INCORPORATED EFFECTS OF SURFACE STRESS AND NONLOCAL ELASTICITY ON BENDING ANALYSIS OF NANOPLATES EMBEDDED IN AN ELASTIC MEDIUM

Monchai Panyatong*, Boonme Chinnaboon, and Somchai Chucheepsakul

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Abstract

This paper studies the bending behavior of nanoplates, incorporating the effects of surface stress and nonlocal elasticity. Thin and moderately thick nanoplates embedded in an elastic medium are analyzed. The complete governing equations, including both the effects of surface stress and nonlocal elasticity, are derived, while the solution for bending is solved by Navier's approach. The analytical solution in this study could serve as a benchmark in the evaluation of future research. Based on the results of this study, the opposite influence of surface stress and nonlocal elasticity on the bending of nanoplates is observed. Various parametric studies are investigated to elucidate the combined effects of surface stress and nonlocal elasticity.

Keywords: Nanoplates, surface sStress, nonlocal elasticity, ending, analytical solution

Introduction

Due to the dominant mechanical, thermal, chemical, nd electronic performances of nanostructures, nanoplates have been widely used as an important part of nanosensors and nanoactuators in the development of nanoelectromechanical systems (NEMs) in recent decades (Craighead, 2000). A thorough understanding of the bending behavior of nanoplates is important in the design of NEMs materials. Plates on an elastic foundation are commonly found in civil, mechanical, and aerospace engineering. However, nanostructures, such as single-layered graphene sheets, are often found embedded in an elastic medium (polymer composites) in order to enhance the strength of the parent material. Thus, it can be modeled by the plate on an elastic foundation problem. Many researchers have investigated the mechanical behavior of nanoscale multilayers, especially embedded graphene sheets in various surroundings, which can be modeled with different types of elastic foundations (Samaei *et al*., 2011; Asemi and Farajpour, 2014; Sobhy, 2014). The structure at a nanoscale level shows a significant size-dependent behavior. It is a well-known fact that the classical continuum theory does not take into account the so-called size effects, which are manifested in the response

Department of Civil Engineering, Faculty of Engineering, King Mongkut's University of Technology Thonburi, Bangkok, 10140, Thailand. Tel. 0-2470-9146; Fax. 0-2427-9063; E-mail: art_gear33@hotmail.com
* *Corresponding author*

of nanostructures. The high surface-to-bulk ratio of nanostructures leads to the exhibition of different behaviors compared with the conventional structures in macroscopic elements because of the considerable influence of surface stress.

The effect of surface stress on the behavior of nanoplates has been investigated by many researchers. Lim and He (2004) developed a continuum model to analyze the bending, buckling, and vibration behaviors of thin elastic films with a nanoscale thickness. Lu *et al.* (2006) proposed a continuum model including surface effects for plate-like thin film structures, which could be used for size-dependent static and dynamic analysis. Assadi and Farshi (2011) studied the stability and self-instability of circular nanoplates, including surface effects. Ansari and Sahmani (2011) investigated the free vibration characteristics of nanoplates, including surface stress effects, based on the continuum model. Assadi (2013) examined the effect of surface properties on the forced vibration of rectangular nanoplates. Wang and Wang (2013) evaluated the influence of surface energy on the post-buckling behavior of nanoplates by applying Galerkin's method to solve the problems. Shaat *et al.* (2013) studied the bending behavior of ultra-thin functionally graded plates, focusing on the influence of surface energy.

Considering small-scale effects, nonlocal elastic theory has presented reliable and accurate results in revealing the mechanical behaviors of nanostructures. For example, Pradhan and Murmu (2009) studied the buckling of single-layer graphene sheets subjected to a biaxial compression load, and solved the problems using the differential quadrature method. Murmu and Pradhan (2009) investigated the influence of small-scale effect on free in-plane vibration by employing a nonlocal continuum model. Aksencer and Aydogdu (2011) analyzed the buckling and vibration of nanoplates using Navier- and Levy-type solutions. Malekzadeh *et al.* (2011) determined the thermal buckling behavior of orthotropic arbitrary straight-sided quadrilateral nanoplates in an elastic medium by employing the differential quadrature method. Farajpour *et al.* (2011) investigated the buckling behavior of variable thickness nanoplates under biaxial compression, and solved the problems by Galerkin's method. Satish *et al.* (2012) analyzed the thermal vibration of orthotropic nanoplates by using the 2 variable refined plate theory and nonlocal continuum mechanics. Wang and Li (2012) studied the bending behavior of a nanoplate embedded in an elastic matrix. Pouresmaeeli *et al.* (2013) examined the vibration of viscoelastic orthotropic nanoplates embedded in a viscoelastic medium. Zenkour and Sobhy (2013) investigated thermal buckling of nanoplates resting on a Winkler-Pasternak elastic medium, based on the sinusoidal shear deformation plate theory. Chakraverty and Behera (2014) studied the free vibration of rectangular nanoplates, and solved the problems by the Rayleigh–Ritz method. Moreover, some researchers considered a combination of both surface effects and nonlocal elasticity. For example, Wang and Wang (2011) studied the vibration of rectangular nanoplates, which combined the effects of surface energy and nonlocal elasticity. Narendar and Gopalakrishnan (2012) investigated the wave propagation characteristics of nanoplates, using nonlocal plate theory together with surface effects. Juntarasaid *et a*l. (2012) analyzed the bending and buckling load of nanowires, including surface stress and the nonlocal elasticity effect with various boundary conditions. Farajpour *et al.* (2013) investigated the influence of temperature change, surface parameters, and nonlocal effects on the buckling of single-layer graphene sheets, using the differential quadrature method. Asemi and Farajpour (2014) studied thermo-mechanical vibration of circular graphene sheets, when both surface and nonlocal effects are taken into account, and solved the formulation by using Galerkin's method.

To the best of the authors' knowledge, based on a review of past literature, no studies have been performed on the bending behavior of nanoplates embedded in an elastic medium and incorporating both the surface effect and nonlocal elasticity. Therefore, the main purpose of this study is to investigate the bending behavior of nanoplates, combining the 2 above-mentioned small-scale effects, based on classical plate theory and the Mindlin plate theory. Thus, the complete governing equations for the bending analysis of nanoplates are achieved. The present work also studies the influence of the surface effect and nonlocal parameter on the displacement ratio when the sizes of the nanoplates are varied.

Formulation of the Problem

A rectangular nanoplate is considered by the thickness h, length a, width b, and an embedded elastic medium that is shown in Figure1. The elastic medium is characterized as a 2-parameter elastic foundation. The perfect chemical bonds are assumed to form between the nanoplates and the elastic medium. The nanoplates maintain continuous contact with the elastic medium, and there are no friction forces at the interface. At the contact surface, the effects of the surface stress of the elastic medium surrounding the nanoplates will be ignored, while only the influence of the surface stress of the nanoplates is considered. The formulation presented in this study is concerned with the application of nonlocal constitutive relations and surface stress at the upper and lower surfaces of the nanoplates. Thus, it is limited to 2-dimensional cases and the plate theories, which are employed to derive the governing equations for the bending problem with the displacement at any material point depending only on the displacement middle plane of the nanoplates.

Nonlocal Elasticity Theory

In nonlocal elasticity theory, it is assumed that the stress at a point depends not only on the strain at that point but also on the strains at all other points in the body. According to Eringen (2002), the nonlocal constitutive relations of a Hookean body can be expressed as

$$\left(1-\mu\nabla^2\right)\sigma_{xx}^{nl} = \frac{E}{1-\upsilon^2}\left(\varepsilon_{xx}+\upsilon\varepsilon_{yy}\right), \tag{1a}$$

$$\left(1-\mu\nabla^2\right)\sigma_{yy}^{nl} = \frac{E}{1-\upsilon^2}\left(\varepsilon_{yy}+\upsilon\varepsilon_{xx}\right), \tag{1b}$$

$$\left(1-\mu\nabla^2\right)\sigma_{xy}^{nl} = 2G\varepsilon_{xy}, \tag{1c}$$

$$\left(1-\mu\nabla^2\right)\sigma_{xz}^{nl} = 2G\varepsilon_{xz}, \tag{1d}$$

$$\left(1-\mu\nabla^2\right)\sigma_{yz}^{nl} = 2G\varepsilon_{yz}, \tag{1e}$$

where E, G and υ are the elastic modulus, the shear modulus, and Poisson's ratio, respectively. The scale factor $\mu = (e_0 l_i)^2$ is a nonlocal parameter, where l_i is an internal characteristic length (such as lattice spacing, granular distance, distance between C-C bonds) and e_0 is a material constant which is determined to calibrate the nonlocal model with experimental results or the results of molecular dynamics simulations.

Surface Stress

For nanostructures, the surface-to-bulk ratio is significant. Therefore, the surface effect

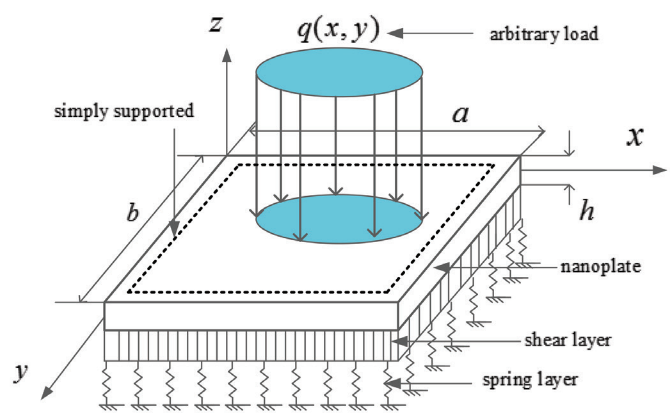

Figure 1. Geometric of a uniform rectangular nanoplate embedded in elastic medium

cannot be ignored. Recently, the Gurtin–Murdoch theory of an elastic solid surface (Gurtin and Murdoch, 1975, 1978) has been widely employed for the investigation of various mechanical responses of the nanoplates. Based on this theory, the original surface stress tensor is expressed by the following equation:

$$\sigma^{s} = \tau_0 \mathrm{I} + (\lambda_0 + \tau_0)(\mathrm{tr}\varepsilon^{sur})\mathrm{I} + 2 (u_0 - \tau_0)\varepsilon^{sur} + \tau_0 \nabla_{\Sigma}\mathrm{u}, \tag{2}$$

where τ_0 is the residual surface tension, λ_0 and u_0 are the surface Lame constants, I is the unit tangent tensor, $\mathrm{tr}\varepsilon^{sur}$ is the trace of the surface strain tensor ε^{sur}, and $\nabla_{\Sigma}\mathrm{u}$ is the surface gradient of the displacement field describing its deformation. Equation (2) refers to the constitutive relation of a tensor tangent to the surface and a component normal to this surface. In previous works (Sharma and Ganti, 2002; Sharma *et al*., 2003; Duan *et al*., 2005a, 2005b), the original surface stress tensor has been simplified by ignoring the last term in Equation (2). However, if the residual surface tension is important, the last term in Equation (2) should be included (Mogilevskaya *et al*., 2008). Especially, for the bending problem, the residual surface tension is significant and contributes to the bending stiffness of structures (Miller and Shenoy, 2000). Therefore, in the present work, the original surface stress equation is developed without any simplifications. Let the upper and lower surfaces of the nanoplates be denoted by S^+ and S^-, respectively. Using Equation (2), the surface stresses at the upper and lower surfaces of the nanoplates can be expressed as follows:

$$\sigma_{xx}^{s\pm} = \tau_0 + (2u_0 + \lambda_0)\frac{\partial u}{\partial x} + (\lambda_0 + \tau_0)\frac{\partial v}{\partial y}, \tag{3a}$$

$$\sigma_{yy}^{s\pm} = \tau_0 + (2u_0 + \lambda_0)\frac{\partial v}{\partial y} + (\lambda_0 + \tau_0)\frac{\partial u}{\partial x}, \tag{3b}$$

$$\sigma_{xy}^{s\pm} = u_0\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \tau_0\frac{\partial v}{\partial x}, \tag{3c}$$

$$\sigma_{yx}^{s\pm} = u_0\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \tau_0\frac{\partial u}{\partial y}, \tag{3d}$$

$$\sigma_{xz}^{s\pm} = \tau_0\frac{\partial w}{\partial x}, \tag{3e}$$

$$\sigma_{yz}^{s\pm} = \tau_0\frac{\partial w}{\partial y}. \tag{3f}$$

Classical Plate Theory

The displacement field of the classical or Kirchhoff plate theory can be written as

$$v(x,y,z) = v(x,y) - z\frac{\partial w}{\partial y}, \tag{4a}$$

$$u(x,y,z) = u(x,y) - z\frac{\partial w}{\partial x}, \tag{4b}$$

$$w(x,y,z) = w(x,y), \tag{4c}$$

where $u(x, y)$, $v(x, y)$ and $w(x, y)$ are the displacement components of the material point at the middle plane of the plate. The strain-displacement relations can be expressed as

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}, \tag{5a}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \tag{5b}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - z\frac{\partial^2 w}{\partial x \partial y}, \tag{5c}$$

$$\varepsilon_{xz} = 0, \varepsilon_{yz} = 0, \ \varepsilon_{zz} = 0. \tag{5d)–(5f}$$

The bending moment resultants incorporating the effects of surface stress and nonlocal elasticity can be written as

$$\mathrm{M}_{xx} = (\sigma_{xx}^{s+} - \sigma_{xx}^{s-})\frac{h}{2} + \int_{-h/2}^{h/2} \sigma_{xx}^{nl} z dz, \tag{6a}$$

$$\mathrm{M}_{yy} = (\sigma_{yy}^{s+} - \sigma_{yy}^{s-})\frac{h}{2} + \int_{-h/2}^{h/2} \sigma_{yy}^{nl} z dz, \tag{6b}$$

$$\mathrm{M}_{xy} = (\sigma_{xy}^{s+} - \sigma_{xy}^{s-})\frac{h}{2} + \int_{-h/2}^{h/2} \sigma_{xy}^{nl} z dz. \tag{6c}$$

Considering Equations (6a)–(6c), the first and second parts of the right-hand side result from the surface stress effect and the nonlocal elasticity, respectively. Let the upper and lower surfaces of the nanoplates have the same properties. Then, using Equations (1a) – (1c), (3a) – (3c), (4a) – (4c), (5a) – (5c), and (6a) – (6c), we obtain the bending moment resultants, as follows:

$$\left(1-\mu\nabla^2\right)\mathrm{M}_{xx}=\frac{-h^2}{2}\left(1-\mu\nabla^2\right)$$
$$\left\{\left(2u_0+\lambda_0\right)\frac{\partial^2 w}{\partial x^2}+\left(\lambda_0+\tau_0\right)\frac{\partial^2 w}{\partial y^2}\right\}-D\left(\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial^2 w}{\partial y^2}\right), \quad (7a)$$

$$\left(1-\mu\nabla^2\right)\mathrm{M}_{yy}=\frac{-h^2}{2}\left(1-\mu\nabla^2\right)\left\{\left(2u_0+\lambda_0\right)\frac{\partial^2 w}{\partial y^2}+\right.$$
$$\left.\left(\lambda_0+\tau_0\right)\frac{\partial^2 w}{\partial x^2}\right\}-D\left(\frac{\partial^2 w}{\partial y^2}+\upsilon\frac{\partial^2 w}{\partial x^2}\right), \quad (7b)$$

$$\left(1-\mu\nabla^2\right)\mathrm{M}_{xy}=\frac{-h^2}{2}\left(1-\mu\nabla^2\right)\left\{\left(2u_0-\tau_0\right)\frac{\partial^2 w}{\partial x\partial y}\right\} \quad (7c)$$
$$-D\left(1-\upsilon\right)\frac{\partial^2 w}{\partial x\partial y},$$

where $D = Eh^3 / 12(1-\upsilon^2)$ is the flexural rigidity of the nanoplates. To derive the equilibrium equations of the classical plate theory, it is necessary to find the shear forces (S_x, S_y) acting on the cross sections; both lower and upper surfaces of the nanoplates must be considered, as follows (Assadi, 2013):

$$S_x=\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}+\sigma_{xz}^{s+}+\sigma_{xz}^{s-} \quad (8a)$$
$$=\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}+2\tau_0\frac{\partial w}{\partial x},$$

$$S_y=\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x}+\sigma_{yz}^{s+}+\sigma_{yz}^{s-} \quad (8b)$$
$$=\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x}+2\tau_0\frac{\partial w}{\partial y}.$$

From the sum of the resultant forces in the transverse direction, which contain the external loading $q(x, y)$ and reaction force of an elastic medium, then, applying Equations (7a)–(7c) and (8a)–(8b), the following governing equation can be obtained:

$$\frac{-\mu h^2}{2}\left(2u_0+\lambda_0\right)\nabla^6 w+\left\{D+\frac{h^2}{2}\left(2u_0+\lambda_0\right)+2\tau_0\mu+\mu G_b\right\}\nabla^4 w$$
$$-\left(2\tau_0+G_b+\mu k_w\right)\nabla^2 w+k_w w=\left(1-\mu\nabla^2\right)q, \quad (9)$$

where k_w and G_b are the Winkler foundation stiffness and the shear layer stiffness of the foundation, respectively.

Mindlin Plate Theory

The Mindlin plate theory is known as the first-order shear deformation theory, and is based on the displacement field that can be expressed as

$$u(x,y,z)=u(x,y)+z\psi_x(x,y), \quad (10a)$$
$$v(x,y,z)=v(x,y)+z\psi_y(x,y), \quad (10b)$$
$$w(x,y,z)=w(x,y), \quad (10c)$$

where $u(x, y)$, $v(x, y)$ and $w(x, y)$ are the displacement components of the material point at the middle plane, and ψ_x and ψ_y are the rotations about the y and x axes, respectively. Furthermore, the strain can be expressed as

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}+z\frac{\partial\psi_x}{\partial x}, \quad (11a)$$
$$\varepsilon_{yy}=\frac{\partial v}{\partial y}+z\frac{\partial\psi_y}{\partial y}, \quad (11b)$$
$$\varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{z}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right), \quad (11c)$$
$$\varepsilon_{xz}=\frac{1}{2}\left(\frac{\partial w}{\partial x}+\psi_x\right), \quad (11d)$$
$$\varepsilon_{yz}=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\psi_y\right). \quad (11e)$$

The bending moment and shear force resultants, including surface stress and nonlocal effects, can be written by the following equations:

$$\mathrm{M}_{xx}=\left(\sigma_{xx}^{s+}-\sigma_{xx}^{s-}\right)\frac{h}{2}+\int_{-h/2}^{h/2}\sigma_{xx}^{nl}zdz, \quad (12a)$$
$$\mathrm{M}_{yy}=\left(\sigma_{yy}^{s+}-\sigma_{yy}^{s-}\right)\frac{h}{2}+\int_{-h/2}^{h/2}\sigma_{yy}^{nl}zdz, \quad (12b)$$
$$\mathrm{M}_{xy}=\left(\sigma_{xy}^{s+}-\sigma_{xy}^{s-}\right)\frac{h}{2}+\int_{-h/2}^{h/2}\sigma_{xy}^{nl}zdz, \quad (12c)$$
$$\mathrm{M}_{yx}=\left(\sigma_{yx}^{s+}-\sigma_{yx}^{s-}\right)\frac{h}{2}+\int_{-h/2}^{h/2}\sigma_{yx}^{nl}zdz, \quad (12d)$$
$$S_x=\left(\sigma_{xz}^{s+}+\sigma_{xz}^{s-}\right)+k^2\int_{-h/2}^{h/2}\sigma_{xz}^{nl}dz, \quad (12e)$$
$$S_y=\left(\sigma_{yz}^{s+}+\sigma_{yz}^{s-}\right)+k^2\int_{-h/2}^{h/2}\sigma_{yz}^{nl}dz, \quad (12f)$$

where k^2 is the shear correction factor. Consequently, by using Equations (1a) – (1e), (3a) – (3f), (10a)– (10c), (11a) – (11e), and (12a) – (12f), assuming that the top and bottom surfaces have the same material properties, the bending moment and shear force resultants can be simplified to the following relations:

$$\left(1-\mu\nabla^2\right)\mathrm{M}_{xx}=\frac{h^2}{2}\left(1-\mu\nabla^2\right)\left\{\left(2u_0+\lambda_0\right)\frac{\partial\psi_x}{\partial x}+\left(\lambda_0+\tau_0\right)\frac{\partial\psi_y}{\partial y}\right\}$$
$$+D\left(\frac{\partial\psi_x}{\partial x}+\upsilon\frac{\partial\psi_y}{\partial y}\right), \tag{13a}$$

$$\left(1-\mu\nabla^2\right)\mathrm{M}_{yy}=\frac{h^2}{2}\left(1-\mu\nabla^2\right)\left\{\left(2u_0+\lambda_0\right)\frac{\partial\psi_y}{\partial y}+\left(\lambda_0+\tau_0\right)\frac{\partial\psi_x}{\partial x}\right\}$$
$$+D\left(\frac{\partial\psi_x}{\partial y}+\upsilon\frac{\partial\psi_y}{\partial x}\right), \tag{13b}$$

$$\left(1-\mu\nabla^2\right)\mathrm{M}_{xy}=\frac{h^2}{2}\left(1-\mu\nabla^2\right)\left\{u_0\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-\tau_0\frac{\partial\psi_y}{\partial x}\right\},$$
$$+\frac{1}{2}D\left(1-\upsilon\right)\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right) \tag{13c}$$

$$\left(1-\mu\nabla^2\right)\mathrm{M}_{yx}=\frac{h^2}{2}\left(1-\mu\nabla^2\right)\left\{u_0\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-\tau_0\frac{\partial\psi_x}{\partial y}\right\},$$
$$+\frac{1}{2}D\left(1-\upsilon\right)\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right) \tag{13d}$$

$$\left(1-\mu\nabla^2\right)S_x=2\tau_0\left(1-\mu\nabla^2\right)\frac{\partial w}{\partial x}+k^2Gh\left(\frac{\partial w}{\partial x}+\psi_x\right), \tag{13e}$$

$$\left(1-\mu\nabla^2\right)S_y=2\tau_0\left(1-\mu\nabla^2\right)\frac{\partial w}{\partial y}+k^2Gh\left(\frac{\partial w}{\partial y}+\psi_y\right). \tag{13f}$$

The equilibrium equations of the Mindlin plate resting on a 2-parameter elastic foundation are given as

$$\frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}+q=k_w w-G_b\nabla^2 w, \tag{14a}$$

$$\frac{\partial \mathrm{M}_{xx}}{\partial x}+\frac{\partial \mathrm{M}_{yx}}{\partial y}-S_x=0, \tag{14b}$$

$$\frac{\partial \mathrm{M}_{yy}}{\partial y}+\frac{\partial \mathrm{M}_{xy}}{\partial x}-S_y=0. \tag{14c}$$

Finally, by substituting Equations (13a) – (13f) into Equations (14a) – (14c), the governing equations of the nanoplates, including the surface and nonlocal effects based on the Mindlin plate theory, can be expressed as

$$k^2Gh\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}+\nabla^2 w\right)+2\tau_0\left(1-\mu\nabla^2\right)\nabla^2 w$$
$$+\left(1-\mu\nabla^2\right)q=\left(1-\mu\nabla^2\right)\left(k_w w-G_b\nabla^2 w\right), \tag{15a}$$

$$\frac{h^2}{2}\left(1-\mu\nabla^2\right)\left\{\left(2u_0+\lambda_0\right)\frac{\partial^2\psi_x}{\partial x^2}+\left(u_0+\lambda_0+\tau_0\right)\frac{\partial^2\psi_y}{\partial x\partial y}+\left(u_0-\tau_0\right)\frac{\partial^2\psi_x}{\partial y^2}\right\}+$$
$$D\left\{\frac{\partial^2\psi_x}{\partial x^2}+\frac{1}{2}\left(1-\upsilon\right)\frac{\partial^2\psi_x}{\partial y^2}+\frac{1}{2}\left(1+\upsilon\right)\frac{\partial^2\psi_y}{\partial x\partial y}\right\}-2\tau_0\left(1-\mu\nabla^2\right)\frac{\partial w}{\partial x}-k^2Gh\left(\frac{\partial w}{\partial x}+\psi_x\right)=0, \tag{15b}$$

$$\frac{h^2}{2}\left(1-\mu\nabla^2\right)\left\{\left(2u_0+\lambda_0\right)\frac{\partial^2\psi_y}{\partial y^2}+\left(u_0+\lambda_0+\tau_0\right)\frac{\partial^2\psi_x}{\partial x\partial y}+\left(u_0-\tau_0\right)\frac{\partial^2\psi_y}{\partial x^2}\right\}+$$
$$D\left\{\frac{\partial^2\psi_y}{\partial y^2}+\frac{1}{2}\left(1-\upsilon\right)\frac{\partial^2\psi_y}{\partial x^2}+\frac{1}{2}\left(1+\upsilon\right)\frac{\partial^2\psi_x}{\partial x\partial y}\right\}-2\tau_0\left(1-\mu\nabla^2\right)\frac{\partial w}{\partial y}-k^2Gh\left(\frac{\partial w}{\partial y}+\psi_y\right)=0. \tag{15c}$$

Note that if the surface effect is ignored (τ_0, u_0 and y_0 are all set to 0), Equations (15a) – (15c) agree with the work of Wang and Li (2012).

Solution to the Problem

In this section, the governing differential equations for the bending behavior of the nanoplates, including surface stress and nonlocal effects, have been solved by Navier's approach for simply supported boundary conditions. The simply supported boundary conditions for a rectangular plate are:-

for the classical plate:

a) $w = 0$ and $M_{xx} = 0$ at $x = 0$ and a,

b) $w = 0$ and $M_y = 0$ at $y = 0$ and b,

for the Mindlin plate:

c) $w = 0$, $M_{xx} = 0$, and ψ_y at $x = 0$ and a,

d) $w = 0$, $M_{yy} = 0$, and ψ_x at $y = 0$ and b,

where a and b are the length and width of the nanoplates, respectively. The displacement solutions can be expressed as
for the classical plate:

$$w^C=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_{mn}^C\sin\alpha_m x\sin\beta \tag{16}$$

for the Mindlin plate:

$$w^M=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_{mn}^M\sin\alpha_m x\sin\beta_n y, \tag{17a}$$

$$\psi_x=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}X_{mn}\cos\alpha_m x\sin\beta_n y, \tag{17b}$$

$$\psi_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Y_{mn} \sin\alpha_m x \cos\beta_n y\,, \tag{17c}$$

where $\alpha_m = m\pi / a$ and $\beta_n = n\pi / b$, and m and n denote the half-wave numbers in the x and y directions, respectively. From Equations (16) and (17a) – (17c), the simply supported boundary conditions for the nanoplates will be satisfied automatically. Furthermore, the external load can be also expressed by the Fourier series as

$$q = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} \sin\alpha_m x \sin\beta_n y\,. \tag{18}$$

The Displacement Coefficient of the Classical Plate

Substituting Equations (16) and (18) into Equation (9), the displacement coefficient of the classical plate is represented by

$$w_{mn}^{C} = \frac{q_{mn}\left\{1+\mu\left(\alpha_m^2+\beta_n^2\right)\right\}}{C_1\left(\alpha_m^2+\beta_n^2\right)^3 + C_2\left(\alpha_m^2+\beta_n^2\right)^2 + C_3\left(\alpha_m^2+\beta_n^2\right) + k_w}, \tag{19}$$

where

$C_1 = \mu h^2(2u_0+\lambda_0)/2$, $C_2 = D + h^2(2u_0+\lambda_0)/2 + 2\mu\tau_0 + \mu G_b$, and $C_3 = 2\tau_0 + G_b + \mu k_w$.

From Equation (19), by setting λ_0, u_0, τ_0 and μ to 0, the coefficient of the traditional solution is obtained, which does not include the effects of surface stress and nonlocal elasticity:

$$w_{mn}^{CO} = \frac{q_{mn}}{D\left(\alpha_m^2+\beta_n^2\right)^2 + G_b\left(\alpha_m^2+\beta_n^2\right) + k_w}\,. \tag{20}$$

The displacement ratio (R_{mn}^{C}) in the classical plate theory is defined as

$$R_{mn}^{C} = \frac{w_{mn}^{C}}{w_{mn}^{CO}}\,. \tag{21}$$

The Displacement Coefficient of the Mindlin Plate

Substituting Equations (17a) – (17c) and (18) into Equations (15a) – (15c) generates a linear system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{Bmatrix} w_{mn}^{M} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} q_{mn} \\ 0 \\ 0 \end{Bmatrix} \tag{22}$$

where $a_{ij}, i, j = 1, 2, 3$ as given in the Appendix. Then, by solving the above equation, the displacement coefficient of the Mindlin plate can be written as

$$w_{mn}^{M} = \frac{q_{mn}\left(a_{22}a_{33}-a_{23}a_{32}\right)}{a_{11}\left(a_{22}a_{33}-a_{23}a_{32}\right)+a_{12}\left(a_{23}a_{31}-a_{21}a_{33}\right)+a_{13}\left(a_{21}a_{32}-a_{22}a_{31}\right)}\,. \tag{23}$$

The displacement coefficient without the effects of surface stress and nonlocal elasticity can be calculated by setting λ_0, u_0, τ_0, and μ in the expressions of a_{ij} to 0. By excluding the effect of size-dependent behavior in Equation (23), the traditional coefficient w_{mn}^{MO} is obtained. Furthermore, the displacement ratio (R_{mn}^{M}) in the Mindlin plate theory is also defined as

$$R_{mn}^{M} = \frac{w_{mn}^{M}}{w_{mn}^{MO}}\,. \tag{24}$$

Numerical Example and Discussion

For the present study, an aluminum nanoplate is embedded in an elastic medium with the following material properties (Assadi, 2013); $E = 68.5\ GPa$, $v = 0.35$, $\tau_0 = 0.910\ N/m$, $\lambda_0 = 5.26\ N/m$ and $u_0 = 2.26\ N/m$. It is assumed that $h = 2\ nm$, the shear correction factor is $k^2 = \pi^2 / 12$, the elastic medium properties (Wang and Li, 2012) are the Winkler foundation modulus, $k_w = 1.13 \times 10^{18}\ Pa/m$, and the stiffness of the shearing layer, $G_b = 2\ N/m$. Nanoplates with the effects of either nonlocal elasticity or surface stress are considered separately. Meanwhile, nanoplates with a combination of these effects are also examined.

The relationships between the displacement ratio (R_{mn}^{C} or R_{mn}^{M}) and the non-dimensional length (a / h) of nanoplates with different half-wave numbers are shown in Figure 2(a) – (h) and Figure 3(a) – (h) for the classical and Mindlin plate theory, respectively. It is apparent that all graphs show a similar trend. For low values of the non-dimensional length (a / h), there is a

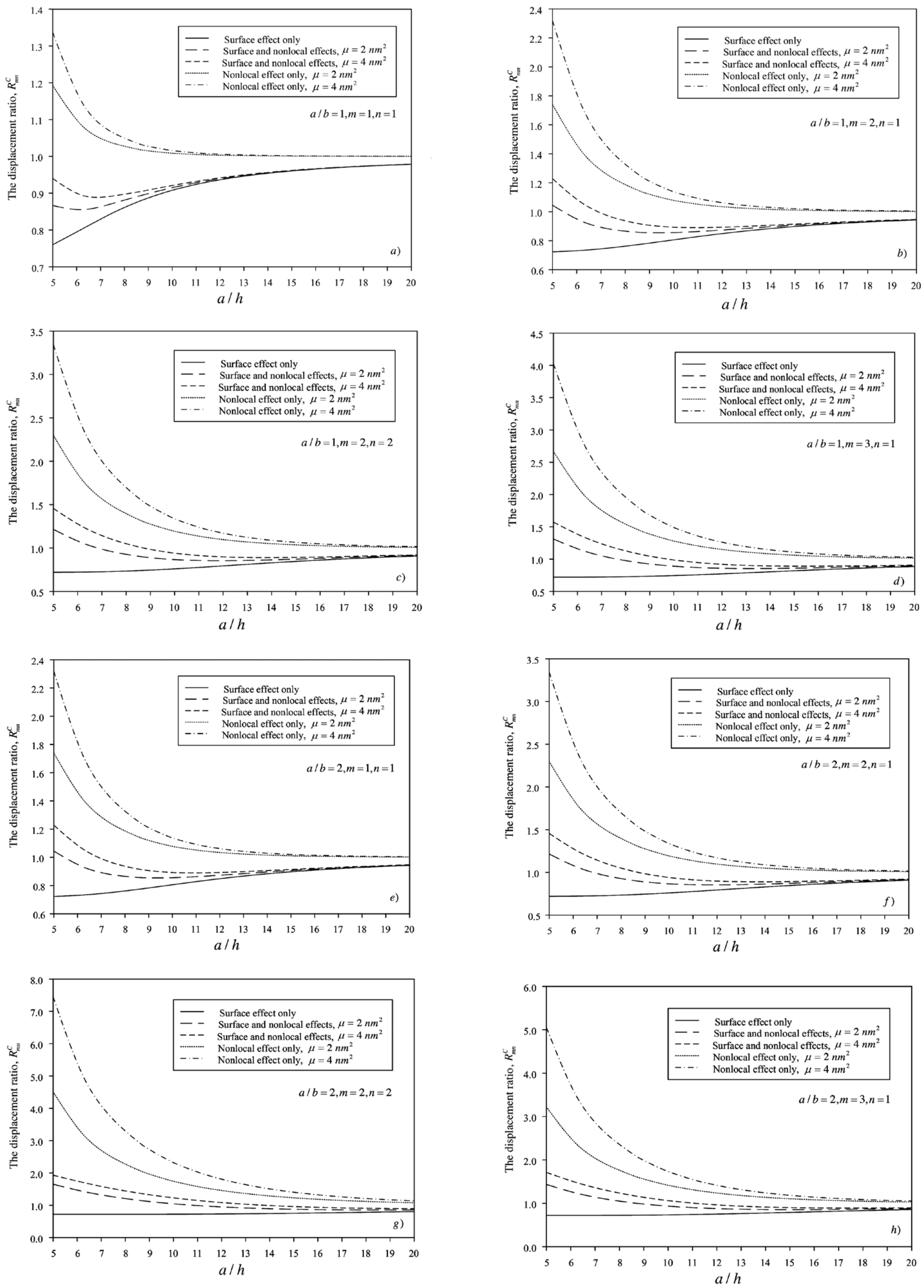

Figure 2. (a)-(h). Variation of the displacement ratio as a non-dimensional length of nanoplates for the classical plate with different length-to-width ratio and half-wave numbers

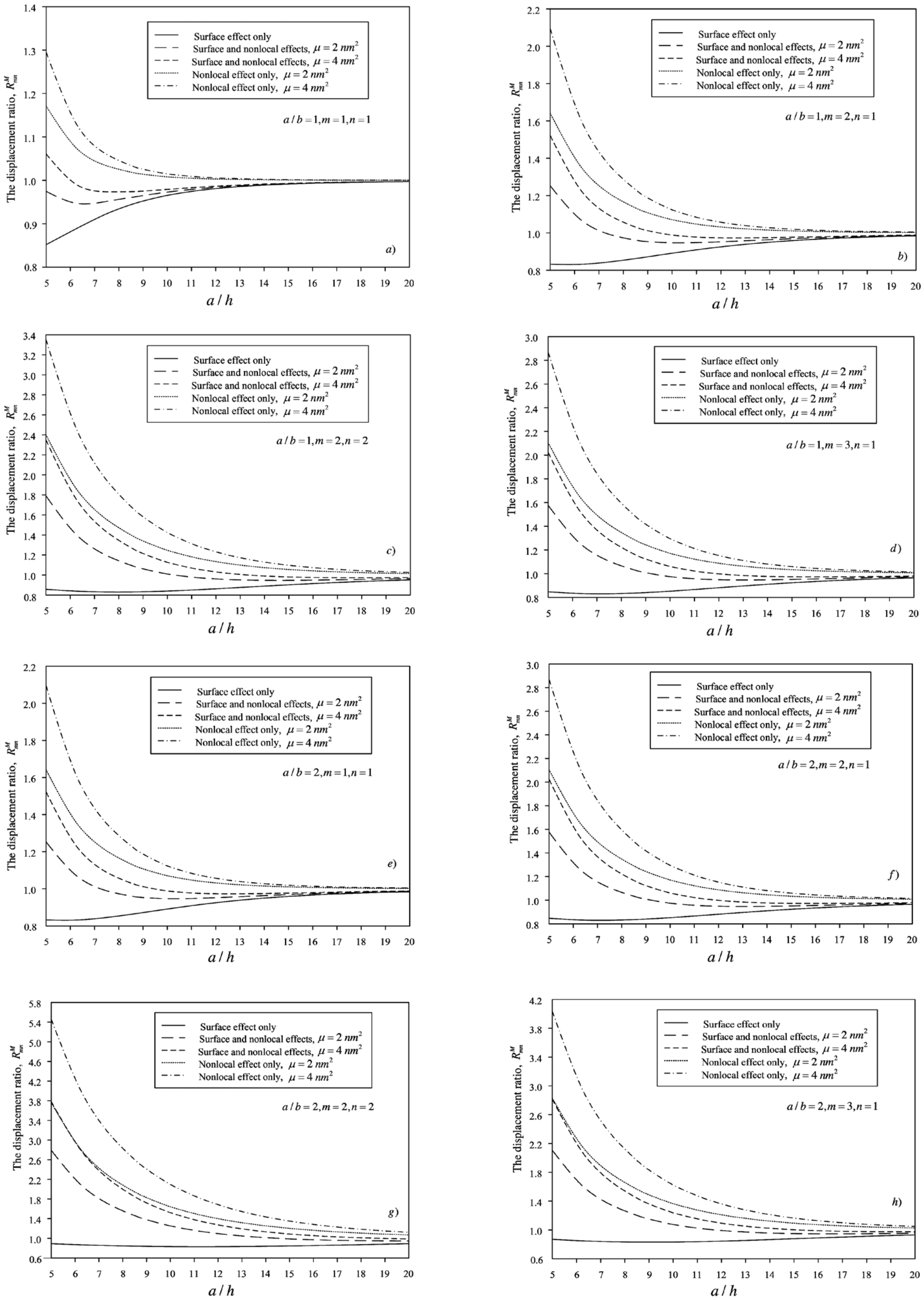

Figure 3(a)-(h). Variation of the displacement ratio as a non-dimensional length of nanoplates for the Mindlin plate with different length-to-width ratio and half-wave numbers

large difference in the displacement ratio between each small-scale effect, whereas the ratio gradually converges to 1, namely to the solution of the classical plate theory, with increasing the non-dimensional length (a / h). Based on these results, both surface stresses and nonlocal elasticity have a greater influence on nanoplates with smaller sizes. Furthermore, if only the effect of surface stress is considered, the displacement ratio will always be less than 1. This means that the surface stresses improve the bending stiffness of nanoplates. In contrast, the displacement ratio will always be higher than 1 when only the effect of nonlocal elasticity is taken into account. It is evident that the influence of the nonlocal parameter is predominant when high values of the half-wave numbers (m, n) are examined. From the aforementioned results, it is interesting to highlight the incorporated effects of surface stress and nonlocal elasticity from the proposed formulations. Due to the opposite results of each individual effect, the displacement ratio tapers off when the combination of both effects is considered.

To evaluate the influence of shear deformation on nanoplates, Figure 3(a–h), plotted according to the Mindlin plate theory, can be compared with Figure 2(a-h), based on the classical plate theory. It is obvious that the displacement ratio R_{mn}^{M} seems to have a value less than the displacement ratio R_{mn}^{C} when the same parameters, the half-wave numbers (m, n), the non-dimensional length (a / h), and the parameters of small scale are concerned. This implies that the small scale of nanoplates has less effect on the Mindlin plates than on classical plates.

To demonstrate the effect of residual surface tension, Figure 4 and Figure 5 are established to compare the results between including and excluding the residual surface effects for both the classical and Mindlin plates, respectively. It is evidently observed that including the residual surface has had a significant effect in increasing the displacement ratios R_{mn}^{C} and R_{mn}^{M}. The effect on the bending of nanoplates has been also reported by Miller and Shenoy (2000).

Finally, the relationships between the displacement ratio and the elastic foundation parameters are presented in Figures 6 and 7 and Figures 8 and 9 for the classical and Mindlin plate theory, respectively. A decreasing trend can be clearly observed in the displacement ratios R_{mn}^{C} and R_{mn}^{M} as the values of the Winkler foundation stiffness (k_w) and shear layer stiffness (G_b) increase.

Conclusions

In this study, the governing equations for the bending analysis of nanoplate models, including both surface stress and nonlocal elasticity effects, are derived. Thin and moderately thick nanoplates

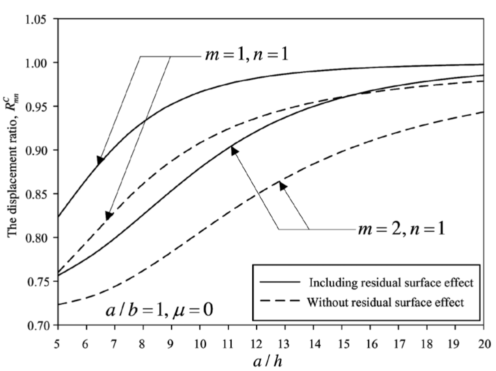

Figure 4. The effect of residual surface tension on the displacement ratio for the classical plate with different half-wave numbers

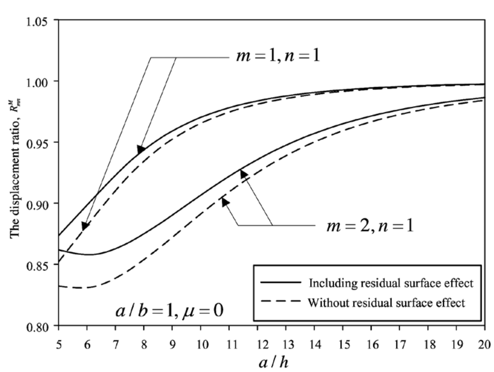

Figure 5. The effect of residual surface tension on the displacement ratio for the Mindlin plate with different half-wave numbers

and the 2-parameter elastic foundation are also considered in the formulations. The analytical solution from the present study could be applied as a benchmark for comparison with other numerical solutions in future work.

The results of this investigation can be summarized as follows:

- When the size of a structure is in the order of nanometers, the surface stress and nonlocal elasticity have a great significance on the bending behavior, which cannot be ignored.
- If the surface stress effect is examined in particular, the displacement ratio will always be less than 1. This means that the surface stresses improve the bending stiffness of nanoplates. In contrast, the displacement ratio will always be higher than 1 when only the effect of nonlocal elasticity is taken into account.
- Because opposite results of each individual effect are observed, the displacement ratio will have a value between those obtained from each effect when the combined surface stress and nonlocal elasticity are considered.
- The small scale of nanoplates has less effect on the Mindlin plates than on classical plates.
- The effect of residual surface tension leads to an improvement in the bending stiffness of nanoplates.
- The influence of surface stress and nonlocal elasticity decreases when nanoplates rest on an elastic foundation.

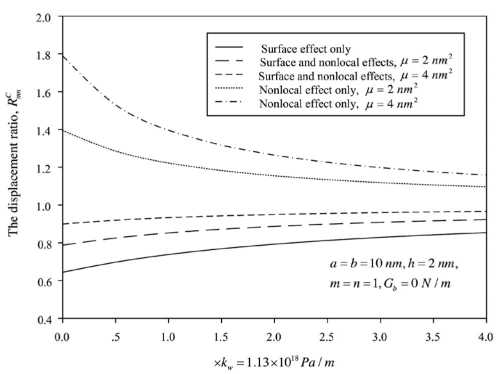

Figure 6. Variation of the displacement ratio as the Winkler foundation stiffness for the classical plate

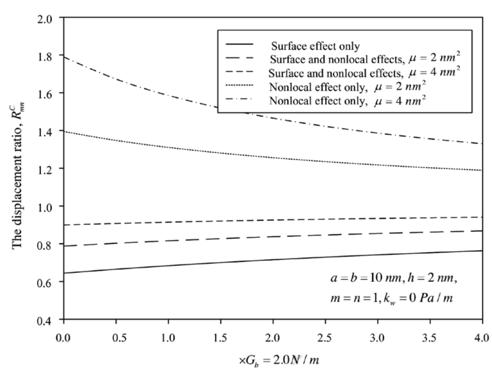

Figure 7. Variation of the displacement ratio as the shear layer stiffness for the classical plate

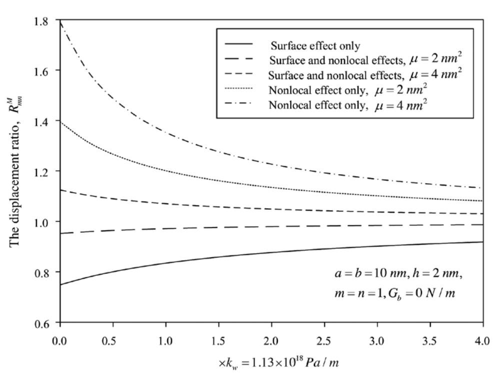

Figure 8. Variation of the displacement ratio as the Winkler foundation stiffness for the Mindlin plate

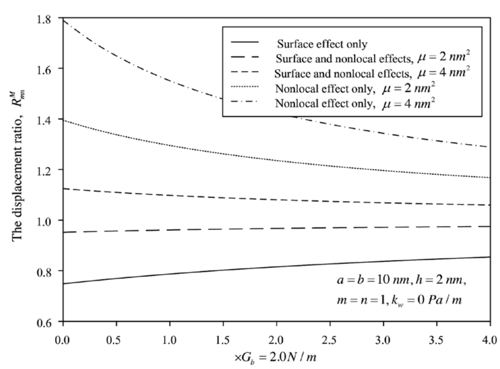

Figure 9. Variation of the displacement ratio as the shear layer stiffness for the Mindlin plate

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Appendix

Coefficients a_{11} through a_{33}:

$$a_{11} = \frac{\left(k^2Gh + 2\tau_0\right)\left(\alpha_m^2 + \beta_n^2\right) + 2\tau_0\mu\left(\alpha_m^2 + \beta_n^2\right)^2}{1 + \mu\left(\alpha_m^2 + \beta_n^2\right)} + G_b\left(\alpha_m^2 + \beta_n^2\right) + k_w, \quad \text{(A1)}$$

$$a_{12} = \frac{k^2Gh\alpha_m}{1 + \mu\left(\alpha_m^2 + \beta_n^2\right)}, \quad \text{(A2)}$$

$$a_{13} = \frac{k^2Gh\beta_n}{1 + \mu\left(\alpha_m^2 + \beta_n^2\right)}, \quad \text{(A3)}$$

$$a_{21} = 2\tau_0\left\{\alpha_m + \mu\left(\alpha_m^3 + \alpha_m\beta_n^2\right)\right\} + k^2Gh\alpha_m, \quad \text{(A4)}$$

$$a_{22} = \frac{h^2}{2}\left(2u_0 + \lambda_0\right)\left\{\alpha_m^2 + \mu\left(\alpha_m^4 + \alpha_m^2\beta_n^2\right)\right\} + \frac{h^2}{2}\left(u_0 - \tau_0\right)\left\{\beta_n^2 + \mu\left(\beta_n^4 + \alpha_m^2\beta_n^2\right)\right\} + D\left\{\frac{(1-\upsilon)}{2}\beta_n^2 + \alpha_m^2\right\} + k^2Gh, \quad \text{(A5)}$$

$$a_{23} = \frac{h^2}{2}\left(u_0 + \lambda_0 + \tau_0\right)\left\{\alpha_m\beta_n + \left(\alpha_m^3\beta_n + \alpha_m\beta_n^3\right)\right\} + \frac{D}{2}(1+\upsilon)\alpha_m\beta_n, \quad \text{(A6)}$$

$$a_{31} = 2\tau_0\left\{\beta_n + \mu\left(\alpha_m^2\beta_n + \beta_n^3\right)\right\} + k^2Gh\beta_n, \quad \text{(A7)}$$

$$a_{32} = a_{23}, \quad \text{(A8)}$$

$$a_{33} = \frac{h^2}{2}\left(2u_0 + \lambda_0\right)\left\{\beta_n^2 + \mu\left(\beta_n^4 + \alpha_m^2\beta_n^2\right)\right\} + \frac{h^2}{2}\left(u_0 - \tau_0\right)\left\{\alpha_m^2 + \mu\left(\alpha_m^4 + \alpha_m^2\beta_n^2\right)\right\} + D\left\{\frac{(1-\upsilon)}{2}\alpha_m^2 + \beta_n^2\right\} + k^2Gh. \quad \text{(A9)}$$

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