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Original Article

Fuzzy dynamic programming problem for single additive constraint with multiplicatively separable return in terms of trapezoidal membership functions

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Abstract

Dynamic programming problems (DP) are multivariable optimization problems that can be decomposed into a series of stages, and optimization is done at each stage with respect to one variable only. DP allows a suitable quantitative study procedure that can be used to assess various optimization problems. The technique offers an efficient procedure for finding optimal decisions. Here, we address a Fuzzy Dynamic Programming Problem with a single additive constraint and multiplicatively separable return, with the support of trapezoidal membership functions and related arithmetic operations. The procedure has been adapted from Fuzzy Dynamic Programming Problem (FDPP). The fuzzified version of the problem is stated and illustrated with a numerical example, and it is shown that the proposed procedure is more efficient in handling the dynamic programming problem than alternative classical procedures. As a final point, the optimal solution is provided in the form of fuzzy numbers with trapezoidal fuzzy membership functions, and also the solution is compared with existing methodology in a numerical example.

Keywords: trapezoidal fuzzy numbers, fuzzy dynamic programming problem, fuzzy additive constraint, fuzzy multiplicatively separable return

1. Introduction

Dynamic programming is different from linear programming on two counts. First, there is no typical mathematical formulation of DPP. Accordingly, there is no procedure, similar to the simplex algorithm, that can be preset to solve all this type of problems. DP is, rather, a procedure that permits us to divide a large problem into a sequence of subproblems, which are then solved stage by stage. This is a general strategy to approach suitable problems, not an algorithm. Moreover, while linear programming is a method operating in a single stage, DP has the power to determine the optimal solutions over, say, a one year time horizon by breaking the problem into twelve smaller one month time horizon problems and solving each of these optimally. Thus, DP uses a multistage approach. There is a wide variety of problems that can be handled using dynamic programming. Dynamic programming problem was first developed in 1950, through efforts of Richard Bellman, and his principle of optimality states that an optimal policy has the property that whatever the initial stages and decisions have been, the remaining decisions must constitute an optimal policy with regards to the state resulting from the preceding decisions. In a dynamic programming problem, no generic mathematical formulation is available. Instead particular equations must be developed for each distinct situation, see Swarup *et al.* (2004). DPP can be classified into different types, such as single

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additive constraint with an additively separable return; or multiplicatively separable return and single multiplicative constraint with additively separable return, Kaliyaperumal (2017). Fuzzy dynamic problems have been dealt with by many researchers (Baldwin, 1982; Esogbue, 1983; Sadina, 2012; Schweickardt, 2007; Xiong, 2005), recently with crisp state transformation function in terms of fuzzified dynamic programming, Zimmerman (1991). Bellman and Zadeh (1970) were the first to suggest a fuzzy approach to the dynamic programming problem. Here we also make an effort to fuzzify a dynamic programming problem and to solve it with a fuzzy optimal solution for single additive constraint with multiplicatively separable return, with the support of trapezoidal membership functions and related arithmetic operations, see Palanivel (2016). The fuzzified version of the problem is stated and illustrated with a numerical example.

2. The Fuzzified Form of Dynamic Programming Problem

2.1. The single additive constraint with a multiplicatively separable return

Fuzzy dynamic programming problem is defined as the problem of determining a fuzzy vector

$\left(\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right],\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right],\dots,\left[u_n^{(1)},u_n^{(2)},u_n^{(3)},u_n^{(4)}\right]\right)$ where

$\left[u_j^{(1)},u_j^{(2)},u_j^{(3)},u_j^{(4)}\right]$ is a trapezoidal fuzzy number for $j=1,2,3,\dots,n$, and we see to optimize by these decisions the fuzzy objective function $\left[z^{(1)},z^{(2)},z^{(3)},z^{(4)}\right]$ that is a separable fuzzy multiplicative function of the 'n' fuzzy variables $\left[u_j^{(1)},u_j^{(2)},u_j^{(3)},u_j^{(4)}\right]$ and is given by $\left[z^{(1)},z^{(2)},z^{(3)},z^{(4)}\right]=\prod_{j=1}^{n}\left(\left[f_j\left(u_j\right)^{(1)},f_j\left(u_j\right)^{(2)},f_j\left(u_j\right)^{(3)},f_j\left(u_j\right)^{(4)}\right]\right)$,

Subject to the fuzzy constraints,

$\sum_{j=1}^{n}\left(\left[a_j\left(u_j\right)^{(1)},a_j\left(u_j\right)^{(2)},a_j\left(u_j\right)^{(3)},a_j\left(u_j\right)^{(4)}\right]\right)\geq\left[b^{(1)},b^{(2)},b^{(3)},b^{(4)}\right]$, Where

$\left[a_j^{(1)},a_j^{(2)},a_j^{(3)},a_j^{(4)}\right]\geq 0,\left[b^{(1)},b^{(2)},b^{(3)},b^{(4)}\right]\geq 0,\left[u_j^{(1)},u_j^{(2)},u_j^{(3)},u_j^{(4)}\right]\geq 0; j=1,2,3,\dots,n$

The problem is an n-stage problem, the suffix j indicating the stage. Now, we need to maximize the fuzzy objective function under the given fuzzy additive constraints. Here each $\left[u_j^{(1)},u_j^{(2)},u_j^{(3)},u_j^{(4)}\right]$ will be called a fuzzy decision variable; and the fuzzy return function $\left[f_j\left(u_j\right)^{(1)},f_j\left(u_j\right)^{(2)},f_j\left(u_j\right)^{(3)},f_j\left(u_j\right)^{(4)}\right]$ is the return at the j^{th} stage. This describes the case of a single additive constraint with multiplicatively separable return.

2.2. Computational procedure

Let us introduce the fuzzy state variables

$\left(\left[x_1^{(1)},x_1^{(2)},x_1^{(3)},x_1^{(4)}\right],\left[x_2^{(1)},x_2^{(2)},x_2^{(3)},x_2^{(4)}\right],\dots,\left[x_n^{(1)},x_n^{(2)},x_n^{(3)},x_n^{(4)}\right]\right)$ defined as follows:

$$\left[x_n^{(1)},x_n^{(2)},x_n^{(3)},x_n^{(4)}\right]=\sum_{j=1}^{n}\left(\left[a_j\left(u_j\right)^{(1)},a_j\left(u_j\right)^{(2)},a_j\left(u_j\right)^{(3)},a_j\left(u_j\right)^{(4)}\right]\right)=\left[b^{(1)},b^{(2)},b^{(3)},b^{(4)}\right]$$

$$\left[x_{n-1}^{(1)},x_{n-1}^{(2)},x_{n-1}^{(3)},x_{n-1}^{(4)}\right]=\sum_{j=1}^{n-1}\left[a_j\left(u_j\right)^{(1)},a_j\left(u_j\right)^{(2)},a_j\left(u_j\right)^{(3)},a_j\left(u_j\right)^{(4)}\right]$$

$$=\left[x_n^{(1)},x_n^{(2)},x_n^{(3)},x_n^{(4)}\right]-\left[a_n\left(u_n\right)^{(1)},a_n\left(u_n\right)^{(2)},a_n\left(u_n\right)^{(3)},a_n\left(u_n\right)^{(4)}\right]$$

$$\left[x_{n-2}^{(1)}, x_{n-2}^{(2)}, x_{n-2}^{(3)}, x_{n-2}^{(4)}\right] = \sum_{j=1}^{n-2}\left(\left[a_j\,(u_j)^{(1)}, a_j\,(u_j)^{(2)}, a_j\,(u_j)^{(3)}, a_j\,(u_j)^{(4)}\right]\right)$$

$$= \left[x_{n-1}^{(1)}, x_{n-1}^{(2)}, x_{n-1}^{(3)}, x_{n-1}^{(4)}\right] - \left[a_{n-1}\,(u_{n-1})^{(1)}, a_{n-1}\,(u_{n-1})^{(2)}, a_{n-1}\,(u_{n-1})^{(3)}, a_{n-1}\,(u_{n-1})^{(4)}\right]$$

$$\left[x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, x_1^{(4)}\right] = \left[a_1(u_1)^{(1)}, a_1(u_1)^{(2)}, a_1(u_1)^{(3)}, a_1(u_1)^{(4)}\right]$$

$$= \left[x_2^{(1)}, x_2^{(2)}, x_2^{(3)}, x_2^{(4)}\right] - \left[a_2\,(u_2)^{(1)}, a_2\,(u_2)^{(2)}, a_2\,(u_2)^{(3)}, a_2\,(u_2)^{(4)}\right]$$

Here each of the fuzzy state variables is a function of the next state and the fuzzy decision variables.

$$\left[x_{j-1}^{(1)}, x_{j-1}^{(2)}, x_{j-1}^{(3)}, x_{j-1}^{(4)}\right] = \left[t_j\,(x_j,u_j)^{(1)}, t_j\,(x_j,u_j)^{(2)}, t(x_j,u_j)^{(3)}, t_j\,(x_j,u_j)^{(4)}\right] \forall j = 1,2,3,\ldots,n$$

that is, each state variable is a function of the next state and decision variables. This is the state transformation function. Since $\left[x_n^{(1)}, x_n^{(2)}, x_n^{(3)}, x_n^{(4)}\right]$ is a function of all the fuzzy decision variables, the maximum fuzzified value of the objective function is denoted by

$$\left[F_n\,(x_n)^{(1)}, F_n\,(x_n)^{(2)}, F_n\,(x_n)^{(3)}, F_n\,(x_n)^{(4)}\right] = \underset{\left[u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, u_n^{(4)}\right]}{max}\left[\prod_{j=1}^{n}\left(\left[f_j\,(u_j)^{(1)}, f_j\,(u_j)^{(2)}, f_j\,(u_j)^{(3)}, f_j\,(u_j)^{(4)}\right]\right)\right]$$

where $\left[u_j^{(1)}, u_j^{(2)}, u_j^{(3)}, u_j^{(4)}\right] \geq 0 \ and \left[x_n^{(1)}, x_n^{(2)}, x_n^{(3)}, x_n^{(4)}\right] \geq \left[b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}\right]$

A particular fuzzy value of $A = \left[u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, u_n^{(4)}\right]$ and its fuzzy membership function is given by

$$\mu_A(x) = \begin{cases} \dfrac{x - u_n^{(1)}}{u_n^{(2)} - u_n^{(1)}} & for \quad u_n^{(1)} \leq x \leq u_n^{(2)} \\ 1 & for \quad u_n^{(2)} \leq x \leq u_n^{(3)} \\ \dfrac{x - u_n^{(4)}}{u_n^{(3)} - u_n^{(4)}} & for \quad u_n^{(3)} \leq x \leq u_n^{(4)} \\ 0, otherwise & \end{cases}$$

$\mu_A(x)$ is selected and assumed to be fixed, and we maximize $\left[z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}\right]$ over the remaining fuzzy variables.

This maximum is given by

$$\left[f_n\,(u_n)^{(1)}, f_n\,(u_n)^{(2)}, f_n\,(u_n)^{(3)}, f_n\,(u_n)^{(4)}\right](*)$$

$$\underset{\left[u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, u_n^{(4)}\right]}{max}\left[\prod_{j=1}^{n}\left(\left[f_j\,(u_j)^{(1)}, f_j\,(u_j)^{(2)}, f_j\,(u_j)^{(3)}, f_j\,(u_j)^{(4)}\right]\right)\right] =$$

$$\left[f_n\,(u_n)^{(1)}, f_n\,(u_n)^{(2)}, f_n\,(u_n)^{(3)}, f_n\,(u_n)^{(4)}\right](*)$$

$$\left[F_{n-1}\,(x_{n-1})^{(1)}, F_{n-1}\,(x_{n-1})^{(2)}, F_{n-1}\,(x_{n-1})^{(3)}, F_{n-1}\,(x_{n-1})^{(4)}\right]$$

The corresponding fuzzy membership function is given by

$$\mu_A(x) = \begin{cases} \dfrac{x - f_n(u_n)^{(1)} F_{n-1}(x_{n-1})^{(1)}}{f_n(u_n)^{(2)} F_{n-1}(x_{n-1})^{(2)} - f_n(u_n)^{(1)} F_{n-1}(x_{n-1})^{(1)}} \\ \quad for\ f_n(u_n)^{(1)} F_{n-1}(x_{n-1})^{(1)} \le x \le f_n(u_n)^{(2)} F_{n-1}(x_{n-1})^{(2)} \\ 1 \quad for\ f_n(u_n)^{(2)} F_{n-1}(x_{n-1})^{(2)} \le x \le f_n(u_n)^{(3)} F_{n-1}(x_{n-1})^{(3)} \\ \dfrac{x - f_n(u_n)^{(4)} F_{n-1}(x_{n-1})^{(4)}}{f_n(u_n)^{(3)} F_{n-1}(x_{n-1})^{(3)} - f_n(u_n)^{(4)} F_{n-1}(x_{n-1})^{(4)}} \\ \quad for\ f_n(u_n)^{(3)} F_{n-1}(x_{n-1})^{(3)} \le x \le f_n(u_n)^{(4)} F_{n-1}(x_{n-1})^{(4)} \\ 0, otherwise \end{cases}$$

The fuzzified values of $\left[u_j^{(1)}, u_j^{(2)}, u_j^{(3)}, u_j^{(4)}\right] \forall j = 1,2,3,\ldots,n-1$ should make

$\prod_{j=1}^{n}\left(\left[f_j(u_j)^{(1)}, f_j(u_j)^{(2)}, f_j(u_j)^{(3)}, f_j(u_j)^{(4)}\right]\right)$ maximum for a fixed value $\left[u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, u_n^{(4)}\right]$

This depends on $\left[x_{n-1}^{(1)}, x_{n-1}^{(2)}, x_{n-1}^{(3)}, x_{n-1}^{(4)}\right]$, which is a function of $\left[x_n^{(1)}, x_n^{(2)}, x_n^{(3)}, x_n^{(4)}\right]$ and $\left[u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, u_n^{(4)}\right]$.

The maximum $\left[z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}\right]$ over all $\left[u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, u_n^{(4)}\right]$ for any feasible $\left[x_n^{(1)}, x_n^{(2)}, x_n^{(3)}, x_n^{(4)}\right]$ is

$$\left[F_n(x_n)^{(1)}, F_n(x_n)^{(2)}, F_n(x_n)^{(3)}, F_n(x_n)^{(4)}\right] = \max_{\left[u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, u_n^{(4)}\right]} \left[\left[f_n(u_n)^{(1)}, f_n(u_n)^{(2)}, f_n(u_n)^{(3)}, f_n(u_n)^{(4)}\right](*) \left[F_{n-1}(x_{n-1})^{(1)}, F_{n-1}(x_{n-1})^{(2)}, F_{n-1}(x_{n-1})^{(3)}, F_{n-1}(x_{n-1})^{(4)}\right]\right].$$

The recursion formula which defines a typical dynamic programming problem in fuzzified form is

$$\left[F_j(x_j)^{(1)}, F_j(x_j)^{(2)}, F_j(x_j)^{(3)}, F_j(x_j)^{(4)}\right] = \max_{\left[u_j^{(1)}, u_j^{(2)}, u_j^{(3)}, u_j^{(4)}\right]} \left[\left[f_j(u_j)^{(1)}, f_j(u_j)^{(2)}, f_j(u_j)^{(3)}, f_j(u_j)^{(4)}\right](*) \left[F_{j-1}(x_{j-1})^{(1)}, F_{j-1}(x_{j-1})^{(2)}, F_{j-1}(x_{j-1})^{(3)}, F_{j-1}(x_{j-1})^{(4)}\right]\right]$$ where $j = 2,3,\ldots,n$ and

$$\left[F_1(x_1)^{(1)}, F_1(x_1)^{(2)}, F_1(x_1)^{(3)}, F_1(x_1)^{(4)}\right] = \left[f_1(u_1)^{(1)}, f_1(u_1)^{(2)}, f_1(u_1)^{(3)}, f_1(u_1)^{(4)}\right].$$

Thus we start with $\left[F_1(x_1)^{(1)}, F_1(x_1)^{(2)}, F_1(x_1)^{(3)}, F_1(x_1)^{(4)}\right]$ and

recursively optimize each time over a single fuzzified variable, to get $\left[F_1(x_1)^{(1)}, F_1(x_1)^{(2)}, F_1(x_1)^{(3)}, F_1(x_1)^{(4)}\right]$,

$\left[F_2(x_2)^{(1)}, F_2(x_2)^{(2)}, F_2(x_2)^{(3)}, F_2(x_2)^{(4)}\right], \ldots, \left[F_n(x_n)^{(1)}, F_n(x_n)^{(2)}, F_n(x_n)^{(3)}, F_n(x_n)^{(4)}\right]$.

Hence, $\left[F_n(x_n)^{(1)}, F_n(x_n)^{(2)}, F_n(x_n)^{(3)}, F_n(x_n)^{(4)}\right]$ is maximized over the fuzzified decision $\left[x_n^{(1)}, x_n^{(2)}, x_n^{(3)}, x_n^{(4)}\right]$.

We get the required fuzzy optimal solution to the problem.

3. Numerical Example

The fuzzified version of the problem is now discussed with the support of a computational procedure for solving single additive constraint with multiplicatively separable return.

The objective function in fuzzified form to be maximized, is

$$\left(\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right](*)\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right](*)\left[u_3^{(1)},u_3^{(2)},u_3^{(3)},u_3^{(4)}\right]\right)$$

subject to the constraints

$$\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right](+)\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right](+)\left[u_3^{(1)},u_3^{(2)},u_3^{(3)},u_3^{(4)}\right]=[2,4,6,8]$$

$$\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right],\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right],\left[u_3^{(1)},u_3^{(2)},u_3^{(3)},u_3^{(4)}\right]\geq 0$$

Here the fuzzy decision variables are $\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right],\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right],$ $\left[u_3^{(1)},u_3^{(2)},u_3^{(3)},u_3^{(4)}\right]$ and the fuzzy state variables are $\left[x_3^{(1)},x_3^{(2)},x_3^{(3)},x_3^{(4)}\right]=$

$$\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right](+)\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right](+)\left[u_3^{(1)},u_3^{(2)},u_3^{(3)},u_3^{(4)}\right]=[2,4,6,8]$$

$$\left[x_2^{(1)},x_2^{(2)},x_2^{(3)},x_2^{(4)}\right]=\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right](+)\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right]$$

$$=\left[x_3^{(1)},x_3^{(2)},x_3^{(3)},x_3^{(4)}\right](-)\left[u_3^{(1)},u_3^{(2)},u_3^{(3)},u_3^{(4)}\right]$$

$$\left[x_1^{(1)},x_1^{(2)},x_1^{(3)},x_1^{(4)}\right]=\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right]=\left[x_2^{(1)},x_2^{(2)},x_2^{(3)},x_2^{(4)}\right](-)\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right]$$

Now we may denote the minimum fuzzified value of the fuzzy objective function $\left[z^{(1)},z^{(2)},z^{(3)},z^{(4)}\right]$ for any feasible fuzzified value of $\left[x_3^{(1)},x_3^{(2)},x_3^{(3)},x_3^{(4)}\right]$ as

$$[F_3(x_3)^{(1)},F_3(x_3)^{(2)},F_3(x_3)^{(3)},F_3(x_3)^{(4)}]=\max_{\left[u_3^{(1)},u_3^{(2)},u_3^{(3)},u_3^{(4)}\right]}\left[\left[u_3^{(1)},u_3^{(2)},u_3^{(3)},u_3^{(4)}\right](*)\,[F_2(x_2)^{(1)},F_2(x_2)^{(2)},F_2(x_2)^{(3)},F_2(x_2)^{(4)}]\right] \quad (1)$$

$$[F_2(x_2)^{(1)},F_2(x_2)^{(2)},F_2(x_2)^{(3)},F_2(x_2)^{(4)}]=\max_{\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right]}\left[\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right](*)\,[F_1(x_1)^{(1)},F_1(x_1)^{(2)},F_1(x_1)^{(3)},F_1(x_1)^{(4)}]\right] \quad (2)$$

$$and\ [F_1(x_1)^{(1)},F_1(x_1)^{(2)},F_1(x_1)^{(3)},F_1(x_1)^{(4)}]=\left(\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right]\right)$$
$$=\left[x_2^{(1)},x_2^{(2)},x_2^{(3)},x_2^{(4)}\right](-)\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right] \quad (3)$$

Now the term, $[F_1(x_1)^{(1)},F_1(x_1)^{(2)},F_1(x_1)^{(3)},F_1(x_1)^{(4)}]=\left(\left[u_1^{(1)},u_1^{(2)},u_1^{(3)},u_1^{(4)}\right]\right)$

$$=\left[x_2^{(1)},x_2^{(2)},x_2^{(3)},x_2^{(4)}\right](-)\left[u_2^{(1)},u_2^{(2)},u_2^{(3)},u_2^{(4)}\right]$$

$$=\left[\left(x_2^{(1)}-u_2^{(4)}\right),\left(x_2^{(2)}-u_2^{(3)}\right),\left(x_2^{(3)}-u_2^{(2)}\right),\left(x_2^{(4)}-u_2^{(1)}\right)\right]$$

We now compute the corresponding fuzzy membership function:

$$\mu_{A-B}(x) = \begin{cases} \dfrac{x-(x_2^{(1)}-u_2^{(4)})}{(x_2^{(2)}-u_2^{(3)})-(x_2^{(1)}-u_2^{(4)})} & for\ (x_2^{(1)}-u_2^{(4)}) \le x \le (x_2^{(2)}-u_2^{(3)}) \\ 1 & for\ (x_2^{(2)}-u_2^{(3)}) \le x \le (x_2^{(3)}-u_2^{(2)}) \\ \dfrac{x-(x_2^{(4)}-u_2^{(1)})}{(x_2^{(3)}-u_2^{(2)})-(x_2^{(4)}-u_2^{(1)})} & for\ (x_2^{(3)}-u_2^{(2)}) \le x \le (x_2^{(4)}-u_2^{(1)}) \\ 0, otherwise & \end{cases}$$

Here $A = \left[x_2^{(1)}, x_2^{(2)}, x_2^{(3)}, x_2^{(4)}\right] and\ B = \left[u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, u_2^{(4)}\right]$, From (2) we have,

$$\left[F_2(x_2)^{(1)}, F_2(x_2)^{(2)}, F_2(x_2)^{(3)}, F_2(x_2)^{(4)}\right] = \underset{\left[u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, u_2^{(4)}\right]}{max}(*) \left[\left[u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, u_2^{(4)}\right](*)\left[\left(x_2^{(1)}-u_2^{(4)}\right), \left(x_2^{(2)}-u_2^{(3)}\right), \left(x_2^{(3)}-u_2^{(2)}\right), \left(x_2^{(4)}-u_2^{(1)}\right)\right]\right] \quad (4)$$

Now compute the maximum of above expression using calculus. Let the fuzzy decision variables be denoted by $\left[u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, u_2^{(4)}\right]$. Taking derivatives of each term with respect to $\left[u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, u_2^{(4)}\right]$, respectively, and equating to zero, we get the maximum when $\left[u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, u_2^{(4)}\right]$ takes the value $\left[\frac{x_2^{(1)}}{2}, \frac{x_2^{(2)}}{2}, \frac{x_2^{(3)}}{2}, \frac{x_2^{(4)}}{2}\right]$. Now substituting these values in expression (4) we get $\left[F_2(x_2)^{(1)}, F_2(x_2)^{(2)}, F_2(x_2)^{(3)}, F_2(x_2)^{(4)}\right] = \left[\frac{(x_2^{(1)})^2}{4}, \frac{(x_2^{(2)})^2}{4}, \frac{(x_2^{(3)})^2}{4}, \frac{(x_2^{(4)})^2}{4}\right]$.

Similarly, from (1) we have,

$$\left[F_3(x_3)^{(1)}, F_3(x_3)^{(2)}, F_3(x_3)^{(3)}, F_3(x_3)^{(4)}\right] = \underset{\left[u_3^{(1)}, u_3^{(2)}, u_3^{(3)}, u_3^{(4)}\right]}{max} \left[\left[u_3^{(1)}, u_3^{(2)}, u_3^{(3)}, u_3^{(4)}\right]^2 (*) \left[\frac{(x_2^{(1)})^2}{4}, \frac{(x_2^{(2)})^2}{4}, \frac{(x_2^{(3)})^2}{4}, \frac{(x_2^{(4)})^2}{4}\right]\right] \quad (5)$$

Now we compute the maximum of the above expression using calculus. Let the fuzzy decision variables be denoted by $\left[u_3^{(1)}, u_3^{(2)}, u_3^{(3)}, u_3^{(4)}\right]$. Taking derivatives of each term with respect to $\left[u_3^{(1)}, u_3^{(2)}, u_3^{(3)}, u_3^{(4)}\right]$ and equating to zero, we get the maximum when $\left[u_3^{(1)}, u_3^{(2)}, u_3^{(3)}, u_3^{(4)}\right]$ takes the value $\left[\frac{x_3^{(1)}}{3}, \frac{x_3^{(2)}}{3}, \frac{x_3^{(3)}}{3}, \frac{x_3^{(4)}}{3}\right]$. Now substituting these values in expression (5) we get $\left[F_3(x_3)^{(1)}, F_3(x_3)^{(2)}, F_3(x_3)^{(3)}, F_3(x_3)^{(4)}\right] = \left[\frac{(x_3^{(1)})^3}{9}, \frac{(x_3^{(2)})^3}{9}, \frac{(x_3^{(3)})^3}{9}, \frac{(x_3^{(4)})^3}{9}\right]$ with its fuzzy membership function

$$\mu(x) = \begin{cases} \dfrac{9x-(x_3^{(1)})^3}{(x_3^{(2)})^3-(x_3^{(1)})^3} & for\ \dfrac{(x_3^{(1)})^3}{9} \le x \le \dfrac{(x_3^{(2)})^3}{9} \\ 1 & for\ \dfrac{(x_3^{(2)})^2}{9} \le x \le \dfrac{(x_3^{(3)})^2}{9} \\ \dfrac{9x-(x_3^{(4)})^2}{(x_3^{(3)})^3-(x_3^{(4)})^3} & for\ \dfrac{(x_3^{(3)})^2}{9} \le x \le \dfrac{(x_3^{(4)})^2}{9} \\ 0, otherwise & \end{cases}$$

Obviously, $[F_3\,(x_3)^{(1)}, F_3\,(x_3)^{(2)}, F_3\,(x_3)^{(3)}, F_3\,(x_3)^{(4)}]$ is maximum for $\left[x_3^{(1)}, x_3^{(2)}, x_3^{(3)}, x_3^{(4)}\right] = [2, 4, 6, 8]$.

Therefore, $\left[u_3^{(1)}, u_3^{(2)}, u_3^{(3)}, u_3^{(4)}\right] = \left[\frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{8}{3}\right]$

Similarly $\left[u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, u_2^{(4)}\right] = \left[\frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{8}{3}\right]$ and $\left[u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, u_1^{(4)}\right] = \left[\frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{8}{3}\right]$

Hence the required maximum of fuzzy optimal solution is,

$$[z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}] = [F_3\,(x_3)^{(1)}, F_3\,(x_3)^{(2)}, F_3\,(x_3)^{(3)}, F_3\,(x_3)^{(4)}] = \left[\frac{50}{27}, \frac{100}{27}, \frac{150}{27}, \frac{200}{27}\right].$$

3.1. Computation of optimal solution with existing and proposed methods

To compare the existing and the proposed methodologies, the results of fuzzy dynamic programming problem in the above numerical example were obtained also by existing methods, as shown in Table 1.

Table 1. Optimal solution comparison between classical and proposed methodologies.

Existing methodology based on the classical methodology Swarup *et al*. (2004).	Proposed methodology based on the fuzzy nature
The optimal solution is $\frac{125}{27}$ with all the decision variables are $\frac{5}{3}$.	The fuzzy optimal solution is $\left[\frac{50}{27}, \frac{100}{27}, \frac{150}{27}, \frac{200}{27}\right] = R\left[\frac{50}{27}, \frac{100}{27}, \frac{150}{27}, \frac{200}{27}\right] = \frac{125}{27}$ with all the decision variables are $\left[\frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{8}{3}\right] = R\left[\frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{8}{3}\right] = \frac{5}{3}$

The methods gave the same results for this numerical example. The proposed methodology is competitive with existing methodology, Swarup *et al*. (2004).

3.2. Results and discussion

The fuzzy optimal solution is $[z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}] = \left[\frac{50}{27}, \frac{100}{27}, \frac{150}{27}, \frac{200}{27}\right]$, which may be physically understood as follows.

The fuzzy optimal solution is going to be continuously bigger than $\frac{50}{27}$ and less than $\frac{200}{27}$ and most likely the return is going to be between $\frac{100}{27}$ and $\frac{150}{27}$. The return is shown below in Figure 1 and the fuzzy optimum solution $\mu_z(x)$ in terms trapezoidal fuzzy membership functions is as follows:

$$\mu_z(x) = \begin{cases} \frac{27x - 50}{50} & for\ \frac{50}{27} \le x \le \frac{100}{27} \\ 1 & for\ \frac{100}{27} \le x \le \frac{150}{27} \\ \frac{200 - 27x}{50} & for\ \frac{150}{27} \le x \le \frac{200}{27} \\ 0, otherwise \end{cases}$$

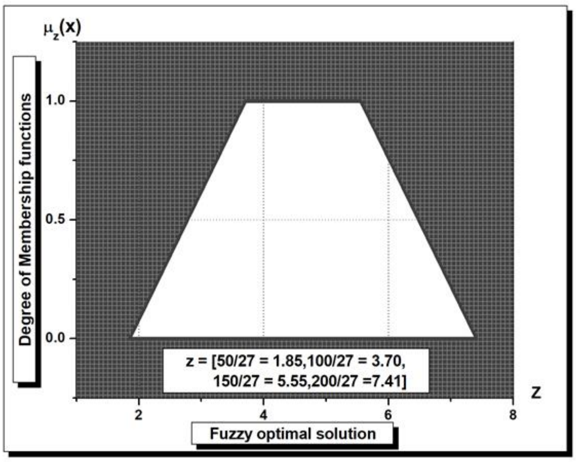

Figure 1. Trapezoidal Membership function for the fuzzy optimum solution $\mu_z(x)$

4. Conclusions

In this article, the fuzzified version of a problem was discussed along with a numerical example, showing that the proposed procedure offers an efficient tool for handling the dynamic programming problem in comparison to classical procedures. As a final point the optimal solution in the form of fuzzy numbers with trapezoidal fuzzy membership functions was explained and discussed, in this new approach to solve dynamic programming problems of fuzzy nature.

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