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Original Article

Characterizations of completely ordered *k*-regular semirings

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Abstract

We introduce the notion of a completely ordered *k*-regular semiring as a generalization of a completely regular ordered semiring and characterize it using its ordered *k*-ideals. Moreover, we show that an ordered semiring S is completely ordered *k*-regular if and only if every ordered *k*-bi-ideal of S is semiprime.

Keywords: ordered semiring, ordered *k*-regular semiring, completely regular semiring, completely ordered *k*-regular semiring

1. Introduction

Bourne (1951) defined a semiring $(S,+,\cdot)$ to be regular if for every $a \in S$ there are $x, y \in S$ such that $a + axa = aya$. Later, Adhikari, Sen and Weinert (1996) renamed the Bourne regularity to be *k*-regular and studied some of its properties. Then many authors; for example, Bhuniya (2011) and Jana (2011) investigated and gave some characterizations of *k*-regular semirings by their *k*-ideals.

An ordered semiring is a semiring together with a partially ordered relation. It was introduced by Gan and Jiang (2011) as an algebraic structure which is a generalization of a semiring. Then Mandal (2014) introduced the notions of regular and *k*-regular ordered semirings. In 2017, Patchakhieo and Pibaljommee (2017) defined the notion of an ordered *k*-regular semiring as a generalization of *k*-regular ordered semiring defined by Mandal and introduced the notions of left and right ordered *k*-regular semirings.

The concept of completely regular on an ordered algebraic structure was introduced by Kehayopulu (1998) on an ordered semigroup. Kehayopulu called an ordered semigroup S to be completely regular if S is regular, left regular and right regular.

In this paper, we introduce the notion of a completely ordered *k*-regular semiring as an ordered semiring S such that S is ordered *k*-regular, left ordered *k*-regular and right ordered *k*-regular. Then we study some properties of completely ordered *k*-regular semirings and give some of their characterizations by their ordered *k*-ideals.

2. Preliminaries

An *ordered semirings* $(S,+,\cdot,\leq)$ is a semiring $(S,+,\cdot)$ together with a binary relation $\leq$ on S such that the relation $\leq$ is compatible with the operations $+$ and $\cdot$ of S. We simply write S for an ordered semiring $(S,+,\cdot,\leq)$ and ab instead of $a \cdot b$ for all $a, b \in S$. An ordered semiring S is said to be *additively commutative* if $a + b = b + a$ for any $a, b \in S$. Throughout this paper, we assume that S is additively commutative.

For any nonempty subsets A, B of S, we denote

$$AB = \{ab \in S \mid a \in A, b \in B\},$$
$$A + B = \{a + b \in S \mid a \in A, b \in B\},$$
$$\Sigma A = \{\textstyle\sum_{i=1}^{n} a_i \in S \mid a_i \in A, n \in \mathbb{N}\},$$
$$\Sigma AB = \{\textstyle\sum_{i=1}^{n} a_i b_i \in S \mid a_i \in A, b_i \in B, n \in \mathbb{N}\}$$

and

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

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In a special case, if $A = \{a\}$ for some $a \in S$, then we write Σa instead of $\Sigma\{a\}$. Clearly, $A \subseteq \Sigma A$ and $\Sigma A = A$ if and only if $A + A \subseteq A$.

Remark 1. Let A, B be nonempty subsets of S. Then the following statements hold:

(i) $A \subseteq \Sigma A$ and $\Sigma(\Sigma A) = \Sigma A$;
(ii) if $A \subseteq B$ then $\Sigma A \subseteq \Sigma B$;
(iii) $A(\Sigma B) \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$ and $(\Sigma A)B \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$;
(iv) $\Sigma(A + B) \subseteq \Sigma A + \Sigma B$;
(v) $\Sigma(A] \subseteq (\Sigma A]$;
(vi) $A \subseteq (A]$ and $((A]] = (A]$;
(vii) if $A \subseteq B$ then $(A] \subseteq (B]$;
(viii) $A(B] \subseteq (A](B] \subseteq (AB]$ and $(A]B \subseteq (A](B] \subseteq (AB]$;
(ix) $A + (B] \subseteq (A] + (B] \subseteq (A + B]$;
(x) $(A \cup B] = (A] \cup (B]$;
(xi) $(A \cap B] \subseteq (A] \cap (B]$.

The *k-closure* (Patchakhieo & Pibaljommee, 2017) of $\emptyset \neq A \subseteq S$ is defined by

$$\overline{A} = \{ x \in S \mid x + a \leq b \text{ for some } a, b, \in A \}.$$

Remark 2. Let A, B be nonempty subsets of S. Then the following statements hold:

(i) $\Sigma\overline{A} \subseteq \overline{\Sigma A}$;
(ii) if $A + A \subseteq A$ then $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{(A]} = \overline{\overline{(A]}}$;
(iii) if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$;
(iv) $A\overline{B} \subseteq \overline{AB}$ and $\overline{A}B \subseteq \overline{AB}$;
(v) $\overline{A} + \overline{B} \subseteq \overline{A + B}$;
(vi) $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$;
(vii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$;
(viii) if $A + A \subseteq A$ then $A \subseteq (A] \subseteq (\overline{A}] = \overline{A} \subseteq \overline{(A]}$.

We note that if $\emptyset \neq A \subseteq S$ is closed under addition then $(A]$, $\overline{A}$ and $\overline{(A]}$ are also closed. As a consequence of Remark 1 and Remark 2, we obtain the following remark.

Remark 3. Let A, B be nonempty subsets of S such that A and B are closed under addition. Then the following statements hold:

(i) $\Sigma\overline{(A]} = \overline{(\Sigma A]}$;
(ii) $\overline{(\overline{(A]}]} = \overline{(A]}$;
(iii) $A\overline{(B]} \subseteq \overline{(A]}\,\overline{(B]} \subseteq \overline{(\Sigma AB]}$ and $\overline{(A]}B \subseteq \overline{(A]}\,\overline{(B]} \subseteq \overline{(\Sigma AB]}$;
(iv) $A + \overline{(B]} \subseteq \overline{(A]} + \overline{(B]} \subseteq \overline{(A + B]}$.

A *left ordered k-ideal* (*right ordered k-ideal*) (Patchakhieo & Pibaljommee, 2017) of S is a subsemigroup $(A, +)$ of $(S, +)$ such that $SA \subseteq A$ ($AS \subseteq A$) and $\overline{A} = A$. An *ordered k-ideal* of S is both a left and a right ordered k-ideal of S. An *ordered quasi k-ideal* (Palakawong na Ayutthaya & Pibaljommee, 2017) of S is a subsemigroup $(Q, +)$ of $(S, +)$ such that $\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq Q$ and $Q = \overline{Q}$. It is easy to see that every ordered quasi k-ideal of S is a subsemiring of S, indeed; $Q^2 \subseteq SQ \cap QS \subseteq Q$. A subsemiring B of S is called an *ordered k-bi-ideal* (*ordered k-interior ideal*) of S if $BSB \subseteq B$ ($SBS \subseteq B$) and $B = \overline{B}$.

For $\emptyset \neq A \subseteq S$, we denote $L_k(A)$ (resp. $R_k(A)$, $J_k(A)$, $Q_k(A)$, $B_k(A)$) as the smallest left ordered k-ideal (resp. right ordered k-ideal, ordered k-ideal, ordered quasi k-ideal, ordered k-bi-ideal) of S containing A. Palakawong na Ayutthaya and Pibaljommee (2017) gave their constructions as the following lemma.

Lemma 2.1 Let $\emptyset \neq A \subseteq S$. Then the following statements hold:

(i) $L_k(A) = \overline{(\Sigma A + \Sigma SA]}$;
(ii) $R_k(A) = \overline{(\Sigma A + \Sigma AS]}$;
(iii) $J_k(A) = \overline{(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]}$;
(iv) $Q_k(A) = \overline{\left(\Sigma A + \overline{(\Sigma SA]} \cap \overline{(\Sigma AS]}\right]}$;
(v) $B_k(A) = \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]}$.

Mandal (2014) defined S to be *regular* if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$. We call S *left regular* (*right regular*) (Palakawong na Ayutthaya & Pibaljommee, 2016) if for every $a \in S$, there exists $x \in S$ such that $a \leq xa^2$ ($a \leq a^2x$).

If S is regular, left regular and right regular then we call S a completely regular ordered semiring. An ordered semiring S is said to be ordered k-regular (Patchakhieo & Pibaljommee, 2017) if $a \in \overline{(aSa]}$ for all $a \in S$ (equivalently, $A \subseteq \overline{(\Sigma ASA]}$ $\forall A \subseteq S$).

In general, every ordered quasi k-ideal of an ordered semiring S is an ordered k-bi-ideal of S but the converse is not true. Palakawong na Ayutthaya & Pibaljommee (2017) gave an example for this case and show that the converse can be true in an ordered k-regular semiring.

3. Ordered *k*-idempotent Semirings

Here, we introduce the notion of fully ordered k-idempotent semirings, give some their characterizations and show that their ordered k-ideals and their ordered k-interior ideals coincide.

Definition 3.1 An ordered k-ideal J of S is called *ordered k-idempotent* if $J = \overline{(\Sigma J^2]}$.

We call S an *fully ordered k-idempotent semiring* if every ordered k-ideal of S is ordered k-idempotent.

Example 3.2 Let $S = \{a, b, c, d, e\}$. Define a binary operation $+$ on S by $x + a = x = a + x$ for all $x \in S$ and $x + y = b$ for all $x, y \in S - \{a\}$. Define a binary operation $\cdot$ on S by $a \cdot a = a$, $x \cdot a = a = a \cdot x$, $d \cdot e = e \cdot d = e \cdot e = c$ and $b \cdot x = x \cdot b = c \cdot x = x \cdot c = d \cdot d = b$ for all $x \in S - \{a\}$. Define a binary relation $\leq$ on S by $\leq := \{(a, a), (b, b), (c, c), (d, d), (b, c)\}$. Then $(S, +, \cdot, \leq)$ forms an additively commutative ordered semiring. We have that S and $\{a\}$ are only two ordered k-ideals of S. It is not difficult to check that $S = \overline{(\Sigma S^2]}$ and $\{a\} = \overline{(\Sigma a^2]}$ and so we obtain that S is a fully ordered k-idempotent semiring.

Theorem 3.3 The following conditions are equivalent:

(i) S is fully ordered k-idempotent;
(ii) $A \subseteq \overline{(\Sigma SASAS]}$ for any $A \subseteq S$;
(iii) $a \in \overline{(\Sigma SaSaS]}$ for any $a \in S$.

Proof. (i)⇒(ii): Assume that S is fully ordered k-idempotent and let $A \subseteq S$. Then $A \subseteq J_k(A) = \overline{(\Sigma J_k(A)^2]}$
$= \overline{\left(\Sigma\overline{(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]} \cdot \overline{(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]}\right]}$
$\subseteq \overline{(\Sigma(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS)(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS)]}$
$\subseteq \overline{(\Sigma A^2 + \Sigma ASA + \Sigma A^2S + \Sigma ASAS + \Sigma SA^2 + \Sigma SASA + \Sigma SA^2S + \Sigma SASAS]}$
$\subseteq \overline{(\Sigma SASAS]}$.

(ii)⇒(iii): It is obvious.

(iii)⇒(i): Assume that (iii) holds and let J be an ordered k-ideal of S. Clearly, $\overline{(\Sigma J^2]} \subseteq \overline{(\Sigma J]} = J$. On the other hand, if $x \in J$ then $x \in \overline{(\Sigma SxSxS]} \subseteq \overline{(\Sigma SJSJS]} \subseteq \overline{(\Sigma J^2]}$. So, $J = \overline{(\Sigma J^2]}$. Therefore, S is fully ordered k-idempotent.

Now, we show that the concept of ordered k-ideals and the concept of ordered k-interior ideals coincide in fully ordered k-idempotent semirings.

Theorem 3.4 If S is fully ordered k-idempotent, then its ordered k-ideals coincide with its ordered k-interior ideals.

Proof. Assume that S is a fully ordered k-idempotent semiring. Since every ordered k-ideal is an ordered k-interior ideal, it is sufficient to show that every ordered k-interior ideal is an ordered k-ideal. Let I be an ordered k-interior ideal of S. Using Theorem 3.2, we obtain that $SI \subseteq \overline{(\Sigma SSISSIS]} \subseteq \overline{(\Sigma SIS]} \subseteq \overline{(\Sigma I]} = I$. Similarly, we can show that $IS \subseteq I$. Hence, I is an ordered k-ideal of S.

Theorem 3.5 An ordered semiring S is fully ordered k-idempotent if and only if $I_1 \cap I_2 = \overline{(\Sigma I_1 I_2]}$ for every two ordered k-interior ideals I_1, I_2 of S.

Proof. Let I_1, I_2 be ordered k-interior ideals of S. By Theorem 3.3, we have that I_1 and I_2 are ordered k-ideals of S. It follows that $\overline{(\Sigma I_1 I_2]} \subseteq \overline{(\Sigma I_1]} = I_1$ and $\overline{(\Sigma I_1 I_2]} \subseteq \overline{(\Sigma I_2]} = I_2$. Thus, $\overline{(\Sigma I_1 I_2]} \subseteq I_1 \cap I_2$. For the opposite inclusion, we have that $I_1 \cap I_2 = \overline{(\Sigma(I_1 \cap I_2)(I_1 \cap I_2)]} \subseteq \overline{(\Sigma I_1 I_2]}$. Hence, $I_1 \cap I_2 = \overline{(\Sigma I_1 I_2]}$.

Conversely, let J be an ordered k-ideal of S. Since J is an ordered k-interior ideal, $J = J \cap J = \overline{(\Sigma JJ]} = \overline{(\Sigma J^2]}$.

4. Left and Right Ordered k-regular Semirings

In this section, we recall the notions of left and right ordered k-regular semirings which defined by Patchakhieo and Pibaljommee (2017) and give some their characterizations.

Definition 4.1 An ordered semiring S is said to be *left ordered k-regular* (*right ordered k-regular*) if $a \in \overline{(Sa^2]}$ ($a \in \overline{(a^2S]}$) for all $a \in S$.

We can also obtain that an ordered semiring S is said to be *left ordered k-regular* (*right ordered k-regular*) if and only if $A \subseteq \overline{(\Sigma SA^2]}$ ($A \subseteq \overline{(\Sigma A^2S]}$) for all $\emptyset \neq A \subseteq S$.

Theorem 4.2 The following conditions are equivalent:

(i) S is left ordered k-regular;
(ii) every left ordered k-ideal of S is a left ordered k-regular subsemiring of S;
(iii) $L_k(A)$ is a left ordered k-regular subsemiring of S for any $A \subseteq S$;
(iv) $L_k(a)$ is a left ordered k-regular subsemiring of S for any $a \in S$.

Proof. (i)⇒(ii): Assume that S is left ordered k-regular and let L be any left ordered k-ideal of S. If $x \in L$ then we obtain

$$x \in \overline{(Sx^2]} \subseteq \overline{\left(S\overline{(Sx^2]}x\right]} \subseteq \overline{(Sx^3]} \subseteq \overline{(SLx^2]} \subseteq \overline{(Lx^2]}.$$

This shows that L is left ordered k-regular.

(ii)⇒(iii) and (iii)⇒(iv) are obvious.

(iv)⇒(i): Assume that (iv) holds and let $a \in S$. It turns out $a \in \overline{(L_k(a)a^2]} \subseteq \overline{(Sa^2]}$. Therefore, S is left ordered k-regular.

As a duality of Theorem 4.2, we obtain the following theorem.

Theorem 4.3 The following conditions are equivalent:

(i) S is right ordered k-regular;
(ii) every right ordered k-ideal of S is a right ordered k-regular subsemiring of S;
(iii) $R_k(A)$ is a right ordered k-regular subsemiring of S for any $A \subseteq S$;
(iv) $R_k(a)$ is a right ordered k-regular subsemiring of S for any $a \in S$.

Definition 4.4 Let $\emptyset \neq T \subseteq S$. Then T is said to be *semiprime* if for any $a \in S$, $a^2 \in T$ implies $a \in T$.

We note that $\emptyset \neq T \subseteq S$ is semiprime if and only if for any $\emptyset \neq A \subseteq S$, $A^2 \subseteq T$ implies $A \subseteq T$.

Now, we give a characterization of an ordered semiring which is both left and right ordered k-regular by its ordered quasi k-ideals as follows.

Theorem 4.5 An ordered semiring S is both left and right ordered k-regular if and only if every ordered quasi k-ideal of S is semiprime.

Proof. Assume that S is left and right ordered k-regular. Let Q be an ordered quasi k-ideal of S and let A be a nonempty subset of S such that $A^2 \subseteq Q$. Then, we have that $A \subseteq \overline{(\Sigma SA^2]}$ and $A \subseteq \overline{(\Sigma A^2S]}$. It follows that

$$A \subseteq \overline{(\Sigma SA^2]} \cap \overline{(\Sigma A^2S]} \subseteq \overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq Q.$$

Therefore, Q is semiprime.

Conversely, assume that every ordered quasi *k*-ideal of S is semiprime. Let $A \subseteq S$. By Lemma 2.1, we have that $Q_k(A^2) = \overline{\left(\Sigma A^2 + \overline{(\Sigma SA^2]} \cap \overline{(\Sigma A^2 S]}\right]}$. Since $A^2 \subseteq Q(A^2)$ is semiprime, $A \subseteq Q(A^2)$. Then, we obtain that

$$A \subseteq Q_k(A^2) = \overline{\left(\Sigma A^2 + \overline{(\Sigma SA^2]} \cap \overline{(\Sigma A^2 S]}\right]} \subseteq \overline{\left(\Sigma A^2 + \overline{(\Sigma SA^2]}\right]}$$
$$\subseteq \overline{\left(\overline{(\Sigma A^2 + \Sigma SA^2]}\right]} = \overline{(\Sigma A^2 + \Sigma SA^2]} \text{ and}$$
$$A \subseteq Q_k(A^2) = \overline{\left(\Sigma A^2 + \overline{(\Sigma SA^2]} \cap \overline{(\Sigma A^2 S]}\right]} \subseteq \overline{\left(\Sigma A^2 + \overline{(\Sigma A^2 S]}\right]}$$
$$\subseteq \overline{\left(\overline{(\Sigma A^2 + \Sigma A^2 S]}\right]} = \overline{(\Sigma A^2 + \Sigma A^2 S]}.$$

In case of $A \subseteq \overline{(\Sigma A^2 + \Sigma SA^2]}$, we obtain that

$$\Sigma A^2 = \Sigma AA \subseteq \Sigma A\overline{(\Sigma A^2 + \Sigma SA^2]} \subseteq \Sigma\overline{(\Sigma A^3 + \Sigma ASA^2]}$$
$$\subseteq \Sigma\overline{(\Sigma A^3 + \Sigma SA^2]} \subseteq \overline{(\Sigma SA^2 + \Sigma SA^2]} = \overline{(\Sigma SA^2]}.$$

Thus, $A \subseteq \overline{(\Sigma A^2 + \Sigma SA^2]} \subseteq \overline{\left(\overline{(\Sigma SA^2]} + \Sigma SA^2\right]} \subseteq \overline{\left(\overline{(\Sigma SA^2 + \Sigma SA^2]}\right]} = \overline{(\Sigma SA^2]}$. In case of $A \subseteq \overline{(\Sigma A^2 + \Sigma A^2 S]}$, we can prove similarly and so we get that $A \subseteq \overline{(\Sigma A^2 S]}$. Therefore, S is left and right ordered *k*-regular.

5. Completely Ordered *k*-regular Semirings

In this section, we introduce the notion of completely ordered *k*-regular semirings, study some of their properties and give some of their characterizations.

Definition 5.1 An ordered semiring S is called *completely ordered k-regular* if S is ordered *k*-regular, left ordered *k*-regular and right ordered *k*-regular.

Example 5.2 Consider the set of all natural numbers $\mathbb{N}$ together with the operations max and min and the natural ordered relation $\leq$. It is easy to see that $(\mathbb{N}, \max, \min, \leq)$ forms a completely ordered *k*-regular semiring.

We know that every regular ordered semiring is an ordered *k*-regular semiring, every left regular ordered semiring is a left ordered *k*-regular semiring and every right regular ordered semiring is a right ordered *k*-regular semiring. These mean that every completely regular ordered semiring is a completely ordered *k*-regular semiring.

We now show that there is a completely ordered *k*-regular semiring which is not completely regular as the following example.

Example 5.3 Let $S = \{a, b, c, d\}$. Define a binary operation $+$ by $a + x = x = x + a$, $b + x = b = x + b$ for all $x \in S$, $c + c = c$ and $c + d = d = d + d$. Define a binary operation $\cdot$ by $x \cdot a = x \cdot c = a$, $x \cdot d = d$ for all $x \in S$, $y \cdot b = d$ for all $y \in S\backslash\{b\}$ and $b \cdot b = b$. Define a binary relation $\leq$ on S by $\leq := \{(a,a), (b,b), (c,c), (d,d), (a,d), (b,d), (c,d)\}$ Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring. Obviously, a, b and d are regular, left regular and right regular. However, we have that $c \notin (cSc] = \{a\}$ and so c is not regular. It leads that S is not regular and thus S is not completely regular.

Since a, b, d are regular, right regular and left regular, we can obtain that a, b, d are ordered *k*-regular, left ordered *k*-regular and right ordered *k*-regular. Moreover, we have that $c \in \overline{(cSc]} = \{a, c\}$, $c \in \overline{(c^2 S]} = S$ and $c \in \overline{(Sc^2]} = \{a, c\}$. This shows that S is completely ordered *k*-regular.

In consequence of Example 5.3, we can conclude that the concept of a completely ordered *k*-regular semiring is a generalization of the concept of a completely regular ordered semiring.

Here, we give some characterizations of completely ordered *k*-regular semirings.

Lemma 5.4 The following conditions are equivalent:

(i) S is completely ordered *k*-regular;
(ii) $A \subseteq \overline{(\Sigma A^2 SA^2]}$ for every $A \subseteq S$;
(iii) $a \in \overline{(a^2 Sa^2]}$ for every $a \in S$.

Proof. (i)⇒(ii): Assume that S is completely ordered *k*-regular and let $A \subseteq S$. We obtain that $A \subseteq \overline{(\Sigma ASA]}$, $A \subseteq \overline{(\Sigma A^2 S]}$ and $A \subseteq \overline{(\Sigma SA^2]}$. It follows that

$$A \subseteq \overline{(\Sigma ASA]} \subseteq \overline{\left(\Sigma\overline{(\Sigma A^2 S]}\, S\overline{(\Sigma SA^2]}\right]}$$
$$\subseteq \overline{\left(\overline{(\Sigma A^2 SA^2]}\right]} = \overline{(\Sigma A^2 SA^2]}.$$

(ii)⇒(iii) and (iii)⇒(i) are obvious.

Theorem 5.5 An ordered semiring S is completely ordered *k*-regular if and only if every ordered quasi *k*-ideal of S is a completely ordered *k*-regular subsemiring of S.

Proof. Assume that S is completely ordered *k*-regular and let Q be an ordered quasi k-ideal of S. Let $A \subseteq Q$. Using Lemma 5.4, we obtain that

$$A \subseteq \overline{(\Sigma A^2 SA^2]} \subseteq \overline{\left(\Sigma A\overline{(\Sigma A^2 S]}S\overline{(\Sigma SA^2]}A\right]}$$
$$\subseteq \overline{\left(\overline{(\Sigma A^3 S]}S\overline{(\Sigma SA^3]}\right]} \subseteq \overline{\left(\overline{(\Sigma A^3 SA^3]}\right]}$$
$$= \overline{(\Sigma A^3 SA^3]} \subseteq \overline{(\Sigma A^2 QSQA^2]} \subseteq \overline{(\Sigma A^2 QA^2]}.$$

Using Lemma 5.4 again, we obtain that Q is completely ordered *k*-regular. The converse is clear, since S itself is an ordered quasi *k*-ideal.

Using the fact that ordered quasi *k*-ideals and ordered *k*-bi-ideals of ordered *k*-regular semirings coincide and Theorem 5.5, we can obtain the following corollary.

Corollary 5.6 An ordered semiring S is completely ordered *k*-regular if and only if every ordered *k*-bi-ideal of S is a completely ordered *k*-regular subsemiring of S.

Theorem 5.7 An ordered semiring S is completely ordered *k*-regular if and only if every ordered *k*-bi-ideal of S is semiprime.

Proof. Since S is completely ordered *k*-regular, S is ordered *k*-regular. So, every ordered quasi *k*-ideal and every ordered *k*-bi-ideal of S coincide (Palakawong na Ayutthaya & Pibaljommee, 2017). Using Theorem 4.5, we obtain that every ordered *k*-bi-ideal of S is also semiprime.

Conversely, let $\emptyset \neq A \subseteq S$. We claim that $\overline{(\Sigma A^2SA^2]}$ is an ordered *k*-bi-ideal of *S*. Since ΣA^2SA^2 is closed under addition, $\overline{(\Sigma A^2SA^2]}$ is also closed. Furthermore, we get that

$$\overline{(\Sigma A^2SA^2]} \cdot \overline{(\Sigma A^2SA^2]} \subseteq \overline{(\Sigma(\Sigma A^2SA^2)(\Sigma A^2SA^2)]} \subseteq \overline{(\Sigma A^2SA^4SA^2)]} \subseteq \overline{(\Sigma A^2SA^2]}.$$

Now, $\overline{(\Sigma A^2SA^2]}$ is a subsemiring of S. We consider

$$\overline{(\Sigma A^2SA^2]}S\overline{(\Sigma A^2SA^2]} \subseteq \overline{(\Sigma A^2S]}S\overline{(\Sigma SA^2]} \subseteq \overline{(\Sigma A^2SS]} \cdot \overline{(\Sigma SA^2]} \subseteq \overline{(\Sigma A^2S]} \cdot \overline{(\Sigma SA^2]} \subseteq \overline{(\Sigma(\Sigma A^2S)(\Sigma SA^2)]} \subseteq \overline{(\Sigma A^2SA^2)]}$$

and $\overline{\overline{(\Sigma A^2SA^2]}} = \overline{(\Sigma A^2SA^2]}$. Hence, $\overline{(\Sigma A^2SA^2]}$ is an ordered *k*-bi-ideal of S. We have that $A^8 = A^2A^4A^2 \subseteq A^2SA^2 \subseteq \Sigma A^2SA^2 \subseteq \overline{(\Sigma A^2SA^2]}$. By assumption, $\overline{(\Sigma A^2SA^2]}$ is semiprime. This implies $A^4 \subseteq \overline{(\Sigma A^2SA^2]}$ and $A^2 \subseteq \overline{(\Sigma A^2SA^2]}$ and hence $A \subseteq \overline{(\Sigma A^2SA^2]}$. Therefore, S is completely ordered *k*-regular by Lemma 5.4.

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