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Certain Integral Formulae Involving Incomplete I-Functions

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ABSTRACT

In this paper, we have determined some integral formulae of incomplete I-functions involving the exponential function, the Legendre polynomials and generalized Laguerre polynomials.

Keywords: Exponential function; Generalized Laguerre polynomials; Incomplete I-functions; Legendre polynomials.

1. Introduction and Definitions

Along with its relevance in engineering and mathematical physics, special functions and their extensions have piqued the interest of numerous researchers, see [1–6]. Certainly, incomplete Gamma functions and their extensions have been observed to play an important role in a variety of situations in the fields of heat conduction and astrophysics. As a result, a great deal of work has been done in this area. See the latest work in [7–13] for further information on potential extensions and applications. Our goal is to look into some new integrals associated with incomplete I-functions that involve a variety of other special functions.

In the year 1997, Rathie [14] discovered the I-function which is defined as follows:

$$I^{m,n}_{p,q}[z] = I^{m,\,n}_{p,\,q}\left[z \,\middle|\, \begin{matrix} (\mathfrak{a}_1,\nu_1;E_1),\ldots,(\mathfrak{a}_p,\nu_p;E_p) \\ (\mathfrak{b}_1,\omega_1;F_1),\ldots,(\mathfrak{b}_p,\omega_p;F_p) \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathfrak{L}} \psi(s)\, z^s\, ds, \tag{1.1}$$

where,

$$\psi(s) = \frac{\prod_{j=1}^{m}[\Gamma(\mathfrak{b}_j-\omega_j s)]^{F_j}\prod_{j=1}^{n}[\Gamma(1-\mathfrak{a}_j+\nu_j s)]^{E_j}}{\prod_{j=m+1}^{q}[\Gamma(1-\mathfrak{b}_j+\omega_j s)]^{F_j}\prod_{j=n+1}^{p}[\Gamma(\mathfrak{a}_j-\nu_j s)]^{E_j}}, \tag{1.2}$$

and $m,n,p,q \in N_0$ with $0 \le n \le p$, $0 \le m \le q$, $\nu_j, E_j(j=1,..,p)$, $\omega_j, F_j(j=$

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$1, \ldots, q) \in \mathbb{R}^+$, $\mathfrak{a}_j, \mathfrak{b}_j \in \mathbb{C}$. The appropriate conditions for the convergence of the $\mathcal{L}$ contour appeared in Eq. (1.1) and the description of the I-function can be found in [14].

We next defined the well-known lower and upper incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ as follows:

$$\gamma(s,x) := \int_0^x y^{s-1}\, e^{-y}\, dy \qquad (1.3)$$

$$\big(\Re(s) > 0;\ x \geqq 0\big)$$

and

$$\Gamma(s,x) := \int_x^\infty y^{s-1}\, e^{-y}\, dy \qquad (1.4)$$

$$\big(x \geqq 0;\ \Re(s) > 0\big)$$

where $\Re(s)$ stands for real part of s. In particular if $x = 0$, then $\Gamma(s,x)$ reduces to the standard gamma function $\Gamma(s)$. These functions satisfy the following decomposition relation:

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s) \quad (\Re(s) > 0). \qquad (1.5)$$

By making use of the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ represented by Eq. (1.3) and Eq. (1.4), Srivastava et al. [15] discovered and defined the incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ as follows:

$$\gamma_{p,\,q}^{m,\,n}(z) = \gamma_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{matrix} (\mathfrak{a}_1, \nu_1, x), (\mathfrak{a}_j, \nu_j)_{2,p} \\ (\mathfrak{b}_j, \omega_j)_{1,q} \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} g(s,x)\, z^{-s}\, ds \qquad (1.6)$$

and

$$\Gamma_{p,\,q}^{m,\,n}(z) = \Gamma_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{matrix} (\mathfrak{a}_1, \nu_1, x), (\mathfrak{a}_j, \nu_j)_{2,p} \\ (\mathfrak{b}_j, \omega_j)_{1,q} \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} G(s,x)\, z^{-s}\, ds, \qquad (1.7)$$

where,

$$g(s,x) = \frac{\gamma(1-\mathfrak{a}_1-\nu_1 s, x)\prod_{j=1}^{m}\Gamma(\mathfrak{b}_j+\omega_j s)\prod_{j=2}^{n}\Gamma(1-\mathfrak{a}_j-\nu_j s)}{\prod_{j=m+1}^{q}\Gamma(1-\mathfrak{b}_j-\omega_j s)\prod_{j=n+1}^{p}\Gamma(\mathfrak{a}_j+\nu_j s)}, \qquad (1.8)$$

and

$$G(s,x) = \frac{\Gamma(1-\mathfrak{a}_1-\nu_1 s, x)\prod_{j=1}^{m}\Gamma(\mathfrak{b}_j+\omega_j s)\prod_{j=2}^{n}\Gamma(1-\mathfrak{a}_j-\nu_j s)}{\prod_{j=m+1}^{q}\Gamma(1-\mathfrak{b}_j-\omega_j s)\prod_{j=n+1}^{p}\Gamma(\mathfrak{a}_j+\nu_j s)}, \qquad (1.9)$$

with the arrangement of conditions established in [15].

These incomplete H-functions satisfy the following decomposition relation:

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z). \qquad (1.10)$$

For $x \geq 0$, the incomplete H-functions presented in Eq. (1.6) and Eq. (1.7) exist.

Jangid et al. [16] described a class of the incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$ which are defined as follow:

$${}^{\gamma}I_{p,q}^{m,n}[z] = {}^{\gamma}I_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{matrix} (\mathfrak{a}_1, \nu_1; E_1 : x), (\mathfrak{a}_j, \nu_j; E_j)_{(2,p)} \\ (\mathfrak{b}_j, \omega_j; F_j)_{(1,q)} \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \phi(s,x)\, z^{s}\, ds, \qquad (1.11)$$

and

$${}^{\Gamma}I_{p,q}^{m,n}[z] = {}^{\Gamma}I_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{matrix} (\mathfrak{a}_1, \nu_1; E_1 : x), (\mathfrak{a}_j, \nu_j; E_j)_{(2,p)} \\ (\mathfrak{b}_j, \omega_j; F_j)_{(1,q)} \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \Phi(s,x)\, z^{s}\, ds, \qquad (1.12)$$

for all $z \neq 0$, where,

$$\phi(s,x) = [\gamma(1-\mathfrak{a}_1+\nu_1 s, x)]^{E_1} \times \frac{\prod_{j=1}^{m}[\Gamma(\mathfrak{b}_j-\omega_j s)]^{F_j}\prod_{j=2}^{n}[\Gamma(1-\mathfrak{a}_j+\nu_j s)]^{E_j}}{\prod_{j=m+1}^{q}[\Gamma(1-\mathfrak{b}_j+\omega_j s)]^{F_j}\prod_{j=n+1}^{p}[\Gamma(\mathfrak{a}_j-\nu_j s)]^{E_j}}, \qquad (1.13)$$

and

$$\Phi(s,x) = [\Gamma(1-\mathfrak{a}_1+\nu_1 s, x)]^{E_1} \times \frac{\prod_{j=1}^{m}[\Gamma(\mathfrak{b}_j-\omega_j s)]^{F_j}\prod_{j=2}^{n}[\Gamma(1-\mathfrak{a}_j+\nu_j s)]^{E_j}}{\prod_{j=m+1}^{q}[\Gamma(1-\mathfrak{b}_j+\omega_j s)]^{F_j}\prod_{j=n+1}^{p}[\Gamma(\mathfrak{a}_j-\nu_j s)]^{E_j}} \tag{1.14}$$

The definitions Eq. (1.11) and Eq. (1.12) at once yield the following decomposition relation:

$${}^{\gamma}I^{m,n}_{p,q}[z] + {}^{\Gamma}I^{m,n}_{p,q}[z] = I^{m,n}_{p,q}[z] \tag{1.15}$$

for the familiar I-function (see [14]).

The incomplete I-functions ${}^{\gamma}I^{m,n}_{p,q}[z]$ and ${}^{\Gamma}I^{m,n}_{p,q}[z]$ defined in Eq. (1.11) and Eq. (1.12) exist for all $x \geq 0$, under the set of conditions given by Rathie [14], with

$$\Delta > 0, |arg\ z| < \Delta\frac{\pi}{2},$$

where

$$\Delta = \sum_{j=1}^{m} F_j\omega_j - \sum_{j=m+1}^{q} F_j\omega_j + \sum_{j=1}^{n} E_j\nu_j - \sum_{j=n+1}^{p} E_j\nu_j.$$

For more recent development in the theory of incomplete H and I-functions, one can see the papers [21–23] and references therein.

2. Main Results

In the following segment, we shall derive certain integrals including the incomplete I-functions with exponential function, Legendre polynomials and generalized Laguerre polynomials, respectively.

2.1 Exponential function and Incomplete I-function

Here, we shall derive the integrals for the product of the incomplete I-function and exponential function with the help of the following Integrals formulae:

$$(i)\quad \int_{-\pi}^{\pi} (\cos\theta)^{\alpha-1}\ \exp(i\beta\theta)d\theta = \frac{\pi\Gamma(\alpha)}{2^{\alpha-1}\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\frac{\alpha-\beta+1}{2}\right)}, \tag{2.1}$$

where $\Re(\alpha) > 0$.

(ii) The following integral was given by Nielson [17],

$$\int_{0}^{\pi} (\sin\theta)^{\alpha}\ e^{-\beta\,\theta}\ d\theta = \frac{\pi\ e^{-\frac{\pi\beta}{2}}\Gamma(\alpha+1)}{2^{\alpha}\Gamma\left(1+\frac{\alpha+i\beta}{2}\right)\ \Gamma\left(1+\frac{\alpha-i\beta}{2}\right)}, \tag{2.2}$$

where $\Re(\alpha) > 1$.

We have established the following results:

Theorem 2.1. *If* $h > 0,\ k,\ \lambda \in C$, $\mu \geq 0$, $\Delta > 0$, $|argz| < \frac{1}{2}\pi\Delta$ *and* $\Re\left(k+\lambda-h\frac{\mathfrak{a}_i}{\nu_i}\right) > 1-\frac{h}{\nu_i}\quad i=1,\cdots,n,$ *then the following integral holds;*

$$\int_{-\pi}^{\pi} (\cos\theta)^{k+\lambda-2}\ \exp\left[i(k-\lambda)\theta\right] \times {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[z\,(e^{i\theta}\cos\theta)^{-h}\ \middle|\ \begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\(\mathfrak{b}_1,\omega_1;F_1)(\mathfrak{b}_j,\omega_j;F_j)_{2,q}\end{matrix}\right]d\theta$$

$$= \frac{\pi}{2^{k+\lambda-2}\Gamma(\lambda)} \times {}^{\Gamma}I^{m+1,\,n}_{p+1,\,q+1}\left[2^h\ z\ \middle|\ \begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),(\mathfrak{a}_j,\nu_j;E_j)_{2,p},(k,h;1)\\(k+\lambda-1,h;1),(\mathfrak{b}_j,\omega_j;F_j)_{1,q}\end{matrix}\right]. \tag{2.3}$$

Proof. To demonstrate the outcome Eq. (2.3), take the left-hand side assertion of Eq. (2.1). Express the incomplete I-function in terms of Mellin-Barnes type integral defined in Eq. (1.12) and then on changing the order of the integrations, we have

$$\int_{-\pi}^{\pi}(\cos\theta)^{k+\lambda-2}\ \exp\left[i(k-\lambda)\theta\right]$$
$$\times\frac{1}{2\pi i}\int z^s\ e^{-i\theta hs}(\cos\theta)^{-hs}\ \Phi(s,x)\ ds\ d\theta$$
$$=\frac{1}{2\pi i}\int z^s\ \Phi(s,x)$$
$$\times\int_{-\pi}^{\pi}(\cos\theta)^{(k+\lambda-hs-1)-1}\ e^{i(k-\lambda-hs)\theta}\ d\theta\ ds$$

using the integral formula Eq. (2.1), we obtain

$$=\frac{1}{2\pi i}\int z^s\ \Phi(s,x)$$
$$\times\frac{\pi\ \Gamma(k+\lambda-hs-1)}{2^{(k+\lambda-hs-2)}\Gamma(k-hs)\Gamma(\lambda)}ds$$
$$=\frac{\pi}{2^{k+\lambda-2}\Gamma(\lambda)}$$
$$\times\frac{1}{2\pi i}\int\frac{z^s\ 2^{hs}\Gamma(k+\lambda-1-hs)\ \Phi(s,x)}{\Gamma(k-hs)}ds \tag{2.4}$$

using Eq. (1.14), we obtain the required R.H.S of Eq. (2.3).

Theorem 2.2. *If* $h>0,\ k,\ \lambda\in C,\ \mu\geq 0,\ \Delta>0,\ |argz|<\frac{1}{2}\pi\Delta$ *and* $\Re\left(k+\lambda-h\frac{\mathfrak{a}_i}{\nu_i}\right)>1-\frac{h}{\nu_i},\quad i=1,\cdots,n,$ *then the following integral holds:*

$$\int_{-\pi}^{\pi}(\cos\theta)^{k+\lambda-2}\ \exp\left[i(k-\lambda)\theta\right]\times$$
$${}^{\Gamma}I_{p,q}^{m,n}\left[ze^{ih\theta}(\sec\theta)^h\ \middle|\ \begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),\ (\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\(\mathfrak{b}_1,\omega_1;F_1)\ (\mathfrak{b}_j,\omega_j;F_j)_{2,q}\end{matrix}\right]d\theta$$
$$=\frac{\pi}{2^{k+\lambda-2}\Gamma(k)}\times$$
$${}^{\Gamma}I_{p+1,q+1}^{m+1,n}\left[2^h z\ \middle|\ \begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),\ (\mathfrak{a}_j,\nu_j;E_j)_{2,p},(\lambda,h;1)\\(k+\lambda-1,h;1),\ (\mathfrak{b}_j,\omega_j;F_j)_{1,q}\end{matrix}\right]. \tag{2.5}$$

Proof. To demonstrate the outcome Eq. (2.5), take the left-hand side assertion of Eq. (2.2). Express the incomplete I-function in terms of Mellin-Barnes type integral defined in Eq. (1.12), we get

$$\int_{-\pi}^{\pi}(\cos\theta)^{k+\lambda-2}\ \exp\left[i(k-\lambda)\theta\right]\times$$
$$\frac{1}{2\pi i}\int z^s\ e^{i\theta hs}(\sec\theta)^{hs}\Phi(s,x)dsd\theta$$

on changing the order of integrations,

$$=\frac{1}{2\pi i}\int z^s\ \Phi(s,x)\times$$
$$\int_{-\pi}^{\pi}(\cos\theta)^{(k+\lambda-hs-1)-1}\ e^{i(k-\lambda+hs)\theta}\ d\theta ds$$

using the integral formula Eq. (2.1), we obtain

$$=\frac{1}{2\pi i}\int z^s\ \Phi(s,x)\times$$
$$\frac{\pi\ \Gamma(k+\lambda-hs-1)}{2^{(k+\lambda-hs-2)}\Gamma(\lambda-hs)\Gamma(k)}ds$$
$$=\frac{\pi}{2^{k+\lambda-2}\Gamma(k)}\times$$
$$\frac{1}{2\pi\ i}\int\frac{z^s\ 2^{hs}\Gamma(k+\lambda-1-hs)\ \Phi(s,x)}{\Gamma(\lambda-hs)}ds \tag{2.6}$$

using Eq. (1.14), we obtain the required R.H.S of Eq. (2.5).

Theorem 2.3. *If* $h > 0$, k, $\lambda \in C$, $\mu \geq 0$, $\Delta > 0$, $|argz| < \frac{1}{2}\pi\Delta$ *and* $\Re\left(k+\lambda-h\frac{\mathfrak{a}_i}{\nu_i}\right) > 1-\frac{h}{\nu_i}$, $i = 1,\cdots,n$, *then the following integral holds:*

$$\int_{-\pi}^{\pi}(\cos\theta)^{k+\lambda-2}\ \exp\left[i(k-\lambda)\theta\right]\ \times$$
$$^{\Gamma}I_{p,\,q}^{m,\,n}\left[z\,(\sec\theta)^{2h}\ \middle|\ \begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),\\(\mathfrak{b}_1,\omega_1;F_1)\end{matrix}\quad \begin{matrix}(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\(\mathfrak{b}_j,\omega_j;F_j)_{2,q}\end{matrix}\right]d\theta$$
$$=\frac{\pi}{2^{k+\lambda-2}}\times$$
$$^{\Gamma}I_{p+2,\,q+1}^{m+1,\,n}\left[2^{2h}\ z\ \middle|\ \begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),(\mathfrak{a}_j,\nu_j;E_j)_{2,p},\\(k+\lambda-1,2h;1),\end{matrix}\quad \begin{matrix}(k,h;1),(\lambda,h;1)\\(\mathfrak{b}_j,\omega_j;F_j)_{1,q}\end{matrix}\right].\tag{2.7}$$

Proof. To prove the result Eq. (2.7), we will take the L.H.S. of Eq. (2.3). Now, write the incomplete I-function in terms of the Mellin-Barness type contour integral defined in Eq. (1.12) and then, on changing the order of the integrals, we get

$$\int_{-\pi}^{\pi}(\cos\theta)^{k+\lambda-2}\ \exp\left[i(k-\lambda)\theta\right]\times$$
$$\frac{1}{2\pi i}\int z^s\ (\sec\theta)^{2hs}\Phi(s,x)\ ds\ d\theta$$
$$=\frac{1}{2\pi i}\int z^s\ \Phi(s,x)\times$$
$$\int_{-\pi}^{\pi}(\cos\theta)^{(k+\lambda-2hs-1)-1}\ e^{[i(k-\lambda)\theta]}\ d\theta\, ds$$

using the integral formula Eq. (2.1)

$$=\frac{1}{2\pi i}\int z^s\ \Phi(s,x)\times$$
$$\frac{\pi\ \ \Gamma(k+\lambda-2hs-1)}{2^{(k+\lambda-2hs-2)}\ \Gamma(\lambda-hs)\ \Gamma(k-hs)}\,ds$$
$$=\frac{\pi}{2^{k+\lambda-2}}\times$$
$$\frac{1}{2\pi\, i}\int\frac{z^s\ 2^{2hs}\Gamma(k+\lambda-1-2hs)\ \Phi(s,x)}{\Gamma(\lambda-hs)\ \Gamma(k-hs)}\,ds\tag{2.8}$$

using definition Eq. (1.14), we will get the desired result Eq. (2.7).

Theorem 2.4. *Let* $\Delta > 0$, $|arg\,z| < \frac{1}{2}\pi\Delta$ *or* $\Delta = 0$, $\Re(\delta) < -1$, *then the following integral formula holds;*

$$\int_{0}^{\pi}(\sin\theta)^{\gamma-1}\ e^{-\eta\,\theta}\times$$
$$^{\Gamma}I_{p,\,q}^{m,\,n}\left[z\,(\sin\theta)^{2h}\ \middle|\ \begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),\\(\mathfrak{b}_1,\omega_1;F_1)\end{matrix}\quad \begin{matrix}(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\(\mathfrak{b}_j,\omega_j;F_j)_{2,q}\end{matrix}\right]d\theta$$
$$=\frac{\pi}{2^{\gamma-1}}\ \exp\left(-\frac{\pi\eta}{2}\right)\times$$
$$^{\Gamma}I_{p+1,\,q+2}^{m,\,n+1}\left[\frac{z}{4^h}\ \middle|\ \begin{matrix}(1-\gamma,2h;1),(\mathfrak{a}_1,\nu_1;E_1;x),\\(\mathfrak{b}_j,\omega_j;F_j)_{1,q},\end{matrix}\quad \begin{matrix}(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\\left(\frac{1-\gamma+i\eta}{2},h;1\right),\left(\frac{1-\gamma-i\eta}{2},h;1\right)\end{matrix}\right],\tag{2.9}$$

where $\gamma,\eta \in C$, $h > 0$ *are such that* $\Re(\gamma)+2h\min\limits_{1\leq j\leq m}\left[\frac{\Re(\mathfrak{b}_j)}{\omega_j}\right] > 0$ *for* $\Delta > 0$, $|arg\ z| < \frac{1}{2}\pi\Delta$ *or* $\Delta = 0$, $\mu \geq 0$, *and* $\Re(\gamma)+2h\min\limits_{1\leq j\leq m}\left[\frac{\Re(\mathfrak{b}_j)}{\omega_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0$ *for* $\Delta = 0$ *and* $\mu < 0$.

Proof. To prove the result Eq. (2.9), take the left-hand side assertion of Eq. (2.4). Now, write the incomplete I-function in terms of the Mellin-Barness contour type integral defined in Eq. (1.12), we get

$$\int_0^{\pi} (\sin\theta)^{\gamma-1}\, e^{-\eta\theta} \times \frac{1}{2\pi i}\int z^s\,(\sin\theta)^{2hs}\Phi(s,x)\; ds\, d\theta$$

On changing the order of the integrations,

$$= \frac{1}{2\pi i}\int z^s\;\Phi(s,x) \times \int_0^{\pi} (\sin\theta)^{\gamma+2hs-1}\, e^{-\eta\theta}\; d\theta\, ds$$

using the integral formula Eq. (2.2)

$$= \frac{1}{2\pi i}\int z^s\;\Phi(s,x)\;\times \frac{\pi\; e^{-\frac{\pi}{2}\eta}}{2^{(\gamma+2hs-1)}} \frac{\Gamma(\gamma+2hs)}{\Gamma\left(1+\frac{\gamma+i\eta-1}{2}+hs\right)\Gamma\left(1+\frac{\gamma-i\eta-1}{2}+hs\right)}\,ds$$

$$= \frac{\pi}{2^{\gamma-1}}\; e^{-\frac{\pi}{2}\eta} \times \frac{1}{2\pi\; i}\int \Phi(s,x)\frac{z^s}{4^{hs}} \times \frac{\Gamma(\gamma+2hs)}{\Gamma\left(1+\frac{\gamma+i\eta-1}{2}+hs\right)\Gamma\left(1+\frac{\gamma-i\eta-1}{2}+hs\right)} ds \tag{2.10}$$

using definition Eq. (1.14), we will get the desired result Eq. (2.9).

Theorem 2.5. *Let* $\gamma, \eta \in C$, $h > 0$, $|arg\;z| < \frac{1}{2}\pi\Delta$ *or* $\Delta > 0$, $\Re(\gamma) > 0$, *then the following integral formula holds;*

$$\int_0^{\pi} (\sin\theta)^{\gamma-1}\, e^{-\eta\,\theta} \times {}^{\Gamma}I_{p,\,q}^{m,\,n}\left[z\, e^{i2h\theta}\;\middle|\; \begin{matrix} (\mathfrak{a}_1,\nu_1;E_1;x), (\mathfrak{a}_j,\nu_j;E_j)_{2,p} \\ (\mathfrak{b}_1,\omega_1;F_1) (\mathfrak{b}_j,\omega_j;F_j)_{2,q} \end{matrix}\right] d\theta$$

$$= \frac{\pi\;\Gamma(\gamma)}{2^{\gamma-1}}\;\exp\left(-\frac{\pi\eta}{2}\right) \times {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n}\left[z\; e^{i\pi h}\;\middle|\; \begin{matrix} (\mathfrak{a}_1,\nu_1;E_1;x),(\mathfrak{a}_j,\nu_j;E_j)_{2,p}, \left(\frac{1+\gamma-i\eta}{2},h;1\right) \\ (\mathfrak{b}_j,\omega_j;F_j)_{1,q}, \left(\frac{1-\gamma-i\eta}{2},h;1\right) \end{matrix}\right]. \tag{2.11}$$

Proof. To demonstrate the result Eq. (2.11), take the left-hand side assertion of Eq. (2.5). Express incomplete I-function in terms of the Mellin-Barness type contour integral defined in Eq. (1.12), we obtain

$$\int_0^{\pi} (\sin\theta)^{\gamma-1}\, e^{-\eta\theta} \times \frac{1}{2\pi i}\int z^s e^{i2h\theta s}\Phi(s,x)dsd\theta$$

On changing the order of the integrations,

$$= \frac{1}{2\pi i}\int z^s\Phi(s,x) \times \int_0^{\pi} (\sin\theta)^{\gamma-1}\, e^{-[\eta-i2hs]\theta}\; d\theta ds$$

using the integral formula Eq. (2.2)

$$= \frac{1}{2\pi i}\int z^s\;\Phi(s,x)\;\frac{\pi\; e^{-\frac{\pi}{2}[\eta-2ihs]}}{2^{(\gamma-1)}} \times \frac{\Gamma(\gamma)}{\Gamma\left(1+\frac{\gamma+i\eta-1+2hs}{2}\right)\Gamma\left(1+\frac{\gamma-i\eta-1-2hs}{2}\right)}\, ds$$

$$= \frac{\pi\,\Gamma(\gamma)}{2^{\gamma-1}}\, e^{-\frac{\pi}{2}\eta} \times \frac{1}{2\pi\, i}\int z^s \Phi(s,x) \times \frac{e^{i\pi hs}}{\Gamma\left(1+\frac{\gamma+i\eta-1+2hs}{2}\right)\,\Gamma\left(1+\frac{\gamma-i\eta-1-2hs}{2}\right)}\, ds \tag{2.12}$$

using definition Eq. (1.14), we will get the desired result Eq. (2.11).

Theorem 2.6. *If* $\Delta > 0$, $\Re(\gamma) > 0$ *or* $\Delta = 0$, $\Re(\delta) < -1$, *then the following integral formula;*

$$\int_0^{\pi} (\sin\theta)^{\gamma-1}\, e^{-\eta\,\theta} \times {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[z\,(\sin\theta)^{2\lambda}\, e^{i2h\theta} \,\middle|\, \begin{matrix} (\mathfrak{a}_1, \nu_1; E_1; x), & (\mathfrak{a}_j, \nu_j; E_j)_{2,p} \\ (\mathfrak{b}_1, \omega_1; F_1) & (\mathfrak{b}_j, \omega_j; F_j)_{2,q} \end{matrix} \right] d\theta$$

$$= \frac{\pi}{2^{\gamma-1}} \exp\left(-\frac{\pi\eta}{2}\right) \times {}^{\Gamma}I^{m,\,n+1}_{p+1,\,q+2}\left[\frac{z\, e^{i\pi h}}{4^{\lambda}} \,\middle|\, \begin{matrix} (1-\gamma, 2\lambda; 1), & (\mathfrak{a}_1, \nu_1; E_1; x), (\mathfrak{a}_j, \nu_j; E_j)_{2,p} \\ (\mathfrak{b}_j, \omega_j; F_j)_{1,q}, & \left(\frac{1-\gamma-i\eta}{2}, \lambda+h; 1\right), \left(\frac{1-\gamma+i\eta}{2}, \lambda-h; 1\right) \end{matrix} \right] \tag{2.13}$$

holds for $h > 0$, $\lambda > h, \gamma, \eta \in C$, *such that* $\Re(\gamma) + 2\lambda \min_{1\le j\le m}\left[\frac{\Re(\mathfrak{b}_j)}{\omega_j}\right] > 0$ *for* $\Delta > 0$ *or* $\Delta = 0$ *and* $\mu \geq 0$; *and* $\Re(\gamma) + 2\lambda \min_{1\le j\le m}\left[\frac{\Re(\mathfrak{b}_j)}{\omega_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0$ *for* $\Delta = 0$ *and* $\mu < 0$.

Proof. To demonstrate the result Eq. (2.13), take the left-hand side assertion of Eq. (2.6). Express incomplete I-function in terms of the Mellin-Barness type contour integral defined in Eq. (1.12), we obtain

$$\int_0^{\pi} (\sin\theta)^{\gamma-1}\, e^{-\eta\theta} \times \frac{1}{2\pi\, i}\int z^s (\sin\theta)^{2\lambda s} e^{2ih\theta s}\Phi(s,x)\, ds\, d\theta$$

On changing the order of the integrations,

$$= \frac{1}{2\pi i}\int z^s\, \Phi(s,x) \times \int_0^{\pi} (\sin\theta)^{\gamma+2\lambda s-1} e^{-[\eta-2hsi]\theta}\, d\theta ds$$

Using the formula Eq. (2.2)

$$= \frac{1}{2\pi\, i}\int z^s\, \Phi(s,x) \times \frac{e^{-\frac{\pi}{2}[\eta-2hsi]}}{2^{\gamma+2\lambda s-1}} \times \frac{\Gamma(\gamma+2\lambda s)}{\Gamma\left(1+\frac{\gamma+i\eta-1}{2}+s(\lambda+h)\right)} \times \frac{1}{\Gamma\left(1+\frac{\gamma-i\eta-1}{2}+s(\lambda+h)\right)}\, ds$$

$$= \frac{\pi}{2^{\gamma-1}} \exp\left(-\frac{\pi\eta}{2}\right) \times \frac{1}{2\pi\, i}\int \frac{z^s\Phi(s,x)}{4^{\lambda s}} \times \frac{e^{\frac{\pi}{2}[2hsi]}}{\Gamma\left(1+\frac{\gamma+i\eta-1}{2}+s(\lambda+h)\right)} \times \frac{\Gamma(\gamma+2\lambda s)}{\Gamma\left(1+\frac{\gamma-i\eta-1}{2}+s(\lambda-h)\right)} ds \tag{2.14}$$

using definition Eq. (1.14), we will get the desired result Eq. (2.13).

2.2 Legendre Polynomials and Incomplete *I*-Function

Here, we shall establish certain integral formulae involving the product of the incomplete I-function and Legendre polynomials defined in [6].

The following known results will be use to prove our theorems.

(i) The following integral has obtained by Erdelyi [18, p. 316];

$$\int_{-1}^{1} (1-x^2)^{\rho-1}\, P_\nu^m(x)\, dx$$

$$= \frac{\pi\, 2^m\, \Gamma\left(\rho + \frac{1}{2}m\right)}{\Gamma\left(1 + \frac{1}{2}(\nu - m)\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}(\nu + m)\right)} \times \frac{\Gamma\left(\rho - \frac{1}{2}m\right)}{\Gamma\left(\rho - \frac{1}{2}\nu\right)\Gamma\left(1 + \rho + \frac{1}{2}\nu\right)} \tag{2.15}$$

provided $2\,\Re(\rho) > |\Re(m)|$.

(*ii*) This integral is given by Milne-Thomson [19, p. 33];

$$E_a f(a) = f(a{+}1),\ E_a^n f(a) = E_a\left[E_a^{n-1} f(a)\right] \tag{2.16}$$

where E denotes the finite difference operator. And, we will use the following notation:

$$(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)} = a(a+1)\cdots(a+r-1). \tag{2.17}$$

Theorem 2.7. *Let* $\rho, z \in C, \Delta > 0, |arg\, z| < \frac{1}{2}\pi\Delta$ *or* $\Delta = 0$, $\Re(\mu) < -1$. *Further, let* $\rho \in C, k > 0$ *satisfy the conditions*

$$\Re(\rho) + k \min_{1\le j\le m}\left[\frac{\Re(\mathfrak{b}_j)}{\omega_j}\right] > \frac{1}{2}|\Re(\mu)|$$

for $\Delta > 0$, $|arg\, z| < \frac{1}{2}\pi\Delta$ *or* $\Delta = 0$, $\mu \ge 0$ *and*

$$\Re(\rho) + k \min_{1\le j\le m}\left[\frac{\Re(\mathfrak{b}_j)}{\omega_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu}\right] > \frac{1}{2}|\Re(\mu)|,$$

For $\Delta = 0, \mu < 0$ *then there holds the formula*

$$\int_{-1}^{1} (1 - x^2)^{\rho-1}\, P_\nu^\lambda(x) \times {}^{\Gamma}I_{p,\,q}^{m,\,n}\left[z\,(1 - x^2)^k \,\middle|\, \begin{matrix} (\mathfrak{a}_1, \nu_1; E_1; x), \\ (\mathfrak{b}_1, \omega_1; F_1) \end{matrix} \begin{matrix} (\mathfrak{a}_j, \nu_j; E_j)_{2,p} \\ (\mathfrak{b}_j, \omega_j; F_j)_{2,q} \end{matrix}\right] dx$$

$$= \frac{2^\lambda\, \pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\, \Gamma\left(\frac{1-\nu-\lambda}{2}\right)} \times {}^{\Gamma}I_{p+2,\, q+2}^{m,\, n+2}\left[z \,\middle|\, \begin{matrix} \left(1 - \rho \pm \frac{\lambda}{2}, k; 1\right), \\ (\mathfrak{b}_j, \omega_j; F_j)_{1,q}, \end{matrix} \begin{matrix} (\mathfrak{a}_1, \nu_1; E_1; x), (\mathfrak{a}_j, \nu_j; E_j)_{2,p} \\ \left(1 - \rho + \frac{\nu}{2}, k; 1\right), \left(-\rho - \frac{\nu}{2}, k; 1\right) \end{matrix}\right]. \tag{2.18}$$

Proof. To prove Eq. (2.18), we first write the incomplete I-function in terms of Mellin-Barnes contour integral form Eq. (1.12), we have

$$I_1(\rho) = \int_{-1}^{1} (1 - x^2)^{\rho-1}\, P_\nu^\lambda(x) \times \frac{1}{2\pi\, i}\int_{\mathcal{L}} z^s (1 - x^2)^{ks}\Phi(s, x)\,ds dx$$

on changing the order of the integration

$$= \frac{1}{2\pi\, i}\int_{\mathcal{L}} z^s\, \Phi(s, x) \times \int_{-1}^{1} (1 - x^2)^{\rho+ks-1} P_\nu^\lambda(x)\, dx ds$$

Now, by making use of the formula Eq. (2.15) evaluate the internal integral

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} z^s \Phi(s, x) \times \frac{2^\lambda \pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\Gamma\left(\frac{1-\nu-\lambda}{2}\right)} \times \frac{\Gamma\left(\rho + \frac{1}{2}\lambda + ks\right)\Gamma\left(\rho - \frac{1}{2}\lambda + ks\right)}{\Gamma\left(\rho - \frac{1}{2}\nu + ks\right)\Gamma\left(1 + \rho + \frac{1}{2}\nu + ks\right)} ds \tag{2.19}$$

and finally reinterpreting the Mellin-Barnes contour integral thus involved by definition of incomplete I-function, we get the desired result.

Theorem 2.8. *Let* k *and* d *are positive integers,* $\mathfrak{U} < \mathfrak{V}$ *or* $\mathfrak{U} = \mathfrak{V} + 1$ *and* $|c| < 1$ *and none of* β_j, $j = 1, \cdots, \mathfrak{V}$ *is a negative integer or zero, then the following integral* *holds,*

$$\int_{-1}^{1}(1-x^2)^{\rho-1}\ P_\nu^\lambda(x) \times {}_{\mathfrak{U}}F_{\mathfrak{V}}(\alpha_1,\cdots,\alpha_{\mathfrak{U}};\beta_1,\cdots,\beta_{\mathfrak{V}};c(1-x^2)^d)\ {}^{\Gamma}I_{p,q}^{m,n}\left[z\,(1-x^2)^k \,\middle|\, \begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),\\(\mathfrak{b}_1,\omega_1;F_1)\end{matrix}\ \begin{matrix}(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\(\mathfrak{b}_j,\omega_j;F_j)_{2,q}\end{matrix}\right]dx$$

$$=\frac{2^\lambda\pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\Gamma\left(\frac{1-\nu-\lambda}{2}\right)}\sum_{r=0}^{\infty}\frac{(\alpha_1)_r\cdots(\alpha_{\mathfrak{U}})_r}{(\beta_1)_r\cdots(\beta_{\mathfrak{V}})_r}\frac{c^r}{r!} \times {}^{\Gamma}I_{p+2,\,q+2}^{m,\,n+2}\left[z\,\middle|\,\begin{matrix}\left(1-\rho-rd\pm\frac{\lambda}{2},k;1\right),(\mathfrak{a}_1,\nu_1;E_1;x),\\(\mathfrak{b}_j,\omega_j;F_j)_{1,q},\left(1-\rho-rd+\frac{\nu}{2},k;1\right),\end{matrix}\ \begin{matrix}(\mathfrak{a}_j,\nu_j;F_j)_{2,p}\\(-\rho-rd-\frac{\nu}{2},k;1)\end{matrix}\right]. \tag{2.20}$$

Proof. On multiplying both sides of Eq. (2.18) by $\frac{\prod_{j=1}^{\mathfrak{U}}\Gamma(\alpha_j+\delta)(c)^\delta}{\prod_{j=1}^{\mathfrak{V}}\Gamma(\beta_j+\delta)}$ and applying the operator $\exp(E^d\,E_\delta)$ yields,

$$\exp(E^d\,E_\delta)\left[I_1(\rho)\ \frac{\prod_{j=1}^{\mathfrak{U}}\Gamma(\alpha_j+\delta)(c)^\delta}{\prod_{j=1}^{\mathfrak{V}}\Gamma(\beta_j+\delta)}\right]$$

$$=\frac{2^\lambda\,\pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\ \Gamma\left(\frac{1-\nu-\lambda}{2}\right)}\exp(E^d\,E_\delta)\frac{\prod_{j=1}^{\mathfrak{U}}\Gamma(\alpha_j+\delta)(c)^\delta}{\prod_{j=1}^{\mathfrak{V}}\Gamma(\beta_j+\delta)} \times {}^{\Gamma}I_{p+2,\,q+2}^{m,\,n+2}\left[z\,\middle|\,\begin{matrix}\left(1-\rho\pm\frac{\lambda}{2},k;1\right),(\mathfrak{a}_1,\nu_1;E_1;x),\\(\mathfrak{b}_j,\omega_j;F_j)_{1,q},\left(1-\rho+\frac{\nu}{2},k;1\right),\end{matrix}\ \begin{matrix}(\mathfrak{a}_j,\nu_j;F_j)_{2,p}\\(-\rho-\frac{\nu}{2},k;1)\end{matrix}\right]. \tag{2.21}$$

Taking summation on both sides of Eq. (2.21) and using the definition of finite difference operator Eq. (2.16), we get

$$\sum_{r=0}^{\infty}\left[\frac{\prod_{j=1}^{\mathfrak{U}}\Gamma(\alpha_j+\delta+r)(c)^{\delta+r}}{\prod_{j=1}^{\mathfrak{V}}\Gamma(\beta_j+\delta+r)\ r!} \times \int_{-1}^{1}(1-x^2)^{\rho+rd-1}P_\nu^\lambda(x)\times{}^{\Gamma}I_{p,\,q}^{m,\,n}\left[z\,(1-x^2)^k\,\middle|\,\begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\((f)_j,\omega_j;F_j)_{1,q}\end{matrix}\right]dx\right]$$

$$=\frac{2^\lambda\,\pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\ \Gamma\left(\frac{1-\nu-\lambda}{2}\right)} \times\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{\mathfrak{U}}\Gamma(\alpha_j+\delta+r)(c)^{\delta+r}}{\prod_{j=1}^{\mathfrak{V}}\Gamma(\beta_j+\delta+r)r!} \times{}^{\Gamma}I_{p+2,\,q+2}^{m,\,n+2}\left[z\,\middle|\,\begin{matrix}\left(1-\rho\pm\frac{\lambda}{2},k;1\right),(\mathfrak{a}_1,\nu_1;E_1;x),\\(\mathfrak{b}_j,\omega_j;F_j)_{1,q},\left(1-\rho+\frac{\nu}{2},k;1\right),\end{matrix}\ \begin{matrix}(\mathfrak{a}_j,\nu_j;F_j)_{2,p}\\(-\rho-\frac{\nu}{2},k;1)\end{matrix}\right]. \tag{2.22}$$

Now, on the left hand side of Eq. (2.22), change the order of the integration and summation which is justified, using the result Eq. (2.17) and finally, replacing $\alpha_j+\delta$ by α_j and $\beta_j+\delta$ by β_j enable us to obtain the value of the integral Eq. (2.20).

Theorem 2.9. *If* $\lambda=0$ *and* $\nu=\lambda$*, where* λ *is a positive integer, then the theorem Eq.* (2.8) *reduces to the following result,*

$$\int_{-1}^{1}(1-x^2)^{\rho-1}\ P_\lambda(x) \times {}_{\mathfrak{U}}F_{\mathfrak{V}}(\alpha_1,\cdots,\alpha_{\mathfrak{U}};\beta_1,\cdots,\beta_{\mathfrak{V}};c(1-x^2)^d)\ {}^{\Gamma}I_{p,q}^{m,n}\left[z\,(1-x^2)^k\,\middle|\,\begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\(\mathfrak{b}_j,\omega_j;F_j)_{1,q}\end{matrix}\right]dx$$

$$=\frac{\pi}{\Gamma\left(\frac{2+\lambda}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)}\sum_{r=0}^{\infty}\frac{(\alpha_1)_r\cdots(\alpha_{\mathfrak{U}})_r}{(\beta_1)_r\cdots(\beta_{\mathfrak{V}})_r}\frac{c^r}{r!}\times$$

$$
{}^{\Gamma}I^{m,n+2}_{p+2,q+2}\left[z\,\middle|\,\begin{matrix}(1-\rho-rd,k;1),(1-\rho-rd,k;1),\\ (\mathfrak{b}_j,\omega_j;F_j)_{1,q},\left(1-\rho-rd+\frac{v}{2},k;1\right),\end{matrix}\right.
$$

$$
\left.\begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),(\mathfrak{a}_j,\nu_j;F_j)_{2,p}\\ \left(-\rho-rd-\frac{v}{2},k;1\right)\end{matrix}\right]. \tag{2.23}
$$

where $P_\lambda(x)$ is the Legendre polynomial defined in [6] and the conditions of the validity are the same as stated in Theorem Eq. (2.8) *with $\lambda = 0$ and v replaced by λ.*

2.3 Generalized Laguerre Polynomial and Incomplete I-functions

Here, we shall derive an integral formula including the incomplete I-function and generalized Laguerre polynomial defined in [6].

Theorem 2.10. *The following formula holds true;*

$$
\int_0^\infty x^\gamma e^{-x}\, L_k^\sigma(x) \times {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[z\,x^\eta\,\middle|\,\begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),\\ (\mathfrak{b}_1,\omega_1;F_1),\end{matrix}\;\begin{matrix}(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\ (\mathfrak{b}_j,\omega_j;F_j)_{2,q}\end{matrix}\right]dx
$$

$$
= \frac{(-1)^k\,(2\pi)^{\frac{1}{2}(1-\eta)}\,\eta^{\gamma+k+\frac{1}{2}}}{k!}
$$

$$
\times\,{}^{\Gamma}I^{m,\,n+2\eta}_{p+2\eta,\,q+\eta}\left[z\,\eta^\eta\,\middle|\,\begin{matrix}(\Delta(\eta,-\gamma),1;1),(\Delta(\eta,\sigma-\gamma),1;1),\\ (\mathfrak{b}_j,\omega_j;F_j)_{1,q},\end{matrix}\;\begin{matrix}(\mathfrak{a}_1,\nu_1;E_1;x),(\mathfrak{a}_j,\nu_j;E_j)_{2,p}\\ (\Delta(\eta,\sigma-\gamma),1;1)\end{matrix}\right], \tag{2.24}
$$

where, η is a positive integer, $\sum_{j=1}^{p}\nu_j - \sum_{j=1}^{q}\omega_j = \rho \le 0$,
$\sum_{i=1}^{m}\omega_i - \sum_{i=m+1}^{q}\omega_i + \sum_{i=1}^{n}\nu_i - \sum_{i=n+1}^{p}\nu_i = \Delta > 0$,
$|arg\ z| < \frac{1}{2}\Delta\,\pi$, *and* $\Re\left[\gamma+1+\eta\left(\frac{\mathfrak{b}_h}{\omega_h}\right)\right] > -1 \quad (h = 1,2,\cdots,m)$.

Proof. To prove Eq. (2.24), we express the incomplete I-function in the form of the Mellin-Barness type of contour integral Eq. (1.12) and change the order of the integrations, we have

$$
\frac{1}{2\pi\,i}\int_{\mathcal{L}} z^s\,\Phi(s,x)\times \left\{\int_0^\infty x^{\gamma+\eta s}\,e^{-x}\,L_k^\sigma(x)\,dx\right\}ds, \tag{2.25}
$$

Now evaluating x-integral with the help of the result [18, p.292,(1)] :

$$
\int_0^\infty x^{\beta-1}\,e^{-x}\,L_n^\alpha(x)dx = \frac{\Gamma(\alpha-\beta+n+1)\Gamma(\beta)}{n!\,\Gamma(\alpha-\beta+1)}, \qquad (\Re(\beta) > 0),
$$

and using the following relations

$$
(\mathfrak{a})_n = \frac{\Gamma(\mathfrak{a}+n)}{\Gamma(\mathfrak{a})}, \quad \frac{\Gamma(1-\mathfrak{a}-n)}{\Gamma(1-\mathfrak{a})} = \frac{(-1)^n}{(\mathfrak{a})_n},
$$

and the Gauss's multiplication theorem for Gamma function [20, p. 26]:

$$
\Gamma(kz) = (2\pi)^{\frac{1}{2}(1-k)}\,K^{kz-\frac{1}{2}}\prod_{s=1}^{k}\Gamma\left(z+\frac{s-1}{K}\right),
$$

Eq. (2.25) reduces to

$$
\frac{(-1)^k}{k!}\,(2\pi)^{\frac{1}{2}(1-\eta)}\,\eta^{\gamma+k+\frac{1}{2}} \times\frac{1}{2\pi i}\int_{\mathcal{L}} z^s\,\eta^{\eta s}\,\Phi(s,x)\times \frac{\prod_{i=0}^{\eta-1}\Gamma\left(\frac{1+\gamma+i}{\eta}-s\right)\prod_{i=0}^{\eta-1}\Gamma\left(\frac{1-\sigma+\gamma+i}{\eta}-s\right)}{\prod_{i=0}^{\eta-1}\Gamma\left(\frac{1-\sigma+\gamma-k+i}{\eta}-s\right)}ds. \tag{2.26}
$$

Therefore, in accordance with the definition Eq. (1.14) of the incomplete I-function, Eq. (2.26) yields the value of the integral Eq. (2.24). □

A family of integrals involving incomplete I-functions has been developed. The exponential function, Legendre polynomials, and modified Laguerre polynomials were used to analyse several integral formulas of incomplete I-functions. Because incomplete I-functions may be reduced to recognisable special functions (such as I-functions, incomplete H-functions, and Fox's H-functions), numerous special cases can be assessed based on our significant inventions by assigning appropriate values to the related parameters.

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