Research Article

Convergence of Three- Step Mean Value Iterative Scheme for a Mapping of Asymptotically Quasi - Nonexpansive Type

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Abstract

In this paper, convergences of three- step mean value iterative scheme are established for a mapping of asymptotically quasi – nonexpansive type in a uniformly convex Banach space. The results obtained in this paper extend and improve the recent ones announced by D. R. Sahu and J. S. Jung [Sahu and Jung, (2003) Fixed – point iteration processes for non- Lipschitzian mappings of asymptotically quasi- nonexpansive type, International Journal of Mathematics and Mathematical Sciences 2003 : 2075 - 2081], and many others.

Key Words: Asymptotically quasi – nonexpansive type; Completely continuous; Uniformly convex Banach space; Three – step mean value iterative scheme

Introduction

Let X be a real Banach space and let C be a nonempty subset of X. Further, for a mapping $T : C \to C$, let $\varnothing \neq F(T)$ be the set of all fixed points of T. A mapping $T : C \to C$ is said to be asymptotically quasi - nonexpansive if there exists a sequence $\{k_n\}$ of real number with $k_n \geq 1$ and $\lim_n k_n = 1$ such that for all $x \in C$, $p \in F(T)$,

$$\|T^n x - p\| \leq k_n \|x - p\|, \text{ for all } n \geq 1.$$

T is called asymptotically quasi- nonexpansive type (Sahu and Jung, 2003) provided T is uniformly continuous and $\limsup_n \{ \sup_{x\in C} (\|T^n x- p\| - \|x - p\|)\} \leq 0$ for all $p \in F(T)$. The mapping T is called uniformly L- Lipschitzian if there exists a positive constant L such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and for all $n \geq 1$. T is completely continuous if for all bounded sequence $\{x_n\} \subset C$ there exists a convergent subsequence of $\{Tx_n\}$ (Schu, 158(1991)).

Recall that a Banach space X is called uniformly convex if for every $0 < \varepsilon \leq 2$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\left\| \frac{x+y}{2} \right\| \leq 1-\delta$ for every $x, y \in S_X$ and $\|x-y\| \geq \varepsilon$, where $S_X = \{ x\in X : \|x\| = 1 \}$ (Megginson, 1998).

In 2003, D. R. Sahu and J. S. Jung (Sahu and Jung, 2003) gave necessary and sufficient conditions for convergences of Mann and Ishikawa iteration processes for mappings of asymptotically quasi - nonexpansive type in Banach spaces. In 2006, J. Quan, S. S. Chang, and X. J. Long (Quan et al., 2006) gave necessary and sufficient conditions for finite- step iterative sequences with mean errors for a family of asymptotically quasi - nonexpansive type mappings in Banach spaces to converge to a common fixed point.

Recently, Nilsrakoo and Saejung (Nilsrakoo and Saejung, 2006) defined a new three-step fixed point iterative scheme which is an extension of Suantai's iterations (Suantai, 2005) and gave some strong convergence theorems of the iterations for asymptotically nonexpansive mappings in a uniformly convex Banach space. The purpose of this paper is to study a strong convergence theorem for a mapping of asymptotically quasi- nonexpansive type of a three- step iteration, defined by Nilsrakoo and Saejung.

Let C be a nonempty convex subset of a real Banach space X and let $T : C \to C$ be a mapping.

Algorithm 1. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1- a_n)x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n, \qquad (1) \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + \gamma_n T^n x_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n, \quad n \geq 1, \end{aligned}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\alpha_n+\beta_n+\gamma_n\}$ are appropriate sequences in [0, 1]. The iterative scheme (1) is called the three- step mean value iterative scheme, defined by Nilsrakoo and Saejung (Nilsrakoo and Saejung, 2006). If $\gamma_n \equiv 0$, then (1) reduces to the modified Noor iterative scheme defined by Suantai (Suantai, 2005) as follows:

Algorithm 2. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1- a_n)x_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n)x_n, \qquad (2) \\ x_n+1 &= \alpha_n T^n y_n + \beta_n T^n z_n + (1- \alpha_n - \beta_n)x_n, \ n \geq 1, \end{aligned}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\alpha_n+\beta_n\}$ are appropriate sequences in [0, 1]. If $c_n = \beta_n = \gamma_n \equiv 0$, then (1) reduces to the Noor iterative scheme defined by Xu and Noor (Xu and Noor, 2002) as follows:

Algorithm 3. For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1- a_n)x_n, \\ y_n &= b_n T^n z_n + (1 - b_n)x_n, \qquad (3) \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{aligned}$$

where $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$ are appropriate sequences in [0, 1].

If $a_n = c_n = \beta_n = \gamma_n \equiv 0$, then Algorithm 1 reduces to Algorithm 4:

Algorithm 4. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= b_n T^n z_n + (1 - b_n)x_n, \qquad (4) \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{aligned}$$

where $\{b_n\}$ and $\{\alpha_n\}$ are appropriate sequences in [0, 1].

Similarly, if $b_n = c_n = \alpha_n = \gamma_n \equiv 0$, then Algorithm 1 reduces to Algorithm 4′:

Algorithm 4′. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1- a_n)x_n, \qquad (5) \\ x_{n+1} &= \beta_n T^n z_n + (1 - \beta_n)x_n, \quad n \geq 1, \end{aligned}$$

where $\{a_n\}$ and $\{\beta_n\}$ are appropriate sequences in [0, 1]. Let us note that schemes (4) and (5) are called the modified Ishikawa iterative scheme defined by Schu (Schu, 1991).

If $a_n = b_n = c_n = \beta_n = \gamma_n \equiv 0$, then (1) reduces to the modified Mann iterative scheme defined by Schu (Schu, 1991) as follows:

Algorithm 5. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ by the iterative scheme

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \ n \geq 1, \qquad (6)$$

where $\{\alpha_n\}$ is an appropriate a sequence in [0, 1].

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1. (Tan and Xu, 1993). Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$, for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

Lemma 1.2. (Nilsrakoo and Saejung, 2007). Let X be a uniformly convex Banach space and $B_r = \{ x \in X$

: $\| x \| < r$ }, $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\| \lambda x + \mu y + \xi z + \nu w\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \nu\|w\|^2 - \tfrac{1}{3}\nu[\lambda g(\|x-w\|) + \mu g(\|y-w\|) + \xi g(\|z-w\|)]$$

for all $x, y, z, w \in B_r$ and $\lambda, \mu, \xi, \nu \in [0, 1]$ with $\lambda + \mu + \xi + \nu = 1$.

Lemma 1.3. (Nilsrakoo and Saejung, 2007). Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in a uniformly convex Banach space X. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0, 1] with $\alpha_n + \beta_n + \gamma_n = 1$, $\limsup_n \|x_n\| \leq d$, $\limsup_n \|y_n\| \leq d$, $\limsup_n \|z_n\| \leq d$ and $\lim_n \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = d$, where $d \geq 0$. If $\liminf_n \alpha_n > 0$ and $\liminf_n \beta_n > 0$, then $\lim_n \|x_n - y_n\| = 0$.

Main Results

In this section, we prove a strong convergence theorem of three-step mean value iterative scheme for a mapping of asymptotically quasi-nonexpansive type. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. Let X be a real Banach space and let C be a nonempty convex subset of X. Let $T : C \rightarrow C$ be a mapping of asymptotically quasi-nonexpansive type with the nonempty fixed-point set F(T). Put

$$G_n = \max \{0, \sup_{x\in C} [\|T^n x - p\| - \|x - p\|]\}, \forall n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1, then we have the following conclusions:

(i) $\lim_n \|x_n - p\|$ exists, for any $p \in F(T)$.

(ii) $\lim_n d(x_n, F(T))$ exists, where $d(x, F(T))$ denotes the distance from x to the fixed-point set F(T).

Proof. Let $p \in F(T)$. Then by using (1), we have

$$\begin{aligned}\|z_n - p\| &\leq a_n \|T^n x_n - p\| + (1 - a_n)\|x_n - p\| \\ &= a_n [\|T^n x_n - p\| - \|x_n - p\|] + \|x_n - p\| \\ &\leq a_n \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + \|x_n - p\| \\ &\leq a_n G_n + \|x_n - p\|; \qquad (7)\end{aligned}$$

$$\begin{aligned}\|y_n - p\| &\leq b_n \|T^n z_n - p\| + c_n \|T^n x_n - p\| + (1 - b_n - c_n) \|x_n - p\| \\ &= b_n [\|T^n z_n - p\| - \|z_n - p\|] + b_n\|z_n - p\| + c_n [\|T^n x_n - p\| - \|x_n - p\|] + (1 - b_n) \|x_n - p\| \\ &\leq b_n \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + a_n b_n \sup_{x\in C}[\|T^n x - p\| - \|x - p\|] + b_n \|x_n - p\| + c_n \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + (1 - b_n) \|x_n - p\| \\ &\leq (b_n + a_n b_n + c_n) \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + \|x_n - p\|.\end{aligned}$$

Hence, $\|y_n - p\| \leq (b_n + a_n b_n + c_n)G_n + \|x_n - p\|$; (8)

and

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n \|T^n y_n - p\| + \beta_n \|T^n z_n - p\| + \gamma_n \|T^n x_n - p\| + (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| = \alpha_n [\|T^n y_n - p\| - \|y_n - p\|] + \alpha_n \|y_n - p\| + \beta_n [\|T^n z_n - p\| - \|z_n - p\|] + \beta_n \|z_n - p\| + \gamma_n [\|T^n x_n - p\| - \|x_n - p\|] + (1 - \alpha_n - \beta_n) \|x_n - p\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + (\alpha_n b_n + \alpha_n a_n b_n + \alpha_n c_n) \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + \alpha_n \|x_n - p\| + \beta_n \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] \\ &\quad + \beta_n a_n \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + \beta_n \|x_n - p\| + \gamma_n \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + \|x_n - p\| + (-\alpha_n - \beta_n) \|x_n - p\| \\ &= (\alpha_n + \alpha_n b_n + \alpha_n a_n b_n + \alpha_n c_n + \beta_n + \beta_n a_n + \gamma_n) \sup_{x\in C} [\|T^n x - p\| - \|x - p\|] + \|x_n - p\| \\ &\leq [(\alpha_n + \beta_n + \gamma_n) + \alpha_n(b_n + c_n) + a_n(\alpha_n b_n + \beta_n)] G_n + \|x_n - p\|.\end{aligned}$$

Hence, $\|x_{n+1} - p\| \leq \|x_n - p\| + 4G_n$. (9)

Since $\sum_{n=1}^{\infty} G_n < \infty$, the lemma follows from Lemma 1.1.

This completes the proof. #

Remark 2.1. For the sequence $\{y_n\}$ and $\{z_n\}$ in C defined by Algorithm 1, it follows by Lemma 2.1 together with inequalities (7) and (8) that $\lim_n \|z_n - p\|$ and $\lim_n \|y_n - p\|$ exist.

Lemma 2.2. Let X be a uniformly convex Banach space and let C be a nonempty convex subset of X. Let $T : C \rightarrow C$ be a mapping of asymptotically quasi-nonexpansive type with the nonempty fixed-point set F(T). Put

$G_n = \max\{0, \sup_{x\in C}[\|T^n x - p\| - \|x - p\|]\}, \forall n \geq 1,$ so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{x_n\}$ be a sequence in C defined by algorithm 1. Then we have the following assertions:

(i) If $0 < \liminf_n \alpha_n \leq \limsup_n(\alpha_n+\beta_n+\gamma_n) < 1$, then $\lim_n \|T^n y_n - x_n\| = 0$.

(ii) If $0 < \liminf_n \beta_n \leq \limsup_n(\alpha_n+\beta_n+\gamma_n) < 1$, then $\lim_n \|T^n z_n - x_n\| = 0$.

(iii) If $0 < \liminf_n \gamma_n \leq \limsup_n(\alpha_n+\beta_n+\gamma_n) < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.

(iv) If $0 < \liminf_n \alpha_n$ and $0 < \liminf_n b_n \leq \limsup_n (b_n + c_n) < 1$, then $\lim_n \|T^n z_n - x_n\| = 0$.

(v) If $0 < \liminf_n \alpha_n$ and $0 < \liminf_n c_n \leq \limsup_n(b_n + c_n) < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.

(vi) If $0 < \liminf_n (\alpha_n b_n + \beta_n)$ and $0 < \liminf_n a_n \leq \limsup_n a_n < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.

Proof. By Lemma 2.1, $\lim_n \|x_n - p\|$ exists for any $p\in F(T)$. If follows that $\{x_n - p\}$, $\{T^n x_n - p\}$, $\{y_n - p\}$, $\{T^n y_n - p\}$, $\{z_n - p\}$ and $\{T^n z_n - p\}$ are all bounded. We may assume that such sequences belong to B_r, where $B_r = \{x \in X : \|x\| < r\}, r > 0$.

By Lemma 1.2, we have

$$\begin{aligned}\|z_n - p\|^2 &\leq a_n \|T^n x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2 - \tfrac{1}{3}(1 - a_n)a_n\, g(\|T^n x_n - x_n\|)\\ &= a_n(\|T^n x_n - p\|^2 - \|x_n - p\|^2) + \|x_n - p\|^2 - \tfrac{1}{3}(1 - a_n)\, a_n\, g(\|T^n x_n - x_n\|);\end{aligned}$$

$$\|y_n - p\|^2 \leq b_n \|T^n z_n - p\|^2 + c_n \|T^n x_n - p\|^2 + (1 - b_n - c_n)\|x_n - p\|^2 - \tfrac{1}{3}(1 - b_n - c_n)\{b_n g(\|T^n z_n - x_n\|) + c_n g(\|T^n x_n - x_n\|)\};$$

and

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n \|T^n y_n - p\|^2 + \beta_n \|T^n z_n - p\|^2 + \gamma_n \|T^n x_n - p\|^2 + (1-\alpha_n - \beta_n - \gamma_n)\|x_n - p\|^2\\ &\quad - \tfrac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)\{\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)\}\\ &= \alpha_n(\|T^n y_n - p\|^2 - \|y_n - p\|^2) + \alpha_n\|y_n - p\|^2 + \beta_n(\|T^n z_n - p\|^2 - \|z_n - p\|^2) + \beta_n\|z_n - p\|^2\\ &\quad + \gamma_n(\|T^n x_n - p\|^2 - \|x_n - p\|^2) + (1-\alpha_n - \beta_n)\|x_n - p\|^2\\ &\quad - \tfrac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)\{\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)\}\\ &\leq \alpha_n(\|T^n y_n - p\|^2 - \|y_n - p\|^2) + \alpha_n b_n \|T^n z_n - p\|^2 + \alpha_n c_n \|T^n x_n - p\|^2 + (1 - b_n - c_n)\alpha_n\|x_n - p\|^2\\ &\quad - \tfrac{1}{3}(1 - b_n - c_n)\alpha_n\{b_n g(\|T^n z_n - x_n\|) + c_n g(\|T^n x_n - x_n\|)\} + \beta_n(\|T^n z_n - p\|^2 - \|z_n - p\|^2)\\ &\quad + \beta_n a_n(\|T^n x_n - p\|^2 - \|x_n - p\|^2) + \beta_n\|x_n - p\|^2 - \tfrac{1}{3}(1 - a_n)a_n\beta_n g(\|T^n x_n - x_n\|)\\ &\quad + \gamma_n(\|T^n x_n - p\|^2 - \|x_n - p\|^2) + (1-\alpha_n - \beta_n)\|x_n - p\|^2\\ &\quad - \tfrac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)\{\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)\}\\ &= \alpha_n(\|T^n y_n - p\|^2 - \|y_n - p\|^2) + \alpha_n b_n(\|T^n z_n - p\|^2 - \|z_n - p\|^2) + \alpha_n b_n\|z_n - p\|^2 + \beta_n(\|T^n z_n - p\|^2\\ &\quad - \|z_n - p\|^2) + \alpha_n c_n(\|T^n x_n - p\|^2 - \|x_n - p\|^2) + \alpha_n c_n\|x_n - p\|^2 + \beta_n a_n(\|T^n x_n - p\|^2 - \|x_n - p\|^2)\\ &\quad + \gamma_n(\|T^n x_n - p\|^2 - \|x_n - p\|^2) + (1 - b_n)\alpha_n\|x_n - p\|^2 - c_n\alpha_n\|x_n - p\|^2 + \beta_n\|x_n - p\|^2\\ &\quad + (1-\alpha_n)\|x_n - p\|^2 - \beta_n\|x_n - p\|^2 - \tfrac{1}{3}(1 - b_n - c_n)\alpha_n\{b_n g(\|T^n z_n - x_n\|) + c_n g(\|T^n x_n - x_n\|)\}\\ &\quad - \tfrac{1}{3}(1 - a_n)a_n\beta_n g(\|T^n x_n - x_n\|) - \tfrac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)\{\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|)\\ &\quad + \gamma_n g(\|T^n x_n - x_n\|)\}\\ &\leq \alpha_n(\|T^n y_n - p\|^2 - \|y_n - p\|^2) + (\alpha_n b_n + \beta_n)(\|T^n z_n - p\|^2 - \|z_n - p\|^2) + \alpha_n b_n a_n(\|T^n x_n - p\|^2\\ &\quad - \|x_n - p\|^2) + \alpha_n b_n\|x_n - p\|^2 - \tfrac{1}{3}(1 - a_n)a_n\alpha_n b_n g(\|T^n x_n - x_n\|) + (\alpha_n c_n + \beta_n a_n + \gamma_n)\\ &\quad (\|T^n x_n - p\|^2 - \|x_n - p\|^2) + (1 - \alpha_n + \alpha_n - \alpha_n b_n)\|x_n - p\|^2 - \tfrac{1}{3}(1 - b_n - c_n)\alpha_n\{b_n g(\|T^n z_n - x_n\|)\\ &\quad + c_n g(\|T^n x_n - x_n\|)\} - \tfrac{1}{3}(1 - a_n)a_n\beta_n g(\|T^n x_n - x_n\|) - \tfrac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)\\ &\quad \{\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)\}\\ &= \alpha_n(\|T^n y_n - p\|^2 - \|y_n - p\|^2) + (\alpha_n b_n + \beta_n)(\|T^n z_n - p\|^2 - \|z_n - p\|^2) + (\alpha_n b_n a_n + \alpha_n c_n\\ &\quad + \beta_n a_n + \gamma_n)(\|T^n x_n - p\|^2 - \|x_n - p\|^2)\end{aligned}$$

$$
\begin{aligned}
& + \|x_n - p\|^2 - \tfrac{1}{3}(1- a_n)a_n(\alpha_n b_n + \beta_n)\, g(\|T^n x_n - x_n\|) - \tfrac{1}{3}(1- b_n - c_n) \\
& \{ \alpha_n b_n g(\|T^n z_n - x_n\|) + \alpha_n c_n g(\|T^n x_n - x_n\|) \} \\
& - \tfrac{1}{3}(1- \alpha_n - \beta_n - \gamma_n)\{\alpha_n g(\|T^n y_n - x_n\|) \\
& + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)\} \\
\le\ & (\|T^n y_n - p\| + \|y_n - p\|)(\|T^n y_n - p\| - \|y_n - p\|) \\
& + 2(\|T^n z_n - p\| + \|z_n - p\|)(\|T^n z_n - p\| - \|z_n - p\|) \\
& + 4(\|T^n x_n - p\| + \|x_n - p\|)(\|T^n x_n - p\| \\
& - \|x_n - p\|) + \|x_n - p\|^2 - \tfrac{1}{3}(1- a_n)a_n(\alpha_n b_n + \beta_n) g \\
& (\|T^n x_n - x_n\|) - \tfrac{1}{3}(1- b_n - c_n)\{ \alpha_n b_n g \\
& (\|T^n z_n - x_n\|) + \alpha_n c_n g(\|T^n x_n - x_n\|) \} \\
& - \tfrac{1}{3}(1- \alpha_n - \beta_n - \gamma_n)\{ \alpha_n g(\|T^n y_n - x_n\|) + \beta_n g \\
& (\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)\} \\
\le\ & (\|T^n y_n - p\| + \|y_n - p\|) \sup_{x\in C} (\|T^n x - p\| - \|x - p\|) \\
& + 2(\|T^n z_n - p\| + \|z_n - p\|) \sup_{x\in C} (\|T^n x - p\| \\
& - \|x - p\|) + 4(\|T^n x_n - p\| + \|x_n - p\|) \sup_{x\in C} \\
& (\|T^n x - p\| - \|x - p\|) + \|x_n - p\|^2 \\
& - \tfrac{1}{3}(1- a_n)a_n(\alpha_n b_n + \beta_n)\, g(\|T^n x_n - x_n\|) \\
& - \tfrac{1}{3}(1- b_n - c_n)\{ \alpha_n b_n g(\|T^n z_n - x_n\|) \\
& + \alpha_n c_n g(\|T^n x_n - x_n\|) \} - \tfrac{1}{3}(1- \alpha_n - \beta_n - \gamma_n) \\
& \{\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) \\
& + \gamma_n g(\|T^n x_n - x_n\|)\} \\
\le\ & \{ \|T^n y_n - p\| + \|y_n - p\| + 2(\|T^n z_n - p\| + \|z_n - p\|) \\
& + 4(\|T^n x_n - p\| + \|x_n - p\|) \}\, G_n + \|x_n - p\|^2 \\
& - \tfrac{1}{3}(1- a_n)a_n(\alpha_n b_n + \beta_n)\, g(\|T^n x_n - x_n\|) \\
& - \tfrac{1}{3}(1- b_n - c_n)\{ \alpha_n b_n g(\|T^n z_n - x_n\|) + \alpha_n c_n g \\
& (\|T^n x_n - x_n\|) \} - \tfrac{1}{3}(1- \alpha_n - \beta_n - \gamma_n) \\
& \{\alpha_n g(\|T^n y_n - x_n\|) + \beta_n g(\|T^n z_n - x_n\|) \\
& + \gamma_n g(\|T^n x_n - x_n\|)\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \le\ & u_n G_n + \|x_n - p\|^2 - \tfrac{1}{3}(1- a_n)a_n \\
& (\alpha_n b_n + \beta_n)\, g(\|T^n x_n - x_n\|) - \tfrac{1}{3}(1- b_n - c_n) \\
& \{ \alpha_n b_n g(\|T^n z_n - x_n\|) + \alpha_n c_n g(\|T^n x_n - x_n\|) \} \\
& - \tfrac{1}{3}(1- \alpha_n - \beta_n - \gamma_n)\{\alpha_n g(\|T^n y_n - x_n\|) \\
& + \beta_n g(\|T^n z_n - x_n\|) + \gamma_n g(\|T^n x_n - x_n\|)\},
\end{aligned}
$$

where $u_n = \|T^n y_n - p\| + \|y_n - p\| + 2(\|T^n z_n - p\| + \|z_n - p\|) + 4(\|T^n x_n - p\| + \|x_n - p\|)$.

Consequently,

$$\alpha_n (1- \alpha_n - \beta_n - \gamma_n) g(\|T^n y_n - x_n\|) \le 3\{ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + u_n G_n\}; \qquad (10)$$

$$\beta_n (1- \alpha_n - \beta_n - \gamma_n) g(\|T^n z_n - x_n\|) \le 3\{ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + u_n G_n\}; \qquad (11)$$

$$\gamma_n (1- \alpha_n - \beta_n - \gamma_n) g(\|T^n x_n - x_n\|) \le 3\{ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + u_n G_n\}; \qquad (12)$$

$$\alpha_n b_n (1- b_n - c_n) g(\|T^n z_n - x_n\|) \le 3\{ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + u_n G_n\}; \qquad (13)$$

$$\alpha_n c_n (1- b_n - c_n) g(\|T^n x_n - x_n\|) \le 3\{ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + u_n G_n\}; \qquad (14)$$

and $(1- a_n)a_n(\alpha_n b_n + \beta_n)\, g(\|T^n x_n - x_n\|) \le 3\{ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + u_n G_n\}$. (15)

We now prove (i).

Since $\sum_{n=1}^{\infty} G_n < \infty$ and $\lim_n \|x_n - p\|$ exists, it follows from (10) that

$$\limsup_n \{\alpha_n (1- \alpha_n - \beta_n - \gamma_n) g(\|T^n y_n - x_n\|)\} = 0.$$

Because g is continuous strictly increasing with $g(0) = 0$ and $0 < \liminf_n \alpha_n \le \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$, we have $\lim_n \|T^n y_n - x_n\| = 0$. By using a similar method, together with inequalities (11), (12), (13), (14) and (15), it can be shown that (ii), (iii), (iv), (v) and (vi) are satisfied, respectively. #

Lemma 2.3. Let X be a uniformly convex Banach space and let C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping of asymptotically quasi-nonexpansive type such that T is uniformly L-Lipschitzian with the nonempty fixed-point set F(T). Put

$$G_n = \max\{0, \sup_{x\in C} [\, \|T^n x - p\| - \|x - p\| \,]\},$$

$\forall n \le 1$, so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{x_n\}$ be a sequence in C defined by algorithm 1.

If $\lim_n \|T^n x_n - x_n\| = \lim_n \|T^n y_n - x_n\| = \lim_n \|T^n z_n - x_n\| = 0$, then $\lim_n \|T x_n - x_n\| = 0$.

Proof. Using (1), we have

$$\|x_{n+1} - x_n\| \le \alpha_n \|T^n y_n - x_n\| + \beta_n \|T^n z_n - x_n\| + \gamma_n \|T^n x_n - x_n\| \to 0 \text{ as } n \to \infty. \qquad (16)$$

Since $\|x_n - Tx_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1}x_{n+1}\|$

$+\|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \le \|x_{n+1} - x_n\|$

$+ \|x_{n+1} - T^{n+1}x_{n+1}\| + L_1\|x_{n+1} - x_n\| + L_2\|T^nx_n - x_n\|$,

for some $L_1, L_2 \in R^+$ and $\|T^nx_n - x_n\| \to 0$ as $n \to \infty$, it follows from (16) and uniform continuity of T that $\lim_n \|Tx_n - x_n\| = 0$. #

Lemma 2.4. Let X be a uniformly convex Banach space and let C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping of asymptotically quasi-nonexpansive type such that T is uniformly L- Lipschitzian with the nonempty fixed-point set F(T). Put

$$G_n = \max \{0, \sup_{x\in C} [\ \|T^nx - p\| - \|x - p\|\ ]\},$$

$\forall n \ge 1$, so that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{x_n\}$ be a sequence in C defined by algorithm 1. Then we have the following assertions:

If $0 < \liminf_n \alpha_n \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ and the parameters satisfy one of the following control conditions:

(C1) $0 < \liminf_n b_n \le \limsup_n(b_n + c_n) < 1$ and one of the following holds:

(a) $0 < \liminf_n a_n \le \limsup_n an < 1$;

(b) $0 < \liminf_n c_n \le \limsup_n(b_n + c_n) < 1$;

(c) $0 < \liminf_n \gamma_n \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$;

(C2) $0 < \liminf_n \beta_n \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ and one of the following holds:

(a) $0 < \liminf_n a_n \le \limsup_n a_n < 1$;

(b) $0 < \liminf_n c_n \le \limsup_n(b_n + c_n) < 1$;

(c) $0 < \liminf_n \gamma_n \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$,

then $\lim_n \|Tx_n - x_n\| = 0$.

Proof. For any $p \in F(T)$, it follows from Lemma 2.1 that $\lim_n \|x_n - p\|$ exists.

Let $\lim_n \|x_n - p\| = d$ for some $d \ge 0$.

Since $0 < \liminf_n \alpha_n \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, by Lemma 2.2(i), we have

$$\lim_n \|T^ny_n - x_n\| = 0. \quad (17)$$

Next, we show that $\lim_n \|z_n - p\| = d$, $\limsup_n \|x_n - p\| \le d$ and $\limsup_n \|T^nx_n - p\| \le d$.

We consider

$\|x_n - p\| \le \|x_n - T^ny_n\| + \|T^ny_n - p\| = \|x_n - T^ny_n\|$

$+ (\|T^ny_n - p\| - \|y_n - p\|) + \|y_n - p\|$

$\le \|x_n - T^ny_n\| + \sup_{x\in C} (\|T^nx - p\| - \|x - p\|) + \|y_n - p\|$

$\le \|x_n - T^ny_n\| + \sup (\|T^nx - p\| - \|x - p\|)$

$+ b_n(\|T^nz_n - p\| - \|z_n - p\|) + \|z_n - p\|$

$+ c_n(\|T^nx_n - p\| - \|x_n - p\|) + (1 - b_n)\|x_n - p\|$

$\le \|x_n - T^ny_n\| + \sup (\|T^nx - p\| - \|x - p\|)$

$+ (\|T^nz_n - p\| - \|z_n - p\|) + \|z_n - p\|$

$+ c_n(\|T^nx_n - p\| - \|x_n - p\|)$

$\le \|x_n - T^ny_n\| + \sup (\|T^nx - p\| - \|x - p\|)$

$+ \sup_{x\in C} (\|T^nx - p\| - \|x - p\|)$

$+ c_n \sup_{x\in C} (\|T^nx - p\| - \|x - p\|) + \|z_n - p\|$

$\le \|x_n - T^ny_n\| + (2 + c_n) \sup (\|T^nx - p\| - \|x - p\|)$

$+ \|z_n - p\|$

$\le \|x_n - T^ny_n\| + (2 + c_n) G_n + \|z_n - p\|$.

Hence $d \le \liminf_n \|z_n - p\|$.

By taking $\limsup_n$ on both sides of the inequality (7), we have $\limsup_n \|z_n - p\| \le d$.

It follows that $\lim_n \|z_n - p\| = d$.

Next, we show that $\limsup_n \|x_n - p\| \le d$ and $\limsup_n \|T^nx_n - p\| \le d$.

Since $\lim_n \|x_n - p\| = d$, $\limsup_n \|x_n - p\| = d$.

We consider

$$\begin{aligned}\|T^nx_n - p\| &= (\|T^nx_n - p\| - \|x_n - p\|) + \|x_n - p\| \\ &\le \sup_{x\in C} (\|T^nx - p\| - \|x - p\|) + \|x_n - p\| \\ &\le G_n + \|x_n - p\|.\end{aligned}$$

Thus $\limsup_n \|T^nx_n - p\| \le \limsup_n G_n + \limsup_n \|x_n - p\| \le d$.

(C1). Let $0 < \liminf_n b_n \le \limsup_n(b_n + c_n) < 1$. Since $0 < \liminf_n \alpha_n$, by Lemma 2.2(iv), we have

$$\lim_n \|T^nz_n - x_n\| = 0. \quad (18)$$

(C1)(a). Let $0 < \liminf_n a_n \le \limsup_n a_n < 1$. From $\lim_n \|z_n - p\| = d$, we obtain

$$d = \lim_n \| a_n(T^n x_n - p) + (1 - a_n)(x_n - p) \|. \quad (19)$$

Since $0 < \liminf_n a_n$, $0 < \liminf_n (1 - a_n)$ and $a_n + (1 - a_n) = 1$, by applying Lemma 1.3, we have

$$\lim_n \|T^n x_n - x_n\| = 0. \quad (20)$$

From (17), (18) and (20), by Lemma 2.3, we have

$$\lim_n \|Tx_n - x_n\| = 0.$$

(C1)(b). Let $0 < \liminf_n c_n \le \limsup_n (b_n + c_n) < 1$, and since $0 < \liminf_n \alpha_n$

By Lemma 2.2(v), we have

$$\lim_n \|T^n x_n - x_n\| = 0. \quad (21)$$

From (17), (18) and (21), by Lemma 2.3, we have

$$\lim_n \|Tx_n - x_n\| = 0.$$

(C1)(c). Let $0 < \liminf_n \gamma_n \le \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$.

By Lemma 2.2(iii), we have

$$\lim_n \|T^n x_n - x_n\| = 0. \quad (22)$$

From (17), (18) and (22), by Lemma 2.3, we have

$$\lim_n \|Tx_n - x_n\| = 0.$$

(C2). Let $0 < \liminf_n \beta_n \le \limsup_n (\alpha_n + \beta_n + \gamma_n) < 1$.

By Lemma 2.2(ii), we have

$$\lim_n \|T^n z_n - x_n\| = 0. \quad (23)$$

Similary, if (C2)(a) or (C2)(b) or (C2)(c) is satisfied, then

$$\lim_n \|Tx_n - x_n\| = 0. \quad \#$$

Theorem 2.5. Let X, C, T and $\{x_n\}$ be defined as in Lemma 2.4. If T is a completely continuous mapping, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$ be fixed point. From Lemma 2.1, we know that $\lim_n \|x_n - p\|$ exists, then $\{x_n\}$ is bounded. By Lemma 2.4, we have

$$\lim_n \|Tx_n - x_n\| = 0. \quad (24)$$

Since T is completely continuous and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Let $\lim_n Tx_{n_k} = q$. Then by (24), $\lim_n x_{n_k} = q$. Thus, by continuity of T, we have $Tq = q$, so q is a fixed point of T.

It follows from Lemma 2.1 that $\lim_n \|x_n - q\|$ exists. But $\lim_k \|x_{n_k} - q\| = 0$, thus $\lim_n \|x_n - q\| = 0$. This completes the proof. #

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