Goodness of Fit of Cumulative Logit Models for Ordinal Response Categories and Nominal Explanatory variables with Two-Factor Interaction

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Abstract

Power and the assessing goodness of fit of cumulative models for ordinal response data with two nominal interaction term of explanatory variables are investigated. The magnitude of goodness-of-fit statistics, the coefficients of determination or R^2 analogs, the likelihood ratio statistic,G_M, AIC (Akaike Information Criterion, Akaike, 1973),and BIC (Bayesian Information Criterion, Schwarz, 1978) are calculated. The simulations have been conducted for the multinomial logit models with K=3 response categories and two random explanatory variables X_1 and X_2 whose joint distribution of (X_1, X_2) is assumed to be multinomial with probabilities π_1 π_2 π_3 and π_4, corresponding to (X_1, X_2) values of (0, 0), (0,1), (1, 0), (1, 1), respectively. Three sets of $(\pi_1, \pi_2, \pi_3, \pi_4)$ are studied to represent different distributional shapes, which were chosen to induce possibly strong effects such that $\beta_1 = \log 2$, $\beta_2 = \log 3$, and $\beta_{12} = 0.0 - 4.5$ (increment 0.3), namely (X_1, X_2)~multinomial (0.10,0.35,0.45,0.10), (X_1, X_2)~ multinomial (0.50,0.30,0.10,0.10), and (X_1, X_2)~multinomial (0.25,0.25, 0.25,0.25). Four sets of the three ordered category distributing corresponding with the (X_1, X_2) were again generated through the models under the proportions of (p_1, p_2, p_3), namely Y~multinomial(p_1, p_2, p_3): (0.05,0.20,0.75), (0.25,0.50,0.25), (0.5,0.20,0.25), and (0.33,0.33,0.33) from which it follows that the true model intercepts are $\alpha_1 = \log \dfrac{p_1}{p_2+p_3}$, $\alpha_2 = \log \dfrac{p_1+p_2}{p_3}$,corresponding to the proportions of Y = 1, 2, 3 respectively. Four sample sizes of 600, 800, 1,000, and 1,500 units were performed. Each condition was carried out for 1,000 repeated simulations using the developed macro program run with the Minitab Release 11.

The results under the distribution conditions of (X_1, X_2)~ multinomial (0.1,0.35,0.45,0.1) and Y ~(0.55, 0.20, 0.25) show that all goodness-of-fit statistics perform better than those of the distribution conditions of which Y~(0.25,0.5,0.25) and Y~(0.33,0.33,0.33) in term of the power of the tests, means and standard deviations of goodness-of-fit statistics. These results are also similar to the condition when (X_1, X_2)~ (0.50,0.30,0.1,0.1). However, when the distribution conditions are symmetric such that (X_1, X_2)~ (0.25,0.25,0.25,0.25) and Y~ (0.33,0.33,0.33) all statistics are much generally improved the model fits. In conclusion it probably is recommended to use large sample sizes in the analysis of ordinal categorical responses when the distributions of variables are asymmetric, except only when the distribution of the response categories is clearly increasing in order. Besides this, there is also a tendency to improve the model fit by using the models with an interaction term when the correlated structures between the explanatory variables are evident.

Key Words: Cumulative logit models; Interaction terms; Goodness of fits; Multinomial ordinal responses.

Introduction

Statistical modeling is generally an iterative process. A minimal/initial model is developed, and fitted to a data set and examined. Further models for the data may then be proposed and specified, with the form of the current model being based on the information provided by the previous models. Throughout the last thirty or so years, statistical modeling has been centered around the classical linear models (LMs) which have focused on the normal distribution properties and homogeneity (constant variance), for example, regression models, ANOVA models (Aitkin et al., 1989). In the present texts, statistical modeling for categorical data analyses are widely used and are developing rapidly so that applying generalized linear models (GLMs) become common and well known statistical models under the impetus of application in disciplines as widely varied as the following: agriculture, demography, ecology, economics, education, engineering, environmental studies and pollution, geography, geology, health science, history, computer, medicine, political science, psychology and sociology (Lindsey, 1997; Lawal, 2003). GLMs appeared on the statistical scene in the path breaking article of Nelder and Wedderburn (1972). They generalize the classical linear models based on the normal distribution to involve two aspects: a variety of distributions from continuous to discrete or categorical, exponential family distribution models and they also involve transformations of mean, through the link functions, linking the systematic part of models to the mean of one of the distributions. Thus, GLMs are now a mature data-analytic methodology. They are developed to handle the correlated structures and overdispersion known as Generalized Estimating Equations (GEEs) by Liang and Seger (1986), as well as, ways to handle Generalized Additive Models (GAMs) by Hastie and Tibshirani (1990). GLMs have also been developed further by, for example, Jorgensen (1997) for modifications of exponential families, Heyde (1997)

for the theory of quasi-likelihood, and McCulloch and Searle, 2001 for Generalized Linear and Mixed Models (GLMMs).

More recently, log-linear models and logit models for discrete, categorical data become well known in the social sciences, applied science, and medicine. The cumulative logit models such as the proportional odds models (Walker and Duncan, 1967; McCullagh, 1980) and the continuation-ratio models for ordinal response (Fienberg, 1980) have been the primary focus in epidemiological and biomedical applications (Amstrong and Sloan,1989; Peterson and Harrell, 1990; Lipsitz et al., 1996; Cole et al., 2003) while other models for the analysis of ordinal outcomes have received less attention.

This paper presents a synthesized GLMs, generalized logit models for analyzing ordinal responses corresponding to the nominal explanatory variables, with and without two-factor interaction. However, many models for ordinal response are developed under rather strong assumption such as the proportional odds assumption, which becomes a popular model for analyzing studies with an ordered categorical outcome. Departures from these assumptions may well result in the incorrect model formulation. Thus, the purpose of this research is to analyze the performance of the proportional odds ratio models, which contain only the main effects and that the interaction effect using goodness-of-fit-statistics and the power of tests.

The Cumulative Logit Models

The cumulative logit model was originally proposed by Walker and Duncan (1967) and later called the proportional odds model by McCullagh (1980). The cumulative logits are defined (Agresti, 2002) as

$P(\ Y \leq j \mid x\ ) \ = \ p_1 + p_2 + \ldots\ldots + p_j, \ j = 1,\ldots, K.$
Then,

$$\text{logit}\ [P(Y \leq j \mid x\ )] = \log\ [\frac{P(\ Y \leq\ j\ \mid \mathbf{x}\ )}{1 - P(\ Y \leq\ j\ \mid \mathbf{x}\ )}]$$

$$= \log |\frac{P(Y \leq j \mid \mathbf{x})}{P(Y > j \mid \mathbf{x})}]$$

$$= \log [\frac{p_1 + p_2 + \ldots + p_j}{p_{j+1} + p_2 + \ldots + p_K}|,$$

$$j = 1, 2, \ldots, K-1.$$

A model that simultaneously uses all cumulative logit is

$$\text{logit } P(Y \leq j \mid \mathbf{x}) = \alpha_j + \mathbf{x}'\beta, \quad j = 1, \ldots, K-1.$$

This model, which extends the logistic model for binary responses to allow for several ordinal responses, has often involved modeling cumulative logits, generalized cumulative logit models (Cole et al., 2003) and also those models often used in repeated measurement modeling (Mc.Culloch, 2000, Mc. Culloch and Searle, 2001). Consider a multinomial response variable Y with categorical outcomes, denoted by 1, ..., K and let X_i denote a p-dimensional vectors of explanatory variables or covariates. The dependence of the cumulative probabilities of Y on X's for the proportional odds model is often of the form in (1).

$$\log [\frac{P(Y \leq j \mid \mathbf{x})}{P(Y > j \mid \mathbf{x})}] = \alpha_j + \mathbf{x}'\beta, \quad j = 1, \ldots, K-1. \quad \ldots(1)$$

It can be expressed in the form

$$\log [\frac{p_1 + p_2 + \ldots + p_j}{p_{j+1} + p_2 + \ldots + p_K}] = \alpha_j + \mathbf{x}'\beta, \quad j = 1, \ldots, K-1.$$

Each cumulative logit has its own intercept. The $\{\alpha_j\}$ are increasing in j, since $P(Y \leq j \mid \mathbf{x})$ increases in j for fixed x, and the logit is an increasing function of this probability and each cumulative logit uses all K response categories.

Hence, for K=3, and j = 1,...,K-1=2, the model (1) consists of two simultaneously cumulative link-functions for solving the model parameters in the following equations:

$$\log [\frac{p_{i1}}{p_{i2} + p_{i3}}] = \alpha_j + \mathbf{x}'\beta, \quad \text{for } j = 1,$$

$$\log [\frac{p_{i1} + p_{i2}}{p_{i3}}] = \alpha_j + \mathbf{x}'\beta, \quad \text{for } j = 2, i = 1,2,...n$$

Where, α_j are the intercept parameters.

$\beta = (\beta_1, \beta_2, ..., \beta_p)'$ is a vector of coefficients

corresponding to X's, and

$$P(Y \leq j \mid x) = p_1 + p_2 + \ldots + p_j, \text{ and}$$

$$P(Y > j \mid x) = p_{j-1} + p_2 + \ldots + p_K, \quad j = 1, ..., K-1.$$

Similarly to (1), we have (2) and (3).

The proportional odds ratio model (minimal):

$$\log \left(\frac{p_{i,k+1} + \ldots + p_{iK}}{p_{i1} + \ldots + p_{ik}} \right) = \alpha_k + \beta_1 x_{1i} + \beta_2 x_{2i},$$

$$k = 1, 2, K = 3, \quad i = 1, 2, \ldots n \qquad \ldots\ldots(2)$$

The proportional odds ratio with two-factor-interaction model (Interaction):

$$\log \left(\frac{p_{i,k+1} + \ldots + p_{iK}}{p_{i1} + \ldots + p_{ik}} \right) = \alpha_k + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_{12} x_{1i} x_{2i},$$

$$k = 1, 2, K = 3, \quad i = 1, 2, \ldots n \qquad \ldots(3)$$

The model (1) for any K ≥ 3 is often called the proportional odds model (McCullagh, 1980). It is based on the assumption that the effects of the explanatory variables $X_1, ..., X_p$ are the same for all categories, on the logarithmic scale. It probably also represents the most widely used ordinal categorical model at the present time.

Simulation and Statistical Analyses

The simulations have been conducted for the multinomial logit models with K=3 response categories and two random explanatory variables X_1 and X_2 whose joint distribution of (X_1, X_2) is assumed to be multinomial with probabilities $\pi_1, \pi_2, \pi_3,$ and $\pi_4,$ corresponding to (X_1, X_2) values of (0, 0), (0,1), (1, 0), (1, 1), respectively. Three sets of ($\pi_1, \pi_2, \pi_3, \pi_4$) are studies to represent different distributional shapes, which were chosen to induce possibly strong effects such that $\beta_1 = \log2$, $\beta_2 = \log3$, and $\beta_{12} = 0.0 - 4.5$, namely $(X_1, X_2) \sim$ multinomial(0.10, 0.35, 0.45, 0.10), $(X_1, X_2) \sim$ multinomial (0.50,0.30,0.10,0.10), and $(X_1, X_2) \sim$ multinomial (0.25,0.25,0.25,0.25). Four sets of the three ordered category distributing corresponding with the (X_1, X_2) were again generated through the models studies in the form of (1.1)-(1.2)

under the proportions of (p_1, p_2, p_3), namely $Y \sim$ multinomial(p_1, p_2, p_3): (0.05,0.20,0.75), (0.25, 0.50,0.25), (0.5,0.20,0.25), and (0.33,0.33,0.33) from which it follows that the model parameters to be used in each condition are

$$\alpha_2 = \log \frac{p_1 + p_2}{p_3}, \beta_1 = \log 2, \text{ and } \beta_2 = \log 3 \text{ for varied}$$

β_{12} from 0-4.5 (increment 0.3), corresponding to the proportion of $Y = 1, 2, 3$ respectively. Consequently, the categorical responses are corresponded with X's under the true models, will be random at each setting of fixed values of the explanatory variables (X_1, X_2) through the cut points and the specified proportions. Four sample sizes were specified to vary from $n = 600, 800, 1,000$ and $1,500$ units. All results were performed for 768 (=4 x 3 x 4 x 16) conditions. Each of which for each model was carried out 1,000 replicates of data sets.

Statistical analyses in assessing goodness of fit of models consist of several statistics which were computed for each combination of the model conditions. The likelihood ratio statistics, the generalized coefficients of determination or R^2 analogs, the percentage correct classification (PCC) of predictive efficiency, the power of the tests, AIC (Akaike Information Criterion, Akaike, 1973), BIC (Baysian Information Criterion, Schwarz, 1978) are evaluated.

All the statistics were computed using the following formulae:

$G_M = -2 [\ln(L_O)-\ln(L_M)]$ (The model chi-square statistic)

The Coefficients of Determination , R^2 analogs:

$$R^2_C = \frac{G_M}{(G_M + n)} \text{ (The contingency coefficient } R^2,$$
$$\text{Aldrich \& Nelson, 1984.)}$$

$$R^2_L = \frac{[\ln(L_O) - \ln(L_M)]}{\ln(L_O)} = 1 - \left[\frac{\ln(L_M)}{\ln(L_O)}\right]$$

(The log likelihood ratio R^2, McFadden, 1974; Menard, 1995)

$$R^2_M = 1 - \left[\frac{L_O}{L_M}\right]^{\frac{2}{n}} \text{ (The geometric mean squared improvement per observation } R^2, \text{ Cox \& Snell, 1989; Maddala, 1983; Ryan, 1997)}$$

$$R^2_N = \frac{\left[1 - (\frac{L_O}{L_M})^{\frac{2}{n}}\right]}{\left[1 - (L_O)^{\frac{2}{n}}\right]} \text{ (The adjusted geometric mean squared improvement } R^2, \text{ Nagelkerke, 1991; Ryan, 1997)}$$

PCC = The average percentage correct classified of model from 1,000 data sets.

AIC = G_M - 2 (Δdf), BIC = G_M - (log(n))(Δdf), (Lawal, 2003).

The power of the test is the percentage corresponding to the rejection of H_0 when H_0 is false in 1,000 simulations.
Whereas,

n   = sample size

L_O = the likelihood function for the model containing only the intercept.

L_M = the likelihood function for the model containing all of the predictors.

G_M = -2 [ln(L_O) - ln(L_M)] = the model chi-square statistic.

All computer simulation programs were developed using the MINITAB macro language and run by MINITAB release 11 on Pentiums IV.

Research Results

Several models for analyzing data with ordinal responses have been fitted and also are examined their goodness-of-fits. The mean and standard deviation, based on 1,000 simulations, of each goodness-of-fit statistic (R_N, BIC, PCC) are summarized in Table 1 – Table 3 (appendix 5-7). All statistics are classified by Y's and X's distributions, β_{12}, and the different sample sizes. For $\beta_{12} = 0$, it corresponds with the cumulative model with main effects or without interaction, whereas, for $\beta_{12} \neq 0$, it do corresponds with the model with two-factor interaction.

The results are shown that the magnitude of goodness-of-fit statistics, the coefficients of determination or R^2 analogs, and the percentage correct classification (PCC) increase as both the sample sizes and the parameter β_{12} increase. For other R^2 analogs, results are all quite similar. We then report only the R_N or the Nagelkerke's R^2 analog and BIC statistics. The likelihood ratio statistic, and the BIC statistic tend to decrease as the sample sizes and β_{12} are large. Thus, statistics do vary dependently upon the distributions of Y and X's (Table 1-3).

Under the distribution conditions of (X_1, X_2) ~(0.1,0.35,0.45,0.1) and Y~(0.55,0.20,0.25). Most goodness-of-fit statistics perform better than those of distribution conditions of which Y~ (0.25,0.5,0.25) and Y~(0.33,0.33,0.33), except for R_N statistic (Table 1). These results are also similar to the condition when (X_1, X_2) ~ (0.50,0.30,0.1,0.1) (Table2). However when the distribution conditions are symmetric, such that (X_1, X_2) ~ (0.25, 0.25, 0.25, 0.25), all statistics for Y~ (0.33,0.33,0.33) are much generally improved the model fits (Table3).

Therefore, in terms of goodness-of-fit statistics, BIC and R^2 analog, the results indicate that when both distributions of Y's and X's are symmetric, they give good fits, especially when X's are most correlated or when β_{12} is large (Table3). However, when considering in term of PCC, the results are different. This is also possible because the conclusions based on R^2 analogs are not necessarily consistent with the conclusions based on the predictive efficiency, with respect to which of several outcomes is better predicted by a given model.

The comparisons in term of the power of the tests, all results concerning with power plots between power and the parameter β_{12} are compared among the four sample sizes: 600, 800, 1,000 and 1,500 for each combination of Y's and X's distributions (Figure 1-12 in appendix 1-4). It is found that the results do confirm the previous ones that the power of the tests varies according to the distributions of Y's and X's, β_{12} and the increasing sample sizes. Moreover,

the power of the tests when the distributions are symmetric, provide more and rapidly approach to 1.00 than those of the distributions are asymmetric, excepting the case when the distribution of Y is in increasing order, Y~(0.05,0.20,0.75)(Figure 1, 5, 9 in Appendix 1-4).

Conclusion and Recommendation

The results are concluded that the goodness-of-fit statistics and tests perform well and do vary according to different distributions of Y's and that of X_1, X_2, β_{12} and the sample sizes of 600, 800, 1,000, and 1,500 units. In addition, good performance in terms of power of the test and their means and standard deviations occur when the distributions of Y's and (X_1, X_2) probably have symmetric shapes (Figure 12 and Table 3) as well as when the distribution of Y is in increasing order (Figure 9).

Due to the above results, it is recommended that in practice for the situations encountered with the asymmetric distributions of Y's and X's, it is possibly safety to use large sample sizes for analysis of ordered categorical data in order to gain some power of the tests. Moreover, for the correlated structures of the explanatory variables, there is a tendency to improve a model fit with two-factor inter action to get a better fit of a model with ordinal response categories and their corresponding nominal explanatory variables (Figure 9 and Table 3).

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Appendix 1:

Figure 1-3 Power plots versus β_{12} for each condition of X_1, X_2 and Y distributions under 4 sample sizes

Appendix 2:

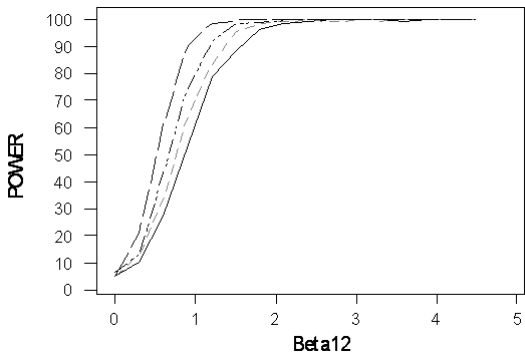**Figure 4-6** Power plots versus β_{12} for each condition of X_1, X_2 and Y distributions under 4 sample sizes

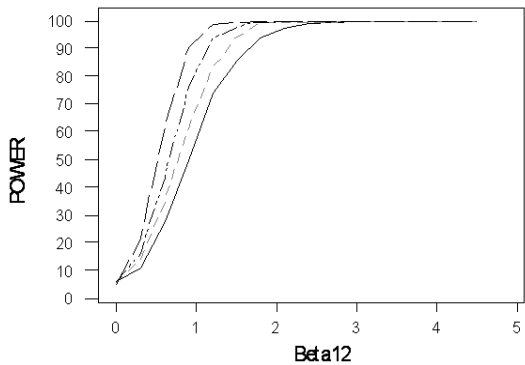**Figure 1** Power under (X1,X2) ~ multinomial (0.10,0.35, 0.45,0.10) and Y~ multinomial (0.05,0.20,0.75)

Figure 4 Power under (X1,X2) ~ multinomial (0.10,0.35, 0.45,0.10) and Y ~ multinomial (0.33,0.33,0.33)

Figure 2 Power under (X1,X2) ~ multinomial (0.10,0.35, 0.45,0.10) and Y~ multinomial(0.55, 0.20, 0.25)

Figure 5 Power under (X1,X2) ~ multinomial (0.50,0.30, 0.10,0.10) and Y~ multinomial (0.05,0.20,0.75)

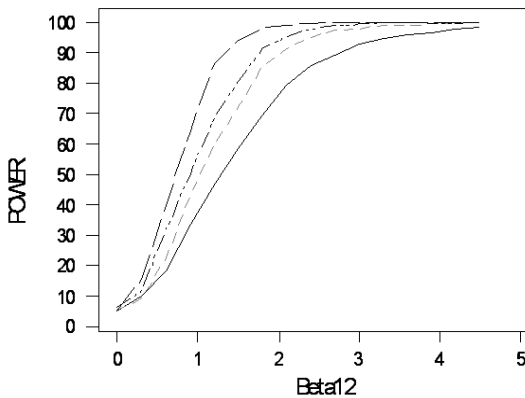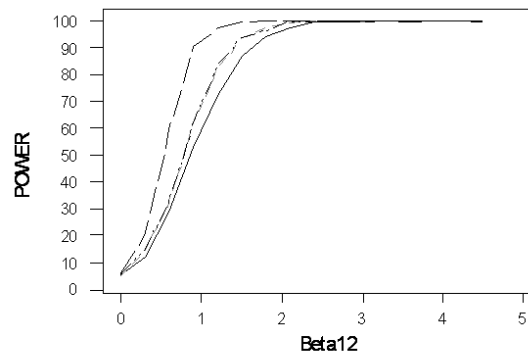**Figure 3** Power under (X1,X2) ~ multinomial (0.10,0.35, 0.45,0.10) and Y ~ multinomial (0.25,0.5,0.25)

Figure 6 Power under (X1,X2) ~ multinomial (0.50,0.30, 0.10,0.10) and Y~multinomial(0.55, 0.20, 0.25)

Sample size 600
Sample size 800
Sample size 1000
Sample size 1500

Sample size 600
Sample size 800
Sample size 1000
Sample size 1500

Appendix 3:

Figure 7-9 Power plots versus β_{12} for each condition of X_1, X_2 and Y distributions under 4 sample sizes

Appendix 4:

Figure 10-12 Power plots versus β_{12} for each condition of X_1, X_2 and Y distributions under 4 sample sizes

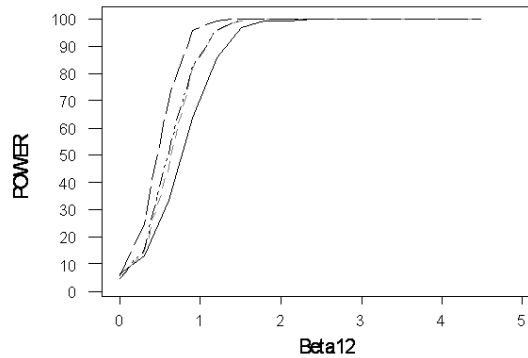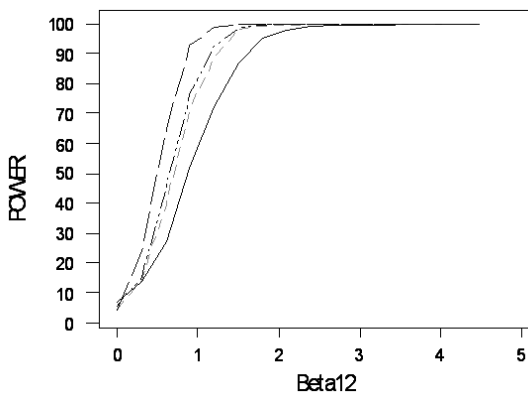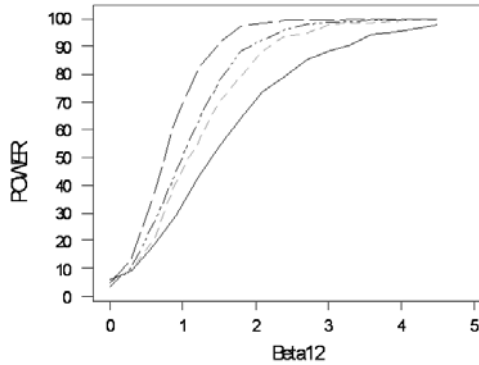**Figure 7**   Power under (X1,X2) ~ multinomial (0.50,0.30, 0.10,0.10) and Y ~ multinomial (0.25,0.5,0.25)

Figure 10 Power under (X1,X2) ~ multinomial (0.25, 0.25, 0.25, 0.25) and Y~multinomial (0.55,0.20, 0.25)

Figure 8   Power under (X1,X2) ~ multinomial (0.50,0.30, 0.10,0.10) and Y ~ multinomial (0.33,0.33,0.33)

Figure 11 Power under (X1,X2) ~ multinomial (0.25,0.25, 0.25,0.25) and Y ~ multinomial (0.25,0.5,0.25)

Figure 9   Power under (X1, X2) ~ multinomial (0.25,0.25, 0.25.0.25) and Y~ multinomial (0.05,0.20,0.75)

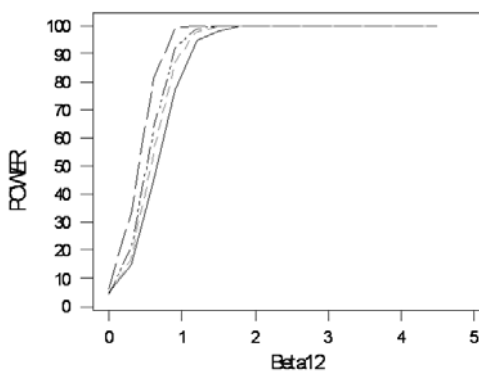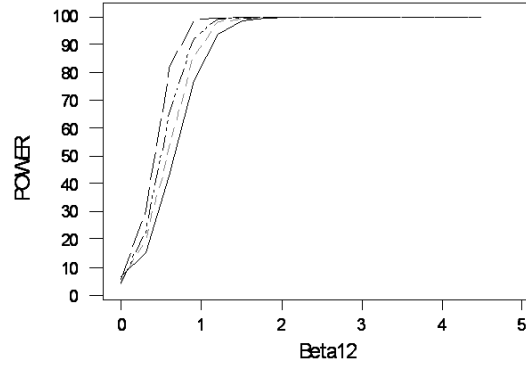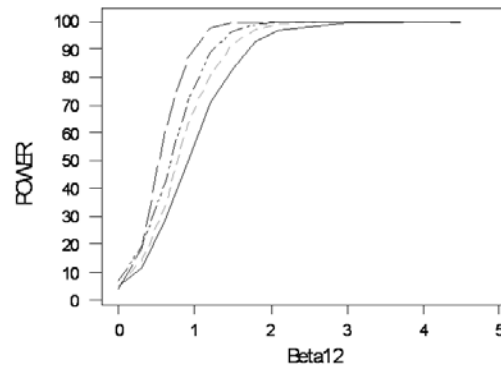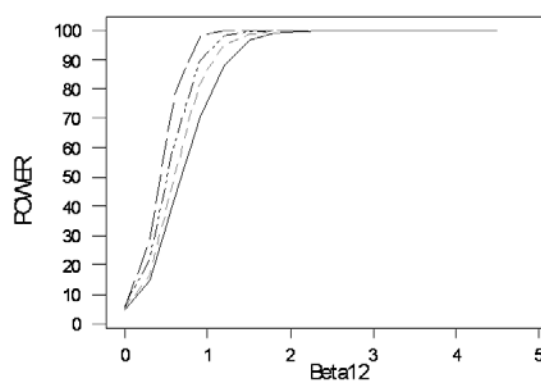**Figure 12** Power under (X1,X2) ~ multinomial (0.25,0.25, 0.25,0.25) and Y ~ multinomial (0.33,0.33,0.33)

Sample size 600
Sample size 800
Sample size 1000
Sample size 1500

Sample size 600
Sample size 800
Sample size 1000
Sample size 1500

Appendix 5: Table 1 Means and standard-deviations of RN, BIC, PCC classified by β_{12}, sample sizes, distributions of Y's and (X1,X2) ~ multinomial (0.10,0.35,0.45,0.10)

Distribution		Y~(0.05, 0.20, 0.75)			Y~(0.25, 0.5, 0.25)			Y~(0.55, 0.20, 0.25)			Y~(0.33, 0.33, 0.33)		
n	β_{12}	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)
Sample size 600	0.0	0.058679 (0.0194721)	1083.98 (25.6156)	56.7395 (1.95122)	0.060107 (0.0194673)	1142.01 (21.6940)	49.6400 (1.92469)	0.052387 (0.0194498)	876.30 (37.7953)	73.4033 (1.83238)	0.063811 (0.0196229)	1141.69 (23.1956)	55.1912 (1.91238)
	2.1	0.212123 (0.0295858)	1051.40 (27.8443)	59.0148 (2.02051)	0.142682 (0.0229250)	1086.00 (22.3357)	51.8870 (1.96330)	0.093214 (0.0207315)	837.13 (935.4929)	74.3300 (1.74841)	0.155567 (0.0228078)	1061.33 (23.9920)	57.5280 (1.89971)
	4.5	0.322947 (0.0235552)	992.86 (27.2899)	61.2027 (1.90744)	0.175407 (0.0163322)	1067.04 (20.4318)	52.0597 (2.03190)	0.103920 (0.0169255)	826.22 (34.6365)	74.5938 (1.73321)	0.185940 (0.0175835)	1037.27 (21.0078)	57.9532 (2.00314)
Sample size 800	0.0	0.057409 (0.0173530)	1463.52 (29.3851)	55.9934 (1.78811)	0.058316 (0.0163818)	1527.46 (26.4722)	49.6187 (1.67847)	0.052828 (0.0170042)	1172.94 (42.3975)	73.3881 (1.55030)	0.058012 (0.0167889)	1528.82 (26.9957)	55.1961 (1.70611)
	2.1	0.226641 (0.0251799)	1412.36 (30.2102)	58.7626 (1.69537)	0.140179 (0.0189626)	1455.15 (26.4239)	51.6619 (1.74502)	0.097928 (0.0171776)	1115.07 (39.8367)	74.4141 (1.46499)	0.136609 (0.0188202)	1438.84 (25.6481)	57.1176 (1.61679)
	4.5	0.346772 (0.0199513)	1325.27 (31.4083)	61.0455 (1.68968)	0.172824 (0.0153550)	1426.87 (23.7003)	52.1307 (1.84894)	0.111255 (0.0144289)	1104.13 (39.3336)	74.4792 (1.50219)	0.164936 (0.0147591)	1412.32 (23.7066)	57.4703 (1.59498)
Sample size 1000	0.0	0.057858 (0.0158284)	1826.36 (33.6037)	56.3663 (1.58360)	0.059951 (0.0151307)	1905.22 (27.2895)	49.8860 (1.45662)	0.049243 (0.0150582)	1458.03 (48.2379)	73.8108 (1.36713)	0.053304 (0.0141273)	1938.15 (29.6395)	54.1414 (1.52324)
	2.1	0.230354 (0.0235285)	1759.94 (35.4838)	59.2585 (1.55560)	0.159612 (0.0184981)	1793.69 (30.2849)	52.4905 (1.67781)	0.093715 (0.0158669)	1384.13 (44.7641)	74.8933 (1.30335)	0.111015 (0.0158306)	1856.88 (28.7404)	55.4964 (1.49588)
	4.5	0.353092 (0.0183483)	1646.95 (35.6761)	61.6079 (1.53995)	0.199785 (0.0141490)	1748.35 (26.3860)	52.8810 (1.79294)	0.106758 (0.0127857)	1368.58 (45.6898)	75.0236 (1.36235)	0.131566 (0.0121908)	1834.23 (26.4756)	55.6743 (1.49857)
Sample size 1500	0.0	0.057337 (0.0126415)	2753.04 (40.3654)	56.1112 (1.28269)	0.058683 (0.0127037)	2869.97 (35.2233)	49.7353 (1.28694)	0.048043 (0.0114475)	2191.54 (58.5107)	73.8900 (1.11701)	0.060091 (0.0128690)	2880.40 (36.6007)	55.0222 (1.25008)
	2.1	0.233535 (0.0189032)	2646.73 (42.4766)	59.1850 (1.22881)	0.156734 (0.0146075)	2704.91 (35.0644)	52.4080 (1.41484)	0.089488 (0.0129840)	2088.94 (54.3407)	74.8891 (1.05884)	0.147409 (0.0144207)	2693.48 (36.1457)	57.1747 (1.17295)
	4.5	0.356469 (0.0148378)	2475.62 (44.9474)	61.6158 (1.25630)	0.194259 (0.0114830)	2642.49 (31.7753)	52.7633 (1.55896)	0.103027 (0.0104999)	2067.81 (53.1326)	74.9665 (1.05052)	0.178252 (0.0109603)	2636.77 (33.3704)	57.5903 (1.23399)

Appendix 6: Table 2 Means and standard-deviations of RN, BIC, PCC classified by β12, sample sizes, distributions of Y's and (X1,X2) ~ multinomial (0.50,0.30,0.10,0.10)

Distribution		Y~(0.05, 0.20, 0.75)			Y~(0.25, 0.5, 0.25)			Y~(0.55, 0.20, 0.25)			Y~(0.33, 0.33, 0.33)		
n	β_{12}	RR_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)
Sample size 600	0.0	0.092791 (0.0249670)	1002.20 (29.4729)	61.9600 (1.93637)	0.096472 (0.0249224)	1164.43 (21.1333)	50.1710 (1.98898)	0.081008 (0.0229454)	971.28 (31.7605)	67.4578 (1.81016)	0.092754 (0.0239711)	1199.72 (19.2019)	48.1515 (1.70600)
	2.1	0.234714 (0.0295569)	972.45 (30.8879)	63.9847 (2.05978)	0.172185 (0.0246705)	1117.85 (20.6846)	51.8897 (2.09187)	0.116830 (0.0223893)	939.52 (30.1370)	68.2015 (1.75682)	0.147693 (0.0240838)	1158.46 (18.9061)	49.3832 (1.78699)
	4.5	0.330918 (0.0231105)	924.25 (30.2801)	65.5607 (2.17406)	0.199730 (0.0198486)	1101.15 (19.8181)	52.0578 (2.21288)	0.124169 (0.0205579)	933.49 (31.0240)	68.3213 (1.84684)	0.164782 (0.0208048)	1146.38 (17.1964)	49.6292 (1.76365)
Sample size 800	0.0	0.095893 (0.0216321)	1346.15 (32.8055)	61.7759 (1.65129)	0.098667 (0.0207859)	1553.21 (23.7019)	50.4211 (1.73304)	0.094352 (0.0221618)	1296.73 (37.0235)	67.1509 (1.56177)	0.110945 (0.0210603)	1590.57 (22.4673)	48.2617 (1.41924)
	2.1	0.265161 (0.0272259)	1295.56 (36.4604)	64.2446 (1.83537)	0.193330 (0.0220534)	1474.51 (25.3943)	52.4574 (2.01253)	0.140867 (0.0210232)	1242.37 (36.5469)	68.0613 (1.60113)	0.186667 (0.0222109)	1511.54 (22.5013)	49.9919 (1.48101)
	4.5	0.376119 (0.0214634)	1215.86 (36.3283)	66.1289 (2.07333)	0.226971 (0.0178454)	1443.93 (22.5117)	52.7596 (1.96677)	0.154078 (0.0176620)	1227.56 (34.7135)	68.2813 (1.58109)	0.211336 (0.0185546)	1487.02 (19.7665)	50.3789 (1.54499)
Sample size 1000	0.0	0.117160 (0.0212601)	1672.38 (38.8353)	61.9885 (1.46849)	0.119330 (0.0204980)	1938.14 (28.1741)	51.6015 (1.54358)	0.092974 (0.0183353)	1644.05 (42.6551)	66.4587 (1.47894)	0.113630 (0.0202277)	1984.51 (26.9013)	48.5029 (1.36610)
	2.1	0.311074 (0.0254571)	1597.65 (42.0316)	64.5490 (2.03852)	0.232552 (0.0218374)	1818.12 (29.0475)	54.0941 (1.55212)	0.138355 (0.0188405)	1576.73 (38.8972)	67.4059 (1.42174)	0.196920 (0.0195033)	1874.05 (24.9741)	50.4194 (1.36017)
	4.5	0.437604 (0.0194316)	1467.59 (40.4446)	65.5483 (2.10619)	0.273434 (0.0159401)	1772.13 (25.1155)	54.6431 (1.45673)	0.151304 (0.0159873)	1561.95 (34.8796)	67.5155 (1.30967)	0.224653 (0.0166215)	1839.20 (22.4471)	50.7908 (1.41253)
Sample size 1500	0.0	0.094653 (0.0152618)	2517.92 (44.7128)	62.2669 (1.18737)	0.103104 (0.0155899)	2933.07 (32.0622)	50.6379 (1.27516)	0.078734 (0.0144625)	2452.18 (52.1990)	67.4458 (1.16080)	0.095011 (0.0153593)	3011.83 (31.5818)	48.0757 (1.14890)
	2.1	0.257192 (0.0195606)	2434.43 (49.0412)	64.6352 (1.34606)	0.202582 (0.0167081)	2776.60 (34.9614)	52.5267 (1.51676)	0.115188 (0.0146244)	2368.67 (47.4650)	68.2650 (1.11806)	0.159471 (0.0155562)	2888.93 (30.4591)	49.4591 (1.16885)
	4.5	0.366935 (0.0153704)	2286.15 (48.5000)	66.4197 (1.67674)	0.238673 (0.0128577)	2716.21 (31.0798)	52.8142 (1.43480)	0.125701 (0.0124415)	2349.83 (47.1991)	68.3757 (1.13745)	0.180292 (0.0128069)	2851.71 (26.9369)	49.6029 (1.20484)

Appendix 7: Table 3 Means and standard-deviations of RN, BIC, PCC classified by β_{12}, sample sizes, distributions of Y's and (X1,X2) ~ multinomial (0.25,0.25,0.25,0.25)

Distribution		Y~(0.05, 0.20, 0.75)			Y~(0.25, 0.5, 0.25)			Y~(0.55, 0.20, 0.25)			Y~(0.33, 0.33, 0.33)		
n	β_{12}	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)	R_N Mean(SD)	BIC Mean(SD)	PCC Mean(SD)
Sample size 600	0.0	0.116262 (0.0259284)	1073.23 (25.9330)	57.0148 (1.96910)	0.119802 (0.0258256)	1115.17 (22.6035)	52.8920 (1.91225)	0.097560 (0.0258256)	871.77 (22.6035)	72.4897 (1.91225)	0.119847 (0.0261005)	1118.68 (24.1767)	55.1137 (1.77703)
	2.1	0.408356 (0.0305820)	978.37 (29.4688)	63.5365 (2.31997)	0.311012 (0.0285471)	963.17 (24.6236)	58.5035 (2.11907)	0.189795 (0.0285471)	776.29 (24.6236)	74.8463 (2.11907)	0.282095 (0.0292620)	961.67 (25.6108)	59.8110 (1.83640)
	4.5	0.584123 (0.0236428)	817.59 (30.8072)	69.0462 (2.42113)	0.380440 (0.0223560)	906.53 (19.7519)	59.4167 (1.99051)	0.216386 (0.0223560)	755.09 (19.7519)	75.1660 (1.99051)	0.335963 (0.0217310)	914.03 (18.6720)	60.6235 (1.75896)
Sample size 800	0.0	0.124660 (0.0238710)	1430.32 (29.9003)	57.2569 (1.68988)	0.117021 (0.0232368)	1495.08 (26.4045)	52.7146 (1.71580)	0.099318 (0.0232368)	1167.42 (26.4045)	72.4154 (1.71580)	0.120732 (0.0237038)	1504.52 (29.3145)	54.4674 (1.60880)
	2.1	0.429295 (0.0281143)	1292.72 (36.4556)	64.2092 (2.19489)	0.311231 (0.0255582)	1287.55 (29.8291)	58.4423 (1.82024)	0.193366 (0.0255582)	1038.69 (29.8291)	74.7671 (1.82024)	0.276914 (0.0242378)	1307.51 (26.8691)	58.8363 (1.53391)
	4.5	0.608017 (0.0208170)	1059.36 (37.2755)	70.0811 (2.35731)	0.382902 (0.0190446)	1209.89 (22.9146)	59.3836 (1.76076)	0.220855 (0.0190446)	1011.17 (22.9146)	75.0383 (1.76076)	0.329991 (0.0184614)	1243.65 (21.0025)	59.6665 (1.52066)
Sample size 1000	0.0	0.113219 (0.0203673)	1810.38 (33.3417)	56.5247 (1.56050)	0.118671 (0.0201180)	1872.84 (29.7068)	52.7988 (1.53319)	0.102035 (0.0201180)	1438.73 (29.7068)	73.0800 (1.53319)	0.122004 (0.0203022)	1875.77 (31.8457)	54.7174 (1.42317)
	2.1	0.414791 (0.0247812)	1642.97 (39.2765)	63.5953 (1.76897)	0.305482 (0.0223053)	1626.84 (31.2972)	57.9488 (1.61880)	0.204218 (0.0223053)	1260.43 (31.2972)	75.6898 (1.61880)	0.288977 (0.0229061)	1603.70 (30.9311)	59.6693 (1.36628)
	4.5	0.594478 (0.0185410)	1359.05 (39.9071)	69.3568 (2.06045)	0.375111 (0.0168851)	1532.73 (27.1699)	58.8085 (1.58313)	0.237176 (0.0168851)	1216.43 (27.1699)	76.0926 (1.58313)	0.345035 (0.0175971)	1520.09 (23.1825)	60.6283 (1.29242)
Sample size 1500	0.0	0.118020 (0.0164119)	2703.23 (41.0660)	57.1033 (1.19603)	0.122105 (0.0166625)	2814.05 (35.9175)	53.0209 (1.26396)	0.096457 (0.0166625)	2187.08 (35.9175)	72.7799 (1.26396)	0.115583 (0.0163160)	2823.25 (38.4614)	54.7628 (1.15201)
	2.1	0.420350 (0.0198970)	2452.16 (47.3426)	64.0603 (1.58752)	0.321496 (0.0181893)	2415.86 (38.4014)	58.5497 (1.26988)	0.194758 (0.0181893)	1933.23 (38.4014)	75.2520 (1.26988)	0.274387 (0.0183760)	2445.30 (39.7569)	59.3437 (1.14981)
	4.5	0.599675 (0.0145161)	2024.14 (47.1927)	69.6568 (1.94135)	0.391742 (0.0146808)	2267.54 (30.8648)	59.6535 (1.10450)	0.223310 (0.0146808)	1873.44 (30.8648)	75.6543 (1.10450)	0.328945 (0.0138756)	2327.39 (29.3471)	60.2329 (1.08923)