Research Article

A Characterization of Groups Whose Lattices of Subgroups are n–M_{p+1} Chains for All Primes p

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Abstract

Whitman, P.M. and Birkhoff, G. answered a well-known open question that for each lattice **L** there exists a group G such that **L** can be embedded into the lattice Sub(G) of all subgroups of G. Gratzer, G. has characterized that G is a finite cyclic group if and only if Sub(G) is a finite distributive lattice. Ratanaprasert, C. and Chantasartrassmee, A. extended a similar result to a subclass of modular lattices M_m by characterizing all integers $m \geq 3$ such that there exists a group G whose Sub(G) is isomorphic to M_m and also have characterized all groups G whose Sub(G) is isomorphic to M_m for some integers m. On the other hand, a very well-known open question in Group Theory asked for the number of all subgroups of a group. In this paper, we consider the extension of the subclass M_m for all integers $m \geq 3$ of modular lattices, the class of n–M_{p+1} chains for all primes p, and all $n \geq 1$ and characterized all groups G whose Sub(G) is an n–M_{p+1} chain. It happens that G is a group whose Sub(G) is an n–M_{p+1} chain if and only if G is an abelian p-group of the form $Z_{p^n} \times Z_p$. Moreover, we can tell numbers of all subgroups of order p_i for each $1 \leq i \leq n$ of the special class of p-groups.

Key Words: Modular lattice; Lattice of subgroups; p-group

Introduction

A lattice **L** is a non-empty ordered set in which each pair of elements a, b of L has the least upper bound denoted by $a \vee b$ and the greatest lower bound denoted by $a \wedge b$. Whitman, P.M. (1946) proved that for each lattice **L** there exists a set X such that **L** can be embedded into the lattice of all equivalence relations on X. One can show that the set Sub(G) of all subgroups of a group G forms a lattice in which $H \vee K = \langle H \cup K \rangle$ and $H \wedge K = H \cap K$ for each pair of elements H, K of Sub(G). We call Sub(G), the lattice of subgroups. Birkhoff, G. (1967) proved that every lattice of all equivalence relations on a set X is isomorphic to the lattice Sub(G) of a group G. These results answered a well-known open question that for each lattice **L** whether there exists a group G such that **L** can be embedded into Sub(G).

A lattice **L** is said to be *distributive* if it satisfies the distributive law; that is, $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$ for all $a,b,c \in L$. Zembery, I. (1973) answered the open question in a special class of lattices by proving that every finite distributive lattice can be embedded into Sub(G) for some abelian group G. Further, Gratzer, G.(1978) has characterized that G is a finite cyclic group if and only if Sub(G) is a finite distributive lattice; and he also proved that Sub(G) of a finite cyclic group G is isomorphic to a product of finite chains. We can conclude that for each finite distributive lattice **L** there exists a finite cyclic group G such that **L** can be embedded into Sub(G). A lattice **L** is said to be *modular* if it satisfies the modular law; that is, $a \geqslant c$ implies that $a \wedge (b \vee c) = (a \wedge b) \vee c$ for all $a,b,c \in L$. It is well-known that if **L** is distributive then **L** is modular. Let $m \geqslant 3$ be a positive integer and let M_m be the set $\{0,1,a_1,a_2 \ldots, a_m\}$ satisfying $0 \leqslant x \leqslant 1$ for all $x \in M_m$ and has no other comparabilities. It is obvious that M_m is a finite modular lattice which is not distributive for each $m \geqslant 3$. It is also proved by Fraleigh, J. B. (1982) that if G is a group whose Sub(G) is isomorphic to M_m for some $m \geqslant 3$ then G is not cyclic. It is known that if G is an abelian group then Sub(G) is modular; but the converse is not always true; for instance, Sub(D_3) the set of all subgroups of the dihedral group D_3 is isomorphic to M_4. Ratanaprasert, C. and Chantasartrassmee, A. (2004) have characterized all groups G whose Sub(G) is isomorphic to M_m for some $m \geqslant 3$. We proved the following theorems.

Theorem 1.1 : Let $m \geqslant 3$ be a positive integer. Then there is a group G whose Sub(G) is isomorphic to M_m if and only if $m = p+1$ for some prime p.

Theorem 1.2 : Let G be a group. Then Sub(G) is isomorphic to M_3 if and only if G is isomorphic to $Z_2 \times Z_2$.

Theorem 1.3 : Let G be a group and p be a prime number. Then Sub(G) is isomorphic to M_{p+1} if and only if either G is isomorphic to $Z_p \times Z_p$ or G is a non-abelian group of order pq, where q is a prime number with q divides p–1, generated by elements c, d such that $c^p = d^q = e$, where e denotes the identity of G and $dc = c^s d$ where s is not congruence to 1 modulo p and $s^q \equiv 1 \pmod p$.

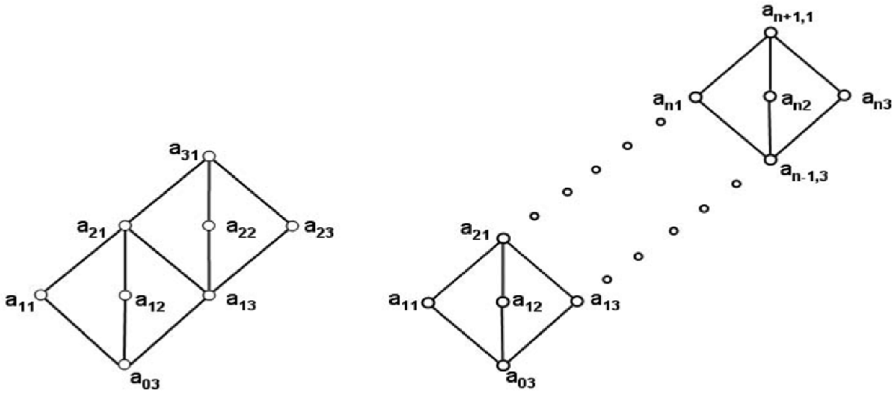
Corollary 1.4 : Let G be a non-abelian group whose Sub(G) is isomorphic to M_{p+1} for some prime p. Then (i) p is an odd prime and (ii) G is of order pq where q is a prime number with q divides p–1 and G contains exactly one subgroup of order p and p subgroups of order q.

Groups whose lattices of subgroups are n-M_3 chains

By the Structure Theorems for Finite Abelian Groups and Theorem 1.2, we look for the diagram of the lattice Sub($Z_{2^2} \times Z_2$) of all subgroups of the abelian p-group $Z_{2^2} \times Z_2$ where $Z_{2^2} := \{0, 1, 2, 3\}$ be the (additive) group of integers modulo 4. One can see that all subgroups of the direct product $Z_{2^2} \times Z_2 = \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1), (3,0), (3,1)\}$ are $a_{03} := \{(0,0)\} = \langle(0,0)\rangle$, $a_{11} := \{(0,0), (0,1)\} = \langle(0,1)\rangle$, $a_{12} := \{(0,0), (2,1)\} = \langle(2,1)\rangle$, $a_{13} := \{(0,0), (2,0)\} = \langle(2,0)\rangle$, $a_{21} := \{(0,0), (0,1), (2,0), (2,1)\} = \langle(0,1), (2,0)\rangle$, $a_{22} := \{(0,0), (1,1), (2,0), (3,1)\} = \langle(1,1)\rangle$, $a_{23} := \{(0,0), (1,0), (2,0), (2,1)\} = \langle(1,0)\rangle$ and $a_{31} := Z_{2^2} \times Z_2 = \langle(1,0), (0,1)\rangle$; and the diagram of the lattice Sub($Z_{2^2} \times Z_2$) is shown in Figure 1(a). For general case, we have the following proposition.

Proposition 2.1 : For each integer $n \geqslant 2$, all subgroups of $Z_{2^n} \times Z_2$ are (a) $\langle(1,0), (0,1)\rangle$, (b) $\langle(1,0)\rangle$, (c) $\langle(1,1)\rangle$, (d) $\langle(2,0), (0,1)\rangle$ or (e) a subgroup of $\langle(2,0), (0,1)\rangle$.

Proof : Let T be a subgroup of $Z_{2^n} \times Z_2$ and for $i \in \{1,2\}$ let p_i be the projection maps of $Z_{2^n} \times Z_2$ on Z_{2^n} and Z_2, respectively. Then each p_i for $i \in \{1,2\}$ is a homomorphism; hence, $p_1(T)$ and $p_2(T)$ are



(a) (b)

Figure 1

subgroups of Z_{2^n} and Z_2, respectively. If $|T| = 2^{n+1}$ then $T = Z_{2^n} \times Z_2 = \langle(1,0), (0,1)\rangle$. Now, we consider the case $|T| = 2^n$. If $p_2(T) = \{0\}$ then $p_1(T) = Z_{2^n}$; hence, $T = Z_{2^n} \times \{0\} = \langle(1,0)\rangle$. We assume that $p_2(T) = \{0,1\} = Z_2$. If $1 \in p_1(T)$ then $(1,1) \in T$; so, T is a cyclic subgroup $\langle(1,1)\rangle$ since the order of (1,1) is $2^n = |T|$. But, if $1 \notin p_1(T)$ then $p_1(T)$ is a subgroup of $\langle 2\rangle = \{2a \mid a \in Z_{2^n}\}$ since every odd integer in Z_{2^n} is its generator. Now, $p_1(T) = \langle 2\rangle$ and $p_2(T) = \langle 1\rangle$ imply that (2,0) and (0,1) are in T; so, $\langle(2,0), (0,1)\rangle$ is a subgroup of T. Since each element of $\langle(2,0),(0,1)\rangle$ is a linear combination of the form s(2,0) + t(0,1) where $1 \leqslant s \leqslant 2^{n-1}$ and $1 \leqslant t \leqslant 2$, the subgroup $\langle(2,0),(0,1)\rangle$ contains $2(2^{n-1}) = 2^n$ distinct elements. So, $|T| = 2^n = |\langle(2,0),(0,1)\rangle|$. Therefore, $T = \langle(2,0), (0,1)\rangle$. Finally, if $|T| < 2^n$ then $p_1(T)$ is a subgroup of $\langle 2\rangle$ since $p_1(T) =$ implies that $T = Z_{2^n} \times \{0\}$ or $T = Z_{2^n} \times Z_2$ in which cases imply $|T| = 2^n > |T|$, a contradiction. Therefore, T is a subgroup of $\langle (2,0), (0,1) \rangle$.

We will generalize the lattice in Figure 1(b) in the following proposition.

Proposition 2.2 : Let n be a positive integer and let $\leqslant^*$ be the usual order on the set $\mathbf{Z}^+ \cup \{0\}$ of all nonnegative integers. If $L := \{a_{ij} \mid 1 \leqslant^* i \leqslant^* n \text{ and } 1 \leqslant^* j \leqslant^* 3\} \cup \{a_{03}, a_{(n+1)1}\}$ and $\leqslant \subseteq L \times L$ is defined by $a_{v3} \leqslant a_{ij} \leqslant a_{ij} \leqslant a_{u1}$ for all $0 \leqslant^* v <^* i <^* u \leqslant^* n+1$ and all $1 \leqslant^* j \leqslant^* 3$ and there are no other comparabilities, then $\mathbf{L} = (L; \leqslant)$ is a lattice.

Proof : It is obvious from the definition of $\leqslant$ that $\leqslant$ is reflexive. Let $x, y \in L$ satisfy $x \leqslant y$ and $y \leqslant x$. Then there are integers $p,q,r,s \in \{0,1,2, ..., n+1\}$ such that $x = a_{pq}$ and $y = a_{rs}$. If q = 3 and since $a_{rs} = y \leqslant x = a_{pq}$, we have s=3; but $p \neq r$ implies by the definition of $\leqslant$ that $r <^* p$ and $p <^* r$ which contradicts to the trichotomy law for $\leqslant^*$; hence, p = r; and so, $x = a_{pq} = a_{rs} = y$. If q = 2 then $a_{p2} = x \leqslant y = a_{rs}$ implies that s=1 or s=2; but s = 1 implies $a_{r1} = y \leqslant x = a_{p2}$ which contradicts to the definition of $\leqslant$; so, s=2 = q. Also, $p \neq r$ implies a similar contradiction as above; hence, p = r. Therefore, x= y. If q=1, then $a_{p1} = x \leqslant y = a_{rs}$ which shows s = 1 and $p \leqslant^* r$. Now, $a_{r1} = y \leqslant x = a_{p1}$ implies that $r \leqslant^* p$. So, p = r. Hence, x = y. In any cases, x = y which shows that $\leqslant$ is anti-symmetric.

Now, let $x,y,z \in L$ satisfy $x \leqslant y$ and $y \leqslant z$. Then there are integers $p,q,r,s,u,v \in \{0,1, ..., n+1\}$ such that $x = a_{pq}$, $y = a_{rs}$ and $z = a_{uv}$; so, $a_{pq} \leqslant a_{rs}$ and $a_{rs} \leqslant a_{uv}$. Since $a_{pq} = a_{rs}$ or $a_{rs} = a_{uv}$ implies that $x \leqslant z$, we consider the case $a_{pq} \neq a_{rs}$ and $a_{rs} \neq a_{uv}$ which implies by the definition of $\leqslant$ that $p <^* r$ and $r <^* u$; so, $p <^* u$. If q = 3 then $x = a_{p3} \leqslant a_{uv} = z$. And if q=2 then s=1; and so $a_{p3} \leqslant a_{uv}$ since $p <^* r$ implies that v = 1 and

$p <^* u$. Finally, if $q = 1$ then $s = v = 1$; and so, $a_{pq} \leqslant a_{uv}$ follows from $p <^* u$. Hence, in which cases, $x \leqslant z$. Therefore, $\leqslant$ is transitive.

To show that **L** is a lattice, let $x, y \in L$. If $x \leqslant y$ or $y \leqslant x$ then $x \vee y$ and $x \wedge y$ are in the set $\{x, y\}$. Let x and y be non-comparable. Then there are integers $p,q,r,s \in \{0,1,2, ..., n+1\}$ such that $x = a_{pq}$ and $y = a_{rs}$. We may assume that $p \leqslant^* r$. Then, since a_{pq} and a_{rs} are non-comparable, $1 \leqslant^* p \leqslant^* n$ and $1 \leqslant^* r \leqslant^* n$.

If $q = 1$ then $s \in \{2,3\}$. Since there are no integers c and d with $p-1 <^* c <^* p$ and $r <^* d <^* r+1$, we have $a_{(p-1)3} \prec a_{p1} = x \leqslant a_{(r+1)1}$ and $a_{(p-1)3} \leqslant a_{rs} = y \prec a_{(r+1)1}$ which shows $x \wedge y = a_{(p-1)3}$ and $x \vee y = a_{(r+1)1}$. If $q = 2$ and $r = p$ then $a_{pq} \wedge a_{ps} = a_{(p-1)3}$ and $a_{pq} \vee a_{ps} = a_{(p+1)1}$; but if $q = 2$ and $p <^* r$ then $s \in \{2,3\}$; so, $x \wedge y$ and $x \vee y$ will be as in the case $q = 1$. And if $q = 3$ then $p = r$; so, $a_{p3} \leqslant a_{ij}$ for all i with $p <^* i$ and for all $1 \leqslant^* j \leqslant^* 3$; so $x \wedge y$ and $x \vee y$ are as in the case $q = 2$ and $r = p$.

Definition : The lattice defined as in Proposition 2.2 is called *n-M_3 chain.*

Figure 1(b) shows the diagram of n-M_3 chain for $n \geqslant 1$. For a special case, we note that M_3 is 1-M_3 chain and Theorem 1.2 showed that Sub($\mathbf{Z}_2 \times \mathbf{Z}_2$) is isomorphic to 1-$M_3$ chain (which is M_3). We now prove in general case that Sub($Z_{2^n} \times Z_2$) is isomorphic to n-M_3 chain for each positive integer n.

Proposition 2.3 : Sub($Z_{2^n} \times Z_2$) is isomorphic to n-M_3 chain for each positive integer n.

Proof : We will prove the proposition by mathematical induction. By Theorem 1.2, Sub($\mathbf{Z}_2 \times \mathbf{Z}_2$) is isomorphic to 1-$M_3$ chain. We may assume that k is a positive integer such that Sub($Z_{2^k} \times Z_2$) is isomorphic to k–M_3 chain and we will prove the proposition for k+1.

By Proposition 2.1, all the subgroups of $Z_{2^{k+1}} \times Z_2$ are $\overline{1}$:= <(1,0),(0,1)>, a := <(2,0),(0,1)>, b := <(1,1)>, c := <(1,0)> or a subgroup of <(2,0),(0,1)>. Since <(2,0),(0,1)> is isomorphic to Sub($Z_{2^k} \times Z_2$), the induction hypothesis implies that Sub(<(2,0),(0,1)>) is isomorphic to k–M_3 chain. It is clear that $\{\overline{1}$, a, b, c, d }, where d = <(2,0)>, is isomorphic to M_3. Hence, Sub($Z_{2^{k+1}} \times Z_2$) is isomorphic to (k+1)–M_3 chain which completes the proof.

Theorem 1.2 and Corollary 1.4(i) also showed that there are no non-abelian groups G such that Sub(G) is isomorphic to n-M_3 chain for all n. We are going to prove in the following theorem that it is also true in the class of n-M_3 chains for all positive integers n.

Theorem 2.4 : Let G be a group and $n \geqslant 3$ be an integer. Then Sub(G) is an n-M_3 chain if and only if G is isomorphic to $Z_{2^n} \times Z_2$.

Proof : The converse of the theorem follows by Proposition 2.3. Let G be a group whose Sub(G) is an n-M_3 chain. Then G is finite and Theorem1.2 implies that G cannot be non-abelian; and also, the Structure Theorem of Finite Abelian Group implies that G is of the form $Z_{p_1^{t_1}} \times Z_{p_2^{t_2}} \times \ldots \times_{p_r^{t_r}}$ where p_i are primes for $1 \leqslant i \leqslant r$. Since an n-$M_3$ chain is not distributive, G is not a cyclic group; so, there exists a prime factor p of $|G|$ such that $Z_p \times Z_p$ is a subgroup of G. So, Theorem 1.1 told us that Sub($\mathbf{Z}_p \times \mathbf{Z}_p$) has at least p+1 atoms. Hence, Cauchy's Theorem implies that all atoms of Sub($\mathbf{Z}_p \times \mathbf{Z}_p$) are atoms of G and there are no other prime q differ from p which is a divisor of $|G|$. So, p+1=3; that is, p=2 is the only prime factor of $|G|$. If $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ is a subgroup of G, then one of M_3 in the n-M_3 chain has at least 7 atoms since $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ contains 7 distinct elements of order 2 which contradicts to the form of an n-M_3 chain that each M_3 in the chain has exactly 3 non-comparable elements. So, G is of the form $Z_{2^n} \times Z_{2^m}$ for some positive integers n and m. Suppose that n>1 and m>1. Then a subgroup $Z_{2^2} \times Z_{2^2}$ of G contains 4 subgroups <(1,0)>, <(0,1)>, <(1,1)> and <(2,0),(0,2)> of order 4 which are non-comparable in Sub($Z_{2^2} \times Z_{2^2}$) and also are in Sub(G). Since G contains only 3 subgroups of the same order which are non-comparable, we get a contradiction. Hence, n =1 or m = 1. Therefore, G is $Z_{2^n} \times Z_2$ for some positive integers n which completes the proof.

Corollary : A lattice **L** is isomorphic to Sub($Z_{2^n} \times Z_2$) for some positive integer n if and only if it is an n-M_3 chain.

Groups whose lattices of subgroups are n-M_{p+1} chains for some odd primes p

Let p be an odd prime number and n be a positive integer. We will now give the definition of n-M_{p+1} chains by extending the definition of n-M_3 chains as follows.

Let L := $\{a_{ij} \mid 1 \leqslant^* i \leqslant^* n \text{ and } 1 \leqslant^* j \leqslant^* p+1\} \cup \{a_{0(p+1)}, a_{(n+1)1}\}$ and $\leqslant \subseteq L \times L$ be defined by $a_{v(p+1)} \leqslant a_{ij} \leqslant a_{ij} \leqslant a_{u1}$ for all $0 \leqslant^* v \leqslant^* i \leqslant^* u \leqslant^* n+1$ and all $1 \leqslant^* j \leqslant^* p+1$ and there are no other comparabilities. Then one can repeat the proof in Proposition 2.2 with p+1 in place of 3 to conclude that **L** = (L; $\leqslant$) is a lattice which will be called an *n-M_{p+1} chain*.

We begin to prove that there is no non-abelian group G whose Sub(G) is isomorphic to an n-M_{p+1} chain if n > 1 and p > 2.

Proposition 3.1 : If G is a group whose Sub(G) is isomorphic to an n-M_{p+1}chain for some odd prime p and some integer n > 1, then G is an abelian group of the form $Z_{p^n} \times Z_p$

Proof : Suppose that there is a non-abelian group G whose Sub(G) is isomorphic to an n-M_{p+1}chain for some integers n >1 and primes p > 2. Then Theorem 1.3 and Corollary 1.4(i) imply that the subgroup H := a_{21} of G which is the top of the first M_{p+1} of the n-M_{p+1} chain must be either $Z_p \times Z_p$ or a non-abelian group of order pq where q is a prime factor of p–1; hence, the prime q must be a factor of $|G|$. If H = $Z_p \times Z_p$, Cauchy's Theorem implies that $|G|$ cannot have other prime factors (except p); that is, G is of order p^t for some positive integer t. Since G is non-abelian, G is not H = $Z_p \times Z_p$; so the subgroup a_{31} of G is of order p^3. If a_{31} is abelian then a_{31} is $Z_p \times Z_p \times Z_p$ (a_{31} cannot be Z_{p^3} since the cyclic group cannot have $Z_p \times Z_p$ as its subgroup) and Sub($Z_p \times Z_p \times Z_p$) is not 2-$M_p$chain since it contains p^3–1 (> p+1) distinct elements of order p and each generates a subgroup which is an atom of Sub(G). So, a_{31} is a non-abelian group of order p^3 which has elements of order p^2 and has no elements of order p^3 ($\because$ if all elements of a_{31} are of order p or there is an element of a_{31} of order p^3 then either Sub(a_{31}) contains p^3–1 atoms which implies that Sub(a_{31}) is not a 2-$M_p$chain or a_{31} is cyclic; in which cases imply a contradiction). Since Sub(a_{31}) contain p+1 co-atoms which are subgroups of order p^2, a_{31} must contain exactly $(p+1)(p^2-1)+1 = p^3+p^2-p$ elements; so, $p^3+p^2-p = p^3$ which implies that p = 0 or p = 1 which contradicts that p is prime. Therefore, H is a non-abelian group of order pq where q is a prime factor of p–1; and also, p and q are the only prime factors of $|G|$. If n>1, Sub(a_{31}) contains p cyclic subgroups of order q and only one cyclic subgroup of order p which is Z_p. Since Sub($Z_p \times Z_p$) is M_{p+1}, the $a_{1(p+1)}$ in Sub(G) must be Z_p and $a_{22}, \ldots, a_{2(p+1)}$ are cyclic subgroups Z_{p^2} . So, a_{31} must contain exactly $pq + p(p^2-p)$ elements. By the First Sylow Theorem and p, q are the only prime factors of $|G|$, we have $pq + p(p^2-p) = p^tq$ where t >1 which implies that $p = q(p^{t-2}+ \ldots +1)$; hence, p = q or $p = q(p^{t-2}+ \ldots +1) > p$ which are impossible in both cases. Therefore, G is an abelian group.

The above argument also shows that there is only one prime number p which is a factor of $|G|$ and G cannot have $Z_p \times Z_p \times Z_p$ as its subgroup; so, G is of the form $Z_{p^n} \times Z_{p^m}$ for some positive integers n and m. Hence, a similar proof in Theorem 2.4 implies that G is of the form $Z_{p^n} \times Z_p$ which completes the proof.

We can state a similar theorem as Theorem 2.4 as follows.

Theorem 3.2 : Let n >1 be an integer and p be a prime number. Then a group G is $Z_{p^n} \times Z_p$ if and only if Sub(G) is an n–M_{p+1} chain.

One can note that both of the class of all n–M_3 chains for all integers n and the class of all n–M_{p+1} chains for all integers n >1 and all odd primes p are subclasses of the class of all modular lattices which are examples answering to the following open problem.

Open Problem : Find a (maximum) subclass M of modular lattices satisfying these 2 conditions :

(i) G is a finite abelian group if and only if Sub(G) is in M, and

(ii) **L** is a lattice in M if and only if **L** is isomorphic to Sub(G) for some finite abelian group G.

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