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Original Article

Ulam stability for fractional differential equations in the sense of Caputo operator

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Abstract

In this paper, we consider the Hyers-Ulam stability for the following fractional differential equations, in the sense of complex Caputo fractional derivative defined, in the unit disk: ${}^{c}D^{\beta}_{z} f(z)=G(f(z),{}^{c}D^{\alpha}_{z} f(z),zf'(z);z)$ $0<\alpha<1<\beta<2$. Furthermore, a generalization of the admissible functions in complex Banach spaces is imposed and applications are illustrated.

Keywords: analytic function, unit disk, Hyers-Ulam stability, admissible functions, fractional calculus, complex fractional differential equation, Caputo fractional derivative

1. Introduction

A classical problem in the theory of functional equations is that: If a function f approximately satisfies functional Equation E, when does there exist an exact solution of E which f approximates. Ulam (1964) imposed the question of the stability of Cauchy equation and in 1941, solved it (Hyers, 1957). Rassias (1978) provided a generalization of Hyers theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (Hyers, 1983; Hyers and Rassias,1992; Hyers *et al.*,1998). Recently, Li and Hua (2009) discussed and proved the Hyers-Ulam stability of spacial type of finite polynomial equation, and Bidkham *et al.* (2010), introduced the Hyers-Ulam stability of generalized finite polynomial equation. Finally, Rassias (2011) imposed a Cauchy type additive functional equation and investigated the generalized Hyers-Ulam 'product-sum' stability of this equation.

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. There are different fractional operators appeared during the past three decades such as Riemann-Liouville operators, Erdélyi-Kober operators, Weyl-Riesz operators and Grünwald-Letnikov operators (Podlubny, 1999).

The main advantage of Caputo fractional derivative is that the fractional differential equations with Caputo fractional derivative use the initial conditions (including the mixed boundary conditions) on the same character as for the integer-order differential equations (Podlubny, 1999). In the present work, we will show another advantage of Caputo fractional derivative based on admissible functions in complex Banach spaces.

2. Preliminaries

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ denote the space of all analytic functions on U. Here we suppose that $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\mathcal{H}[a, m]$ be the subspace of $\mathcal{H}$ consisting of

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functions of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \ldots, \quad z \in U.$$

Srivastava and Owa (1989) posed definitions for fractional operators (derivative and integral) in the complex z-plane C as follows:

Definition 2.1 The fractional derivative of order $0 < \alpha < 1$ is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane C containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2.2 The fractional integral of order $\alpha > 0$ is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane (C) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Note that Definition 2.1 and 2.2 correspond to the Riemann-Liouville derivative and integral respectively in the real form.

Remark 2.1

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \mu > -1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \mu > -1.$$

It was shown that (Ibrahim and Darus, 2008)

$$I_z^\alpha D_z^\alpha f(z) = D_z^\alpha I_z^\alpha f(z) = f(z), \quad f(0) = 0.$$

Definition 2.3 The Caputo fractional derivative of order $\mu > 0$ is defined, for a function $f(z)$ by

$${}^c D_z^\mu f(z) := \frac{1}{\Gamma(n-\mu)} \int_0^z \frac{f^{(n)}(\zeta)}{(z-\zeta)^{\mu-n+1}} d\zeta,$$

where $n = [\mu]+1$, (the notation $[\mu]$ stands for the largest integer not greater than μ), the function $f(z)$ is analytic in simply-connected region of the complex z-plane C containing the origin and the multiplicity of $(z-\zeta)^{n-\mu-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 2.2 The following relations hold:

(i) Representation

$${}^c D_z^\mu f(z) = I_z^{n-\mu} D_z^n f(z), \quad n-1 < \mu < n;$$

(ii) The Caputo fractional derivative of the power function

$${}^c D_z^\mu z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha} = D_z^\alpha z^\mu;$$

(iii)

$$I_z^\mu \, {}^c D_z^\mu f(z) = f(z), \quad z \in U, f(0) = 0, \mu \in (0,1);$$

(iv) Linearity

$${}^c D_z^\mu (\lambda f(z) + g(z)) = \lambda \, {}^c D_z^\mu f(z) + {}^c D_z^\mu g(z);$$

(v) Non-commutation

$${}^c D_z^\mu D_z^\alpha f(z) \neq D_z^\alpha \, {}^c D_z^\mu f(z).$$

More details on fractional derivatives and their properties and applications can be found in Kilbas *et al.* (2006); Sabatier *et al.* (2007); Li *et al.* (2009) and Li *et al.* (2011).

We next introduce the generalized Hyers-Ulam stability depending on the properties of the fractional operators. Recently the author studied the generalized Hyers-Ulam stability for various types of fractional differential equations (Ibrahim, 2011; Ibrahim, 2012a,b,c,d).

Definition 2.4 Let p be a real number. We say that

$$\sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z) \tag{1}$$

has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0, w \in \overline{U} = U \cup \partial U$, if

$$|\sum_{n=0}^{\infty} a_n w^{n+\alpha}| \leq \varepsilon (\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n})$$

then there exists some $z \in \overline{U}$ that satisfies equation (1) such that

$$|z^i - w^i| \leq \varepsilon K,$$

$$(z, w \in \overline{U}, \quad i \in \mathbb{N}).$$

In the present paper, we study the generalized Hyers-Ulam stability for holomorphic solutions of the fractional differential equation in complex Banach spaces *X* and *Y*

$${}^c D_z^\beta f(z) = G(f(z), z \, {}^c D_z^\alpha f(z), zf'(z); z), \tag{2}$$

where

$$(0 < \alpha < 1 < \beta < 2,)$$

and $G: X^3 \times U \to Y$ and $f: U \to X$ are holomorphic functions such that $f(0) = \Theta$ (Θ is the zero vector in *X*).

3. Generalized Hyers-Ulam stability

In this section we present extensions of the generalized Hyers-Ulam stability to holomorphic vector-valued functions. Let X, Y represent complex Banach spaces. The class of admissible functions $\mathsf{G}(X,Y)$, consists of those functions $g: X^3 \times U \to Y$ that satisfy the admissibility conditions:

$$\|g(r, ks, lt; z\| \geq 1, when \|r\| = \|s\| = \|t\| = 1, \tag{3}$$

We need the following results:

Lemma 3.1 (Hill, 1957) If $f: D \to X$ is holomorphic, then $\|f\|$ is a subharmonic of $z \in D \subset \mathbb{C}$. It follows that $\|f\|$ can have no maximum in *D* unless $\|f\|$ is of constant value throughout *D*.

Lemma 3.2 (Miller and Mocanu, 2000) Let $f: U \to X$ be the holomorphic vector-valued function defined in the unit disk *U* with $f(0) = \Theta$ (the zero element of *X*). If there exists a $z_0 \in U$ such that

$$\|f(z_0)\| = \max_{|z|=|z_0|} \|f\|,$$

then

$$\|z_0 f'(z_0)\| = \kappa \|f(z_0)\|, \quad \kappa \geq 1.$$

Theorem 3.1 Let $G \in \mathsf{G}(X,Y)$. If $f: U \to X$ is a holomorphic vector-valued function defined in the unit disk *U*, with $f(0) = \Theta$, then

$$\|G(f(z), z\,{}^cD_z^\alpha f(z), zf'(z); z)\| < 1$$
$$\Rightarrow \|f(z)\| < 1. \tag{4}$$

Proof From Definition 2.3, we observe that

$$\| z\,{}^cD_z^\alpha f(z) \| = \| \frac{z}{\Gamma(1-\alpha)} \int_0^z \frac{f'(\zeta)}{(z-\zeta)^\alpha} d\zeta \|$$
$$\leq \frac{\| zf'(z) \|}{\Gamma(2-\alpha)} |z|^{1-\alpha}$$
$$\leq \frac{\| zf'(z) \|}{\Gamma(2-\alpha)}, \quad z \in U.$$

Assume that $\|f(z)\| \geq 1$ for $z \in U$. Thus, there exists a point $z_0 \in U$ for which $\|f(z_0)\| = 1$. According to Lemma 3.1, we have $\|f(z)\| < 1$

$$(z \in U_{r_0} = \{z : |z| < |z_0| = r_0\}),$$

and

$$\max_{|z| \leq |z_0|} \|f(z)\| = \|f(z_0)\| = 1.$$

In view of Lemma 3.2, at the point z_0 there is a constant $\kappa \geq 1$ such that

$$\|z_0 f'(z_0)\| = \kappa \|f(z_0)\| = \kappa.$$

Therefore,

$$\|z_0\,{}^cD_{z_0}^\alpha f(z_0)\| = \frac{\|z_0 f'(z_0)\|}{\Gamma(2-\alpha)} = \frac{\kappa \|f(z_0)\|}{\Gamma(2-\alpha)} = \frac{\kappa}{\Gamma(2-\alpha)},$$

consequently, we obtain

$$\|f(z_0)\| = \frac{\Gamma(2-\alpha)}{\kappa} \|z_0\,{}^cD_{z_0}^\alpha f(z_0)\|$$
$$= \frac{1}{\kappa} \|z_0 f'(z_0)\| = 1, \quad \kappa \geq 1.$$

We put $k := \frac{\kappa}{\Gamma(2-\alpha)} \geq 1$ and $l := \kappa$; hence from Equation (3), we deduce

$$\|G(f(z_0), z_0\,{}^cD_{z_0}^\alpha f(z_0), z_0 f'(z_0); z_0)\| =$$
$$\|G(f(z_0), k[z_0\,{}^cD_{z_0}^\alpha f(z_0)/k], l[z_0 f'(z_0)/l]; z_0)\|$$
$$\geq 1,$$

which contradicts the hypothesis in (4), we must have $\|f\| < 1$.

Corollary 3.1 Assume the problem (2). If $G \in \mathsf{G}(X,Y)$ is the holomorphic vector-valued function defined in the unit disk *U* then

$$\| G(f(z), z\,{}^cD_z^\alpha f(z), zf'(z); z) \| < 1$$
$$\Rightarrow \| I_z^\beta G(f(z), z\,{}^cD_z^\alpha f(z), zf'(z); z) \| < 1. \tag{5}$$

Proof By continuity of the fractional differential equation (2) has at least one holomorphic solution f satisfying $(f(0) = f'(0) = 0)$. According to Remark 2.2, the solution *f*(*z*) of the problem (2) takes the form

$$f(z) = I_z^\beta G(f(z), z\,{}^cD_z^\alpha f(z), zf'(z); z).$$

Therefore, in virtue of Theorem 3.1, we obtain the Assertion (5).

Theorem 3.2 Let $G \in \mathsf{G}(X,Y)$ be holomorphic vector-valued functions defined in the unit disk *U* then the Equation (2) has the generalized Hyers-Ulam stability for $z \to \partial U$.

Proof Assume that

$$G(z) := \sum_{n=0}^\infty \varphi_n z^n, \quad z \in U$$

therefore, by Remark 2.1, we have

$$I_z^\alpha G(z) = \sum_{n=0}^\infty a_n z^{n+\alpha} = f(z).$$

Also, $z \to \partial U$. and thus $|z| \to 1$. According to Theorem 3.1, we have

$$\| f(z) \| < 1 = |z|.$$

Let $\varepsilon > 0$ and $w \in \overline{U}$ be such that

$$|\sum_{n=1}^\infty a_n w^{n+\alpha}| \leq \varepsilon(\sum_{n=1}^\infty \frac{|a_n|^p}{2^n}).$$

We will show that there exists a constant *K* independent of ε such that

$$|w^i - u^i| \leq \varepsilon K, \quad w \in \overline{U}, \ u \in U$$

and satisfies (1). We put the function

$$f(w) = \frac{-1}{\lambda a_i} \sum_{n=1, n\neq i}^\infty a_n w^{n+\alpha}, \tag{6}$$

$$(a_i \neq 0, 0 < \lambda < 1)$$

thus, for $w \in \partial U$, we obtain

$$\begin{aligned}|w^i - u^i| &= |w^i - \lambda f(w) + \lambda f(w) - u^i| \\ &\leq |w^i - \lambda f(w)| + \lambda |f(w) - u^i| \\ &< |w^i - \lambda f(w)| + \lambda |w^i - u^i| \\ &= |w^i + \frac{1}{a_i} \sum_{n=1, n\neq i}^{\infty} a_n w^{n+\alpha}| \\ &+ \lambda |w^i - u^i| \\ &\leq \frac{1}{|a_i|} |\sum_{n=1}^{\infty} a_n w^{n+\alpha}| + \lambda |w^i - u^i|.\end{aligned}$$

Without loss of generality, we consider $|a_i| = \max_{n\geq 1}(|a_n|)$ yielding

$$\begin{aligned}|w^i - u^i| &\leq \frac{1}{|a_i|(1-\lambda)} |\sum_{n=1}^{\infty} a_n w^{n+\alpha}| \\ &\leq \frac{\varepsilon}{|a_i|(1-\lambda)} (\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n}) \\ &\leq \frac{\varepsilon |a_i|^{p-1}}{(1-\lambda)} (\sum_{n=0}^{\infty} \frac{1}{2^n}) \\ &\leq \frac{2\varepsilon |a_i|^{p-1}}{(1-\lambda)} \\ &:= K\varepsilon.\end{aligned}$$

This completes the proof.

4. Applications

In this section, we introduce some applications of functions to achieve the generalized Hyers-Ulam stability.

Example 4.1 Consider the function $G: X^3 \times U \to \mathbb{R}$ by

$$G(r,s,t;z) = a(\|r\| + \|s\| + \|t\|) + b|z,|$$

with $a \geq 0.5$, $b \geq 0$ and $G(\Theta,\Theta,\Theta,0) = 0$. Our aim is to apply Corollary 3.1, this follows since

$$\begin{aligned}\|G(r,ks,\kappa t;z)\| &= a(\|r\| + k\|s\| + l\|t\|) + b|z| \\ &= a(1+k+l) + b|z| \geq 1,\end{aligned}$$

when $\|r\| = \|s\| = \|t\| = 1$, $z \in U$. Hence by Corollary 3.1, we have : If $a \geq 0.5$, $b \geq 0$ and $f: U \to X$ is a holomorphic vector-valued function defined in *U*, with $f(0) = \Theta$, then

$$\begin{aligned}&a(\| f(z) \| + \| z\, {}^cD_z^\alpha f(z) \| + \| zf'(z) \|) \\ &+ b|z| < 1 \Rightarrow \| f(z) \| < 1.\end{aligned}$$

Consequently,

$$\| I_z^\alpha G(f(z), z\, {}^cD_z^\alpha f(z), zf'(z); z) \| < 1,$$

thus in view of Theorem 3.2, f has the generalized Hyers-Ulam stability.

Example 4.2 Assume the function $G: X^3 \to X$ by

$$G(r,s,t;z) = G(r,s,t) = re^{\| s \| \| t \| - 1},$$

with $G(\Theta,\Theta,\Theta) = \Theta$. By applying Corollary 3.1, we need to show that $G \in \mathcal{G}(X,X)$. Since

$$\|G(r,ks,lt)\| = \|re^{\|ks\|\|lt\|-1}\| = e^{kl-1} \geq 1,$$

when $\|r\| = \|s\| = \|t\| = 1$, $k \geq 1$ and $l \geq 1$. Hence by Corollary 3.1, we have : For $f: U \to X$ is a holomorphic vector-valued function defined in *U* with $f(0) = \Theta$, then

$$\begin{aligned}&\| f(z) e^{\| z\, {}^cD_z^\alpha f(z) \| \| zf'(z) \| - 1} \| < 1 \\ &\Rightarrow \| f(z) \| < 1.\end{aligned}$$

Consequently,

$$\| I_z^\alpha G(f(z), z\, {}^cD_z^\alpha f(z), zf'(z); z) \| < 1,$$

thus in view of Theorem 3.2, f has the generalized Hyers-Ulam stability.

Example 4.3 Let $a,b,c: U \to \mathbb{C}$ satisfy

$$|a(z) + \mu b(z) + \nu c(z)| \geq 1,$$

for every $\mu \geq 1, \nu > 1$ and $z \in U$. Consider the function G : $X^3 \to Y$ by

$$G(r,s,t;z) = a(z)r + \mu b(z)s + \nu c(z)t,$$

with $G(\Theta,\Theta,\Theta) = \Theta$. Now for $\|r\| = \|s\| = \|t\| = 1$, we have

$$\| G(r,\mu s,\nu t;z) \| = |a(z) + \mu b(z) + \nu c(z)| \geq 1$$

and thus $G \in \mathcal{G}(X,Y)$. If $f: U \to X$ is a holomorphic vector-valued function defined in *U* with $f(0) = \Theta$, then

$$\begin{aligned}&\| a(z)f(z) + b(z)z\, {}^cD_z^\alpha f(z) + zc(z)f'(z) \| < 1 \\ &\Rightarrow \| f(z) \| < 1.\end{aligned}$$

Hence according to Theorem 3.2, *f* has the generalized Hyers-Ulam stability.

References

Bidkham, M., Mezerji, H. A. and Gordji, M. E. 2010. Hyers-Ulam stability of polynomial equations. Abstract and Applied Analysis. 2010, 1-7.

Hill, E. and Phillips, R. S. 1957. Functional Analysis and Semi-group, American Mathematical Society, U.S.A., pp. 110-115.

Hyers, D. H. 1941. On the stability of linear functional equation. Proceedings of the National Academy of Sciences. 27, 222-224.

Hyers, D. H. 1983. The stability of homomorphisms and related topics. Global Analysis-Analysis on Manifolds. 75, 140-153.

Hyers, D. H. and Rassias, Th. M. 1992. Approximate homomorphisms, Aequationes mathematicae. 44, 125-153.

Hyers, D. H., Isac, G. I. and Rassias, Th. M. 1998. Stability of Functional Equations in Several Variables, Birkhauser, Basel, Switzerland, pp. 110-155.

Ibrahim, R. W. and Darus, M. 2008. Subordination and superordination for univalent solutions for fractional differential equations. Journal of Mathematical Analysis and Applications. 345, 871-879.

Ibrahim, R. W. 2011. Approximate solutions for fractional differential equation in the unit disk. Electronic Journal of Qualitative Theory of Differential Equations. 64, 1-11.

Ibrahim, R. W. 2012a. Ulam stability for fractional differential equation in complex domain. Abstract and Applied Analysis. 2012, 1-8.

Ibrahim, R. W. 2012b. Generalized Ulam-Hyers stability for fractional differential equations, International Journal of Mathematics. 2012, 1-9.

Ibrahim, R. W. 2012c. On generalized Hyers-Ulam stability of admissible functions. Abstract and Applied Analysis. 2012, 1-10.

Ibrahim, R. W. 2012d. Ulam-Hyers stability for Cauchy fractional differential equation in the unit disk. Abstract and Applied Analysis. 2012, 1-10.

Kilbas, A.A., Srivastava, H. M. and Trujillo, J.J. 2006. Theory and Applications of Fractional Differential Equations: North-Holland Mathematics Studies, Volume 204, Elsevier Science B.V, Amsterdam, Netherlands, pp. 122-135.

Li, C. P., Dao, X. H. and Guo, P. 2009. Fractional derivatives in complex plane. Nonlinear Analysis: Theory, Methods & Applications. 71, 1857-1869.

Li, C. P., Qian, D. L. and Chen, Y. Q. 2011. On Riemann-Liouville and Caputo derivatives. Discrete Dynamics in Nature and Society. 2011, 1-15.

Li, Y. and Hua, L. 2009. Hyers-Ulam stability of a polynomial equation. Banach Journal of Mathematical Analysis. 3, 86-90.

Miller, S. S. and Mocanu, P. T. 2000. Differential Subordinantions: Theory and Applications, Pure and Applied Mathematics, New York, U.S.A., pp. 132-145.

Podlubny, L. 1999. Fractional Differential Equations, Acadamic Press, London, UK., pp. 52-75.

Rassias, Th. M. 1978. On the stability of the linear mapping in Banach space. Proceedings of the American Mathematical Society. 72, 297-300.

Rassias, M. J. 2011. Generalised Hyers-Ulam product-sum stability of a Cauchy type additive functional equation. European Journal of Pure and Appllied Mathematics. 4, 50-58.

Sabatier, J., Agrawal, O. P. and Machado, J. A. 2007. Advance in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, Netherlands, pp. 133-155.

Srivastava, H.M. and Owa, S. 1989. Univalent Functions, Fractional Calculus, and Their Applications, Halsted Press, John Wiley and Sons, New York, U.S.A., pp. 162-175.

Ulam, S. M. 1964. A Collection of Mathematical Problems, International Science Publications, Problems in Modern Mathematics, Wiley, New York, U.S.A.