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Some identities for (s, t)-Pell and (s, t)-Pell-Lucas numbers and its application to Diophantine equations

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Abstract

In this paper, some new identities for *(s,t)* - Pell and *(s,t)* - Pell-Lucas numbers are obtained by using matrix methods. Moreover, the solutions of some Diophantine equations are presented by applying these identities.

Keywords: Pell-numbers; Pell-Lucas number; *(s,t)*-Pell number; *(s,t)*-Pell-Lucas numbers; Matrix method

1. Introduction

Let s,t be any real number with $s^2+t>0$, $s>0$ and $t\neq 0$. Then the (s,t)-Pell sequences $\{P_n(s,t)\}_{n\in\mathbb{N}}$ [1] is defined by

$$P_n(s,t)=2sP_{n-1}(s,t)+tP_{n-2}(s,t),\quad \text{for all } n\geq 2, \tag{1}$$

with initial conditions $P_0(s,t)=0$ and $P_1(s,t)=1$. The first few terms of $\{P_n(s,t)\}_{n\in\mathbb{N}}$ are $0,\ 1,\ 2s,\ 4s^2+t,\ 8s^3+4st$ and so on. The terms of this sequence are called (s,t)-Pell numbers and we denoted the n^{th} (s,t) - Pell numbers by $P_n(s,t)$. The (s,t) - Pell numbers for negative subscripts can be defined as $P_{-n}(s,t)=\dfrac{-P_n(s,t)}{(-t)^n}$, for all $n\geq 1$. Then it follows that $P_n(s,t)=2sP_{n-1}(s,t)+tP_{n-2}(s,t)$, for all $n\in\mathbb{Z}$. Also, (s,t)-Pell-Lucas sequences $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ [1] is defined by $Q_0(s,t)=2$, $Q_1(s,t)=2s$ and

$$Q_n(s,t)=2sQ_{n-1}(s,t)+tQ_{n-2}(s,t),\quad \text{for all } n\geq 2, \tag{2}$$

The first few terms of $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ are $2,\ 2s,\ 4s^2+2t,\ 8s^3+6st$ and so on. The terms of this sequence are called (s,t)-Pell-Lucas numbers and we denoted the n^{th} (s,t)-Pell-Lucas numbers by $Q_n(s,t)$. The (s,t)-Pell-Lucas numbers for negative subscripts are defined as $Q_{-n}(s,t)=\dfrac{Q_n(s,t)}{(-t)^n}$, for all $n\geq 1$. It can be seen that $Q_n(s,t)=2sP_n(s,t)+2tP_{n-1}(s,t)$ and $Q_n(s,t)=P_{n+1}(s,t)+tP_{n-1}(s,t)$ for all $n\in\mathbb{Z}$ For more detailed information about (s,t)-Pell and (s,t)-Pell-Lucas numbers can be found in [1].

From the definitions of (s,t)-Pell and (s,t)-Pell-Lucas numbers, we have that the characteristic equation of (1) and (2) are in the form

$$x^2=2sx+t \tag{3}$$

and the root of equation (3) are $\alpha=s+\sqrt{s^2+t}$ and $\beta=s-\sqrt{s^2+t}$. We note that $\alpha+\beta=2s$, $\alpha-\beta=2\sqrt{s^2+t}$ and $\alpha\beta=-t$. Also, from the definitions of (s,t)-Pell and (s,t)-Pell-Lucas numbers, we have that if $s=\frac{1}{2}$, $t=1$, then the classical Fibonacci and Lucas sequence are obtained, and if $s=1$, $t=1$, then the classical Pell and Pell-Lucas sequence are obtained. It is well known that the Fibonacci, Lucas, Pell and Pell-Lucas sequences are the famous recursive sequences that have been studied in the literatures by many authors for over several years, because they are extensively used in various research areas such as Engineering, Architecture, Nature and Art (for examples see: [2-7]).

In this paper, we will establish some identities for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using matrix methods. Moreover, we present the solution of some Diophantine equations by applying these identities. In the rest of this paper, for convenience we will use the symbol P_n and Q_n instead of $P_n(s,t)$ and $Q_n(s,t)$ respectively.

2. Main Results

In this section, we will establish some identities for (s,t)-Pell and (s,t)-Pell-Lucas numbers by using the square matrix X which satisfy the property $X^2=2sX+tI$. Now, we begin with the following three Lemmas.

Lemma 2.1. If X is a square matrix with $X^2=2sX+tI$, then $X^n=P_nX+tP_{n-1}I$ for all $n\in\mathbb{Z}$.

Proof. If $n=0$, then the proof is obvious. It can be shown by induction that $X^n=P_nX+tP_{n-1}I$ for all $n\in\mathbb{N}$. Now, we will show that $X^{-n}=P_{-n}X+tP_{-n-1}I$ for all $n\in\mathbb{N}$. Let $Y=2sI-X=-tX^{-1}$. Then we have

$$Y^2=(2sI-X)^2=2s(2sI-X)+tI=2sY+tI.$$

It implies that $Y^n=P_nY+tP_{n-1}I$. That is $(-tX^{-1})^n=P_n(2sI-X)+tP_{n-1}I$. Thus

$$\begin{aligned}(-t)^nX^{-n}&=2sP_nI-P_nX+tP_{n-1}I\\&=-P_nX+(2sP_n+tP_{n-1})I\\&=-P_nX+P_{n+1}I.\end{aligned}$$

Therefore,

$$X^{-n} = -\frac{P_n}{(-t)^n}X + \frac{P_{n+1}}{(-t)^n}I = P_{-n}X + tP_{-(n+1)}I = P_{-n}X + tP_{-n-1}I.$$

This complete the proof.

Lemma 2.2. Let $W = \begin{bmatrix} s & 2(s^2+t) \\ \frac{1}{2} & s \end{bmatrix}$, then $W^n = \begin{bmatrix} \frac{1}{2}Q_n & 2(s^2+t)P_n \\ \frac{1}{2}P_n & \frac{1}{2}Q_n \end{bmatrix}$ for all $n \in \mathbb{Z}$.

Proof. Since $W^2 = 2sW + tI,$ the proof follows from Lemma 2.1 and using $Q_n = 2sP_n + 2tP_{n-1}$.

Lemma 2.3. $Q_n^{\ 2} - 4(s^2 + t)P_n^{\ 2} = 4(-t)^n$ for all $n \in \mathbb{Z}$.

Proof. Since $\det(W^n) = (\det(W))^n = (-t)^n$ and $\det(W^n) = \frac{1}{4}Q_n^2 - (s^2+t)P_n^2$, we get

$Q_n^2 - 4(s^2+t)P_n^2 = 4(-t)^n$.

Lemma 2.4. $2Q_{m+n} = Q_mQ_n + 4(s^2+t)P_mP_n$ for all $m,n \in \mathbb{Z}$.

Proof. Since $W^{m+n} = W^mW^n$, we get the result.

Lemma 2.5. $\alpha^n = \alpha P_n + tP_{n-1}$ and $\beta^n = \beta P_n + tP_{n-1}$ for all $n \in \mathbb{Z}$.

Proof. Take $X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, then $X^2 = 2sX + tI$. By Lemma 2.1, we have $X^n = P_nX + tP_{n-1}I$. It follows that

$$\begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha P_n + tP_{n-1} & 0 \\ 0 & \beta P_n + tP_{n-1} \end{bmatrix}.$$

This implies that $\alpha^n = \alpha P_n + tP_{n-1}$ and $\beta^n = \beta P_n + tP_{n-1}$.

By using Lemma 2.1 and Lemma 2.5, we get the following Theorem.

Theorem 2.6. *Let* $A = \begin{bmatrix} \alpha & 0 \\ t & \beta \end{bmatrix}$, *then* $A^n = \begin{bmatrix} \alpha^n & 0 \\ tP_n & \beta^n \end{bmatrix}$ *for all* $n \in \mathbb{Z}$

Proof. Since $A^2 = \begin{bmatrix} 2s\alpha + t & 0 \\ t(\alpha+\beta) & 2s\beta + t \end{bmatrix} = 2sA + tI$, by Lemma 2.1 and Lemma 2.5, we get that

$A^n = P_n A + tP_{n-1} I = \begin{bmatrix} \alpha^n & 0 \\ tP_n & \beta^n \end{bmatrix}$. Thus, we get the result.

By using Theorem 2.6, we get the following Theorem.

Theorem 2.7. *Let* $m, n \in \mathbb{Z}$. *Then*

$$4(s^2+t)(-t)^m P_n^2 + 4(s^2+t)(-t)^n P_m^2 - Q_{m+n}^2 = -4(s^2+t)P_m P_n Q_{m+n} - 4(-t)^{m+n}.$$

Proof. Let a matrix A as in Theorem 2.6. It can be seen that

$$\frac{1}{t}A^{n+1} + A^{n-1} = \begin{bmatrix} \frac{2\sqrt{s^2+t}}{t}\alpha^n & 0 \\ Q_n & -\frac{2\sqrt{s^2+t}}{t}\beta^n \end{bmatrix}.$$

Since $\left(\frac{1}{t}A^{n+1} + A^{n-1}\right)\left(\frac{1}{t}A^{m+1} + A^{m-1}\right) = \frac{1}{t^2}A^{m+n+2} + \frac{2}{t}A^{m+n} + A^{m+n-2}$, we get that

$$2\sqrt{s^2+t}P_{m+n} = \alpha^m Q_n - \beta^n Q_m.$$

Thus,

$$\begin{aligned} 4(s^2+t)P_{m+n}^2 &= (2\sqrt{s^2+t}P_{m+n})(2\sqrt{s^2+t}P_{m+n}) \\ &= (\alpha^m Q_n - \beta^n Q_m)(\alpha^n Q_m - \beta^m Q_n). \end{aligned}$$

Since $4(s^2+t)P_{m+n}^2 = Q_{m+n}^2 - 4(-t)^{m+n}$, we obtain

$$(-t)^m Q_n^2 + (-t)^n Q_m^2 + Q_{m+n}^2 = Q_m Q_n Q_{m+n} + 4(-t)^{m+n}. \qquad (4)$$

Since $Q_n^2 = 4(s^2+t)P_n^2 + 4(-t)^m$ and $Q_m Q_n = 2Q_{m+n} - 4(s^2+t)P_m P_n$, we get that

$$4(s^2+t)(-t)^m P_n^2 + 4(s^2+t)(-t)^n P_m^2 - Q_{m+n}^2 = -4(s^2+t)P_m P_n Q_{m+n} - 4(-t)^{m+n}, \qquad (5)$$

and so the proof is completed.

Example 2.8. Let $m = 1$ and $n = 2$. Then

$$4(s^2+t)(-t)^1 P_2^2 + 4(s^2+t)(-t)^2 P_1^2 - Q_3^2 = -4(s^2+t)P_1 P_2 Q_3 - 4(-t)^3.$$

Proof. Consider,

$$4(s^2+t)(-t)^1P_2^2+4(s^2+t)(-t)^2P_1^2-Q_3^2=4(s^2+t)(-t)(2s)^2+4(s^2+t)t^2(1)^2-(8s^3+6st)^2$$
$$=-16s^4t-16s^2t^2+4s^2t^2+4t^3-64s^6-96s^4t-36s^2t^2$$
$$=-64s^6-112s^4t-48s^2t^2+4t^3,$$

and

$$-4(s^2+t)P_1P_2Q_3-4(-t)^3=-4(s^2+t)(1)(2s)(8s^3+6st)+4t^3$$
$$=-64s^6-112s^4t-48s^2t^2+4t.^3$$

Thus, $4(s^2+t)(-t)^1P_2^2+4(s^2+t)(-t)^2P_1^2-Q_3^2=-4(s^2+t)P_1P_2Q_3-4(-t)^3$.

Theorem 2.9. Let $m,n\in\mathbb{Z}$. Then

$$(-t)^mQ_n^2-4(s^2+t)(-t)^nP_m^2-4(s^2+t)P_{m+n}^2=-4(s^2+t)Q_nP_mP_{m+n}+4(-t)^{m+n}.$$

Proof. By using a similar argument as in Theorem 2.7 and the property

$$\left(\frac{1}{t}A^{n+1}+A^{n-1}\right)A^m=A^m\left(\frac{1}{t}A^{n+1}+A^{n-1}\right)=\frac{1}{t}A^{m+n+1}+A^{m+n-1},$$

we get that

$Q_{m+n}=\alpha^mQ_n-2\sqrt{s^2+t}\beta^nP_m$ and $Q_{m+n}=2\sqrt{s^2+t}\alpha^nP_m+\beta^mQ_n$.

It follows that

$$Q_{m+n}^2=(\alpha^mQ_n-2\sqrt{s^2+t}\beta^nP_m)(2\sqrt{s^2+t}\alpha^nP_m+\beta^mQ_n)$$
$$=4(s^2+t)Q_nP_mP_{m+n}+(-t)^mQ_n^2-4(s^2+t)(-t)^nP_m^2.$$

Since $Q_{m+n}^2=4(s^2+t)P_{m+n}^2+4(-t)^{m+n}$, we have

$$(-t)^mQ_n^2-4(s^2+t)(-t)^nP_m^2-4(s^2+t)P_{m+n}^2=-4(s^2+t)Q_nP_mP_{m+n}+4(-t)^{m+n}. \quad (6)$$

This completed the proof.

Example 2.10. Let $m=2$ and $n=0$. Then

$$(-t)^2Q_0^2-4(s^2+t)(-t)^0P_2^2-4(s^2+t)P_2^2=-4(s^2+t)Q_0P_2P_2+4(-t)^2.$$

Proof. Consider,

$$(-t)^2Q_0^2-4(s^2+t)(-t)^0P_2^2-4(s^2+t)P_2^2=t^2(2)^2-4(s^2+t)(1)(2s)^2-4(s^2+t)(2s)^2$$
$$=-32s^4-32s^2t+4t^2,$$

and

$$-4(s^2+t)Q_0P_2P_2+4(-t)^2=-4(s^2+t)(2)(2s)(2s)+4t^2$$
$$=-32s^4-32s^2t+4t^2.$$

Thus, $(-t)^2 Q_0^2 - 4(s^2+t)(-t)^0 P_2^2 - 4(s^2+t)P_2^2 = -4(s^2+t)Q_0P_2P_2 + 4(-t)^2$.

3. Applications

In this section, by applying Theorem 2.7 and Theorem 2.9, we give the solutions of some Diophantine equations. We will investigate in two cases:

Case 1: If $s \in \mathbb{Z}^+$ and $t = 1$, then we get the following Theorems

Theorem 3.1. If m and n are even integers, then the integer solutions of the equation $z^2 - 4(s^2+1)x^2 - 4(s^2+1)y^2 = 4(s^2+1)xyz + 4$ are given by $(x,y,z) = (P_m(s,1), P_n(s,1), Q_{m+n}(s,1))$. If m and n are odd integers, then the integer solutions of the equation $z^2 + 4(s^2+1)x^2 + 4(s^2+1)y^2 = 4(s^2+1)xyz + 4$ are given by $(x,y,z) = (P_m(s,1), P_n(s,1), Q_{m+n}(s,1))$ and if m is an odd integer and n is an even integer, then the integer solutions of the equation $z^2 - 4(s^2+1)x^2 + 4(s^2+1)y^2 = 4(s^2+1)xyz - 4$ are given by $(x,y,z) = (P_m(s,1), P_n(s,1), Q_{m+n}(s,1))$.

Proof. The result follows immediately from Theorem 2.7.

Theorem 3.2. If m and n are even integers, then the integer solutions of the equation $z^2 + x^2 + y^2 = xyz + 4$ are given by $(x,y,z) = (Q_m(s,1), Q_n(s,1), Q_{m+n}(s,1))$ If m and n are odd integers, then the integer solutions of the equation $z^2 - x^2 - y^2 = xyz + 4$ are given by $(x,y,z) = (Q_m(s,1), Q_n(s,1), Q_{m+n}(s,1))$ and if m is an odd integer and n is an even integer, then the integer solutions of the equation $z^2 + x^2 - y^2 = xyz - 4$ are given by $(x,y,z) = (Q_m(s,1), Q_n(s,1), Q_{m+n}(s,1))$.

Proof. The result follows directly from (4).

Theorem 3.3. If m and n are even integers, then the integer solutions of the equation $x^2 - 4(s^2+1)y^2 - 4(s^2+1)z^2 = -4(s^2+1)xyz + 4$ are given by $(x,y,z) = (Q_n(s,1), P_m(s,1), P_{m+n}(s,1))$. If m and n are odd integers, then the integer solutions of the equation $x^2 - 4(s^2+1)y^2 + 4(s^2+1)z^2 = 4(s^2+1)xyz - 4$ are given by $(x,y,z) = (Q_n(s,1), P_m(s,1), P_{m+n}(s,1))$ and if m is an odd integer and n is an even integer, then the integer solutions of the equation $x^2 + 4(s^2+1)y^2 + 4(s^2+1)z^2 = 4(s^2+1)xyz + 4$ are given by $(x,y,z) = (Q_n(s,1), P_m(s,1), P_{m+n}(s,1))$.

Proof. The result follows immediately from Theorem 2.9.

Case 2: If $s \in \mathbb{Z}^+$ and $t = -1$, then we get the following Theorems

Theorem 3.4. The integer solutions of the equation $z^2 - 4(s^2 - 1)x^2 - 4(s^2 - 1)y^2 = 4(s^2 + 1)xyz + 4$ are given by $(x, y, z) = (P_m(s, -1), P_n(s, -1), Q_{m+n}(s, -1))$.

Proof. The result follows immediately from Theorem 2.7.

Theorem 3.5. The integer solutions of the equation $z^2 + x^2 + y^2 = xyz + 4$ are given by $(x, y, z) = (Q_m(s, -1), Q_n(s, -1), Q_{m+n}(s, -1))$.

Proof. The result follows directly from (4).

Theorem 3.6. The integer solutions of the equation $4(s^2 - 1)z^2 - x^2 + 4(s^2 - 1)y^2 = 4(s^2 - 1)xyz - 4$ are given by $(x, y, z) = (Q_n(s, -1), P_m(s, -1), P_{m+n}(s, -1))$.

Proof. The result follows immediately from Theorem 2.8.

4. Conclusion

Nowadays, many mathematicians are interested in solving Diophantine equations. We think it is a little hard and interesting to give all integer (positive integer) solutions of the Diophantine equations.

$$z^2 - 4(s^2 + 1)x^2 - 4(s^2 + 1)y^2 = 4(s^2 + 1)xyz + 4$$
$$z^2 + 4(s^2 + 1)x^2 + 4(s^2 + 1)y^2 = 4(s^2 + 1)xyz + 4$$
$$z^2 - 4(s^2 + 1)x^2 + 4(s^2 + 1)y^2 = 4(s^2 + 1)xyz + 4$$
$$z^2 + x^2 + y^2 = xyz + 4$$
$$z^2 - x^2 - y^2 = xyz + 4$$
$$z^2 + x^2 - y^2 = xyz - 4$$
$$x^2 - 4(s^2 + 1)y^2 - 4(s^2 + 1)z^2 = -4(s^2 + 1)xyz + 4$$
$$x^2 - 4(s^2 + 1)y^2 + 4(s^2 + 1)z^2 = 4(s^2 + 1)xyz - 4$$
$$x^2 + 4(s^2 + 1)y^2 + 4(s^2 + 1)z^2 = 4(s^2 + 1)xyz + 4$$
$$z^2 - 4(s^2 - 1)x^2 - 4(s^2 - 1)y^2 = 4(s^2 + 1)xyz + 4$$

and

$$4(s^2 - 1)z^2 - x^2 + 4(s^2 - 1)y^2 = 4(s^2 - 1)xyz - 4.$$

Although they have infinite many integer solutions by the above Theorems.

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