson INAR(1) one-sided EWMA chart with estimated parameters. *International Journal of Production Research*, *52*(18), 5415-5431.