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Original Article

Formulae of the Frobenius number in relatively prime three Lucas numbers

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Abstract

In this paper, we find the explicit formulae of the Frobenius number for numerical semigroups generated by relatively prime three Lucas numbers L_i, L_{i+2} and L_{i+l} for given integers $i \geq 3$, $l \geq 4$.

Keywords: Frobenius number, Lucas numbers, Fibonacci numbers

1. Introduction

Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be integers. Any expression of the form $c_1a_1 + c_2a_2 + \ldots + c_na_n$ where $c_1, c_2, \ldots, c_n$ are integers, is called a linear combination of $a_1, a_2, \ldots, a_n$. Given positive integers $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ with gcd $(a_1, \ldots, a_n) = 1$, the Frobenius Problem is a problem to determine the largest positive integer that cannot be representable as a nonnegative integer combination of $a_1, \ldots, a_n$.

Definition The Frobenius number of $a_1, a_2, \ldots, a_n$, denoted by $g(a_1, a_2, \ldots, a_n)$, is the largest integer Z such that $Z \neq c_1a_1 + c_2a_2 + \ldots + c_na_n$ for all nonnegative integers $c_1, c_2, \ldots, c_n$.

For example, $g(3,5) = 7$, $g(6,9,20) = 43$.

The Frobenius Problem is well known as the coin problem that asks for the largest monetary amount that cannot be obtained using only coins in the set of coin denominations which has no common divisor greater than 1. This problem is also referred to as the McNugget number problem introduced by Henri Picciotto. There are several applications of the Frobenius Problem, for example, in obtaining upper bounds for the running time of the Shell-sort algorithm, studying partitions of vector spaces and investigating algebraic geometric codes; see Ramíres Alfonsín (2005).

The origin of this problem for $n = 2$ was proposed by Sylvester (1884), and this was solved by Sharp (1884) :

$$g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1 = a_1a_2 - a_1 - a_2.$$

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Roberts (1956) found the Frobenius number of an arithmetic sequence:

$$g(a,a+d,\ldots,a+kd)=a\left\lfloor\frac{a-2}{k}\right\rfloor+d(a-1).$$

For $n=3$, Selmer and Beyer (1978) solved the Frobenius Problem by a continued fraction algorithm. Then Rödseth (1978) improved their result. Greenberg (1988) found another algorithm.

In the 21st century, the Frobenius Problem is still an interesting problem. There are several studies associated with this problem, as follows. Marín *et al.* (2007) investigated the Frobenius number of Fibonacci numbers F_i, F_{i+2}, F_{i+k} for integers $i,k\geq 3$ where F_n is the n^{th} term of the Fibonacci sequence defined by $F_n=F_{n-1}+F_{n-2},\ n\geq 3$ with $F_1=1$ and $F_2=1$. They found that

$$g(F_i,F_{i+2},F_{i+k})=\begin{cases}(F_i-1)F_{i+2}-F_i(rF_{k-2}+1), & \text{if } r=0 \text{ or } r\geq 1 \text{ and}\\ & F_{k-2}F_i<(F_i-rF_k)F_{i+2},\\ (rF_k-1)F_{i+2}-F_i((r-1)F_{k-2}+1), & \text{otherwise,}\end{cases}$$

where $r=\left\lfloor\frac{F_i-1}{F_k}\right\rfloor$ for $r,k\geq 3$. Later on, Ýlhan and Kýper (2008) established the Frobenius number involving Lucas numbers L_n defined by $L_n=L_{n-1}+L_{n-2},\ n\geq 3$ with $L_1=1$ and $L_2=3$. They found the following formulae:

$$g(L_i,L_{i+1},L_{i+k})=L_iL_{i+1}-L_i-L_{i+1} \quad \text{for } i,k\geq 2,$$

$$g(L_i,L_{i+2},L_{i+3})=L_i\left\lfloor\frac{L_i-2}{2}\right\rfloor+L_{i+1}(L_i-1) \quad \text{for } i\geq 3,$$

and

$$g\left(L_{3i},L_{3i}+2,2L_{3i}+1\right)=\frac{L_{3i}^2}{2}+L_{3i}-1 \quad \text{for } i\geq 1.$$

Moreover, Ong and Ponomarenko (2008) solved the Frobenius Problem for sets of the form $\{m^k,m^{k-1}n,m^{k-2}n^2,\ldots,n^k\}$, where m,n are relatively prime positive integers:

$$g(m^k,m^{k-1}n,m^{k-2}n^2,\ldots,n^k)=n^{k-1}(mn-m-n)+\frac{(n-1)m^2(m^{k-1}-n^{k-1})}{m-n}$$

for any positive integer k. Gil *et al.* (2015) found the Frobenius number of primitive Pythagorean triples:

$$g(m^2-n^2,2mn,m^2+n^2)=(m-1)(m^2-n^2)+(m-1)(2mn)-(m^2+n^2).$$

Recently, Tripathi (2017) gave an exact formula for $g(a_1,a_2,a_3)$, where a_1,a_2,a_3 are pairwise coprime positive integers. His results are divided into several cases and are complicated, so we do not record them here.

In a recent paper, we investigate the Frobenius number $g(L_i,L_{i+2},L_{i+l})$ for integers $i\geq 3$, $l\geq 4$ by using the idea in Marín *et al.* (2007) and generalize the work of Ýlhan and Kýper (2008). Our work needs the well-known Theorem of Brauer and Shockley (1962) stated as follows:

Theorem A. Let $1<a_1<\ldots<a_n$ be integers such that $\gcd(a_1,\ldots,a_n)=1$.

Let $B=\{a_1x_1+\ldots+a_nx_n \mid x_i\in N\cup\{0\} \text{ for all } i=1,2,...,n\}$. Then

$$g(a_1,\ldots,a_n)=\max_{l\in\{1,2,\ldots,a_1-1\}}\{t_l\}-a_1,$$

where t_l is the smallest positive integers congruent to l modulo a_1 and $t_l\in B.$

Note that Theorem A can give the value for $g(a_1,\ldots,a_n)$; however, the formula is not in closed form and it is difficult to find t_l for each l. In our work, we are able to give an explicit formula for $g(L_i,L_{i+2},L_{i+l})$.

2. Necessary Lemmas

Before investigating the value of $g(L_i,L_{i+2},L_{i+l})$ for $i\geq 3$, $l\geq 4$, we establish some lemmas. By Theorem A, for fixed integers $i\geq 3$, $l\geq 4$, we get

$$g(L_i,L_{i+2},L_{i+l})=\max_{k\in\{1,2,\ldots,L_i-1\}}\{t_k^*\}-L_i$$

where t_k^* is the smallest positive integer congruent to k modulo L_i and $t_k^*=xL_{i+2}+yL_{i+l}$ for some $x,y\geq 0$. Then we shall construct the Table 1, denoted by T_1, having entries $t_{x,y}=xL_{i+2}+yL_{i+l}$ for integers $x,y\geq 0.$ Since

$$L_{i+l}=L_{i+2}F_{l-1}+L_{i+1}F_{l-2}=L_{i+2}(F_l-F_{l-2})+(L_{i+2}-L_i)F_{l-2}=F_lL_{i+2}-F_{l-2}L_i,$$

we get

$$t_{x,y}=xL_{i+2}+yL_{i+l}=xL_{i+2}+y(F_lL_{i+2}-F_{l-2}L_i)=(x+yF_l)L_{i+2}-yF_{l-2}L_i.$$

Thus the table T_1 can be represented as the table T_2.

Table 1. T_1 : $t_{x,y}=xL_{i+2}+yL_{i+l}$ for $x,y\geq 0$

x \ y	0	1	2	$\cdots$	r	$\cdots$
0	0	L_{i+l}	$2L_{i+l}$	$\cdots$	rL_{i+l}	
1	L_{i+2}	$L_{i+2}+L_{i+l}$	$L_{i+2}+2L_{i+l}$	$\cdots$	$L_{i+2}+rL_{i+l}$	
2	$2L_{i+2}$	$2L_{i+2}+L_{i+l}$	$2L_{i+2}+2L_{i+l}$	$\cdots$	$2L_{i+2}+rL_{i+l}$	
3	$3L_{i+2}$	$3L_{i+2}+L_{i+l}$	$3L_{i+2}+2L_{i+l}$	$\cdots$	$3L_{i+2}+rL_{i+l}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
F_l-2	$(F_l-2)L_{i+2}$	$(F_l-2)L_{i+2}+L_{i+l}$	$(F_l-2)L_{i+2}+2L_{i+l}$	$\cdots$	$(F_l-2)L_{i+2}+rL_{i+l}$	
F_l-1	$(F_l-1)L_{i+2}$	$(F_l-1)L_{i+2}+L_{i+l}$	$(F_l-1)L_{i+2}+2L_{i+l}$	$\cdots$	$(F_l-1)L_{i+2}+rL_{i+l}$	
F_l	F_lL_{i+2}	$F_lL_{i+2}+L_{i+l}$	$F_lL_{i+2}+2L_{i+l}$	$\cdots$	$F_lL_{i+2}+rL_{i+l}$	
F_l+1	$(F_l+1)L_{i+2}$	$(F_l+1)L_{i+2}+L_{i+l}$	$(F_l+1)L_{i+2}+2L_{i+l}$	$\cdots$	$(F_l+1)L_{i+2}+rL_{i+l}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

From now on, we define the set $T_{F_l-1,\infty}$ to contain the first F_l-1 entries of all columns in Table 2: T_2. That is

$$T_{F_l-1,\infty}=\{\,t_{x,y}\mid 0\leq x\leq F_l-1 \text{ and } y\geq 0\,\}.$$

Table 2. T_2 : $t_{x,y}=(x+yF_l)L_{i+2}-yF_{l-2}L_i$ for $x,y\ge 0$

$x \backslash y$	0	1	2	$\cdots$	r	$\cdots$
0	0	$F_lL_{i+2}-F_{l-2}L_i$	$2F_lL_{i+2}-2F_{l-2}L_i$	$\cdots$	$rF_lL_{i+2}-rF_{l-2}L_i$	$\cdots$
1	L_{i+2}	$(1+F_l)L_{i+2}-F_{l-2}L_i$	$(1+2F_l)L_{i+2}-2F_{l-2}L_i$	$\cdots$	$(1+rF_l)L_{i+2}-rF_{l-2}L_i$	$\cdots$
2	$2L_{i+2}$	$(2+F_l)L_{i+2}-F_{l-2}L_i$	$(2+2F_l)L_{i+2}-2F_{l-2}L_i$	$\cdots$	$(2+rF_l)L_{i+2}-rF_{l-2}L_i$	$\cdots$
3	$3L_{i+2}$	$(3+F_l)L_{i+2}-F_{l-2}L_i$	$(3+2F_l)L_{i+2}-2F_{l-2}L_i$	$\cdots$	$(3+rF_l)L_{i+2}-rF_{l-2}L_i$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
F_l-1	$(F_l-1)L_{i+2}$	$(2F_l-1)L_{i+2}-F_{l-2}L_i$	$(3F_l-1)L_{i+2}-2F_{l-2}L_i$	$\cdots$	$\left((r+1)F_l-1\right)L_{i+2}-rF_{l-2}L_i$	$\cdots$
F_l	F_lL_{i+2}	$2F_lL_{i+2}-F_{l-2}L_i$	$3F_lL_{i+2}-2F_{l-2}L_i$	$\cdots$	$(r+1)F_lL_{i+2}-rF_{l-2}L_i$	$\cdots$
F_l+1	$(F_l+1)L_{i+2}$	$(2F_l+1)L_{i+2}-F_{l-2}L_i$	$(3F_l+1)L_{i+2}-2F_{l-2}L_i$	$\cdots$	$\left((r+1)F_l+1\right)L_{i+2}-rF_{l-2}L_i$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Throughout the paper, we set $r=\left\lfloor \frac{L_i-1}{F_l}\right\rfloor$ and $L_i-1=rF_l+q$ for some integer $0\le q\le F_l-1$. Let $T_{F_l-1,r}$ be the set that contains the first F_l-1 entries of columns $0,1,2,\ldots,r-1$ and the first q entries of column r, i.e.,

$$T_{F_l-1,r}=\{\, t_{x,y}\mid 0\le x\le F_l-1 \text{ and } 0\le y\le r-1\,\}\cup\{t_{0,r},t_{1,r},\ldots,t_{q,r}\}.$$

Lemma 1. (i) The set $T_{F_l-1,r}$ is a complete system of residues modulo L_i.

(ii) In the table T_1, $t_{m,n}\le t_{j,k}$ for all $m\le j$ and $n\le k$. Moreover, $t_{m+1,n}<t_{m,n+1}$ for all $0\le m,n\le F_l-2$.

Proof. (i) For each $t_{x,y}=(x+yF_l)L_{i+2}-yF_{l-2}L_i\in T_{F_l-1,r}$, we have $0\le x+yF_l\le q+rF_l=L_i-1$. Since $\gcd(L_i,L_{i+2})=1$, $T_{F_l-1,r}$ is a complete system of residues modulo L_i.

(ii) Recall that $t_{m,n}=mL_{i+2}+nL_{i+l}$ and $t_{j,k}=jL_{i+2}+kL_{i+l}$. It is obvious that for $m\le j$, $t_{m,n}\le t_{j,n}$ and for $n\le k$, $t_{m,n}\le t_{m,k}$. Therefore, $t_{m,n}\le t_{j,k}$ for all $m\le j$ and $n\le k$. For $0\le m,n\le F_l-2$, we have $t_{m,n+1}-t_{m+1,n}=(F_l-1)L_{i+2}-F_{l-2}L_i>0.$

We define t_x as follows:

$$\begin{array}{llllll}
t_0=t_{0,0} & t_{F_l}=t_{0,1} & t_{2F_l}=t_{0,2} & \cdots & t_{rF_l}=t_{0,r} & \cdots\\
t_1=t_{1,0} & t_{F_l+1}=t_{1,1} & t_{2F_l+1}=t_{1,2} & \cdots & t_{rF_l+1}=t_{1,r} & \cdots\\
t_2=t_{2,0} & t_{F_l+2}=t_{2,1} & t_{2F_l+2}=t_{2,2} & \cdots & t_{rF_l+2}=t_{2,r} & \cdots\\
\vdots & \vdots & \vdots & & \vdots & \\
t_{F_l-1}=t_{F_l-1,0} & t_{2F_l-1}=t_{F_l-1,1} & t_{3F_l-1}=t_{F_l-1,2} & \cdots & t_{(r+1)F_l-1}=t_{F_l-1,r} & \cdots .
\end{array}$$

The elements of $T_{F_l-1,\infty}$ can be represented as $t_x=xL_{i+2}-\left\lfloor \frac{x}{F_l}\right\rfloor F_{l-2}L_i$ for $x=0,1,\ldots$.

Lemma 2. Let $t_{u,v}$ be an entry of T_1 and $t_{u,v}\notin T_{F_l-1,r}$. Then there exist $t_{x,y}\in T_{F_l-1,r}$ such that $t_{u,v}\equiv t_{x,y}\ (\mathrm{mod}\, L_i)$ and $t_{u,v}>t_{x,y}$.

Proof. By the definition of t_x given above, the set $T_{F_l-1,r}$ can be written as

$$T_{F_l-1,r}=\{t_0,\ldots,t_{F_l-1},t_{F_l},\ldots,t_{2F_l-1},t_{2F_l},\ldots,t_{3F_l-1},\ldots,t_{rF_l},\ldots,t_{rF_l+q}=t_{L_i-1}\}.$$

We will consider two cases as follows.

Case 1: $t_{u,v}\in T_{F_l-1,\infty}\setminus T_{F_l-1,r}$

Then $t_{u,v}=t_{aL_i+b}$ for some integer $a\geq 1$ and $0\leq b\leq L_i-1$. We see that

$$t_{aL_i+b}=(aL_i+b)L_{i+2}-\left\lfloor\frac{aL_i+b}{F_l}\right\rfloor F_{l-2}L_i\equiv bL_{i+2}-\left\lfloor\frac{b}{F_l}\right\rfloor F_{l-2}L_i=t_b \pmod{L_i}.$$

Since $0\leq b\leq L_i-1$, $t_b=t_{x,y}\in T_{F_l-1,r}$ for some x,y. That is, $t_{u,v}\equiv t_{x,y}\pmod{L_i}$. Next, we will show that $t_{u,v}>t_{x,y}$, i.e., $t_{aL_i+b}>t_b$. Since $t_{aL_i+b}\geq t_{L_i+b}$ for $a\geq 1$, it is enough to show only that $t_{L_i+b}>t_b$. Recall that $r=\left\lfloor\frac{L_i-1}{F_l}\right\rfloor$ and $L_i-1=rF_l+q$ for some $0\leq q\leq F_l-1$. We will consider two subcases depending on the value of r.

Subcase 1.1: If $r=0$, then $L_i-1<F_l$, so $L_i+b\leq 2F_l-1$. If $0\leq L_i+b\leq F_l-1$, then both t_b and t_{L_i+b} are in the first column of the table T_1. By Lemma 1(ii), we obtain $t_{L_i+b}>t_b$.

Suppose that $F_l\leq L_i+b\leq 2F_l-1$. Then t_b and t_{L_i+b} are in the first and second columns of the table T_1, respectively. If $L_i<\frac{F_l}{2}$, then $L_i+b<\frac{F_l}{2}+\frac{F_l}{2}=F_l$, a contradiction. Hence we have $F_{l-2}\leq\frac{F_l}{2}\leq L_i$. Finally, we have

$$t_{L_i+b}-t_b=L_iL_{i+2}-F_{l-2}L_i=L_i\left(L_{i+2}-F_{l-2}\right)>L_i\left(L_i-F_{l-2}\right)>0.$$

Subcase 1.2: Suppose that $r\geq 1$. Consider

$$t_{L_i+b}-t_b=L_i\left(L_{i+2}-F_{l-2}\left(\left\lfloor\frac{L_i+b}{F_l}\right\rfloor-\left\lfloor\frac{b}{F_l}\right\rfloor\right)\right).$$

Write $b=mF_l+n$ where $0\leq n\leq F_l-1$. Since $L_i-1=rF_l+q$ with $0\leq q\leq F_l-1$, it follows that

$$\left\lfloor\frac{L_i+b}{F_l}\right\rfloor-\left\lfloor\frac{b}{F_l}\right\rfloor=\left\lfloor\frac{L_i-1+b+1}{F_l}\right\rfloor-m=\left\lfloor\frac{rF_l+q+mF_l+n+1}{F_l}\right\rfloor-m\leq r+1.$$

It is enough to show that $L_{i+2}>\left(r+1\right)F_{l-2}$. To this end, we see that

$$\begin{aligned}L_{i+2}-\left(r+1\right)F_{l-2}&=L_i+L_{i+1}-\left(r+1\right)F_{l-2}\\&=rF_l+q+1+L_{i+1}-\left(r+1\right)F_{l-2}\\&=r\left(F_l-F_{l-2}\right)-F_{l-2}+q+1+L_{i+1}\\&=rF_{l-1}-F_{l-2}+q+1+L_{i+1}>0\end{aligned}$$

since $r\geq 1$.

Case 2: $t_{u,v} \notin T_{F_l-1,\infty}$

Since $T_{F_l-1,r}$ is a complete system of residues modulo L_i, there exists $t_{x,y} \in T_{F_l-1,r}$ such that $t_{u,v} \equiv t_{x,y} \pmod{L_i}$. Then $0 \le x \le F_l - 1 < u$. If $v \ge y$, by Lemma 1(ii), $t_{x,y} \le t_{x,v} < t_{u,v}$. Suppose $v < y$. Then $t_{u,v} \equiv t_{x,y} \pmod{L_i}$ implies $u + vF_l \equiv x + yF_l \pmod{L_i}$. From Lemma 1(i), $0 \le x + yF_l \le L_i - 1$, and thus $u + vF_l = m(x + yF_l)$ for some integer $m \ge 1$. Hence $u + vF_l \ge x + yF_l$. Since $-vF_{l-2}L_i > -yF_{l-2}L_i$, we have $t_{u,v} > t_{x,y}$.

3. Main Theorem

Theorem. Let $i \ge 3$, $l \ge 4$ be integers and $r = \left\lfloor \frac{L_i - 1}{F_l} \right\rfloor$. Then

$$g(L_i, L_{i+2}, L_{i+l}) = \begin{cases} (L_i - 1)L_{i+2} - (1 + rF_{l-2})L_i, & \text{if 1.) } r = 0, \\ & \text{or 2.) } r \ge 1 \text{ and } (L_i - rF_l)L_{i+2} > F_{l-2}L_i, \\ (rF_l - 1)L_{i+2} - (1 + (r-1)F_{l-2})L_i, & \text{otherwise.} \end{cases}$$

Proof. From Theorem A, now we have to consider t_k^* for $k = 1, 2, \ldots, L_i - 1$ when t_k^* is the smallest positive integer congruent to k modulo L_i and t_k^* can be written as $xL_{i+2} + yL_{i+l}$ for some integers $x, y \ge 0$. Since $t_x = xL_{i+2} - \left\lfloor \frac{x}{F_l} \right\rfloor F_{l-2}L_i$ for $x = 0, 1, \ldots$.

If $r = 0$, by Lemma 2, we have that t_x is the smallest positive integer congruent to k modulo L_i for some integer $0 \le k \le L_i - 1$. And we see that t_x can be represented as a linear combination of L_{i+2} and L_{i+l}. Hence $T_{F_l-1,r} = \{t_k^* \mid k = 1, 2, \ldots, L_i - 1\}$. If $r \ge 1$, by Lemma 1(ii), then

$$t_{F_l-1,i} = \max_{0 \le x \le F_l - 1} \{t_{x,i} \mid t_{x,i} \in T_{F_l-1,r}\} \quad \text{for each } i = 0, 1, \ldots, r-1,$$

$$t_{F_l-1,r-1} = \max_{0 \le i \le r-1} \{t_{F_l-1,i} \mid t_{F_l-1,i} \in T_{F_l-1,r}\},$$

and

$$t_{k,r} = \max_{0 \le x \le k} \{t_{x,r} \mid t_{x,r} \in T_{F_l-1,r}\}.$$

We will find the necessary condition for $t_{k,r} > t_{F_l-1,r-1}$. It is true if and only if $(L_i - 1)L_{i+2} - F_{l-2}L_i > (rF_l - 1)L_{i+2} - (r-1)F_{l-2}L_i$ that is $(L_i - rF_l)L_{i+2} > F_{l-2}L_i$. Hence we can conclude the result of this theorem.

Example 1. Let $i = 3$ and $l = 5$. Then $r = \left\lfloor \frac{L_3 - 1}{F_5} \right\rfloor = 0$, and by our main theorem, we have

$$g(L_3, L_5, L_8) = g(4, 11, 47) = (L_3 - 1)L_5 - (1 + (0)F_3)L_3 = 3(11) - 1(4) = 29.$$

We would like to confirm the value of $g(4, 11, 47)$ by the well-known Theorem A. Since $g(L_3, L_5, L_8) = g(4, 11, 47)$ $= \max_{k \in \{1,2,3\}} \{t_k^*\} - 4$. Then we have to find t_k^* for each $k = 1, 2, 3$, that t_k^* is the smallest positive integer congruent to k modulo

$L_3 = 4$ and $t_k^* \in B$. We get $t_1^* = 33, t_2^* = 22$ and $t_3^* = 11$. Thus $g(L_3, L_5, L_8) = \max\{33, 22, 11\} - 4 = 29$ which is the same value obtained by our result.

Example 2. Take $i = 4$ and $l = 4$. Then $r = \left\lfloor \frac{L_4 - 1}{F_4} \right\rfloor = 2$, and $(L_4 - 2F_4)L_6 > F_2 L_4$. Thus

$$g(L_4, L_6, L_8) = g(7, 18, 47) = (L_4 - 1)L_6 - (1 + 2F_2)L_4 = 87.$$

On the other hand, by using Theorem A,

$$g(L_4, L_6, L_8) = g(7, 18, 47) = \max_{k \in \{1,2,3,4,5,6\}} \{t_k^*\} - 7.$$

We get $t_1^* = 36, t_2^* = 65, t_3^* = 94, t_4^* = 18, t_5^* = 47$ and $t_6^* = 83$. Thus $g(L_4, L_6, L_8) = \max\{36, 65, 94, 18, 47, 83\} - 7 = 87$ which is the same value as above.

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