

FINITE ELEMENT METHOD WITH EDGE ELEMENTS FOR ANISOTROPIC WAVEGUIDE PROBLEMS

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Abstract

This paper presents the finite element analysis of eigenmodes in arbitrarily shaped anisotropic waveguides by using high-order edge elements. The method uses the variational expression of the propagation constant, which is expressed in terms of the transverse components. This variational expression is suited for analyzing dispersive waveguides, in which the medium varies with a frequency such as ferrite. The shape function uses an edge element which has the field components along the edge as unknown parameters which satisfy the tangential continuity condition of the fields across material interfaces. In addition, the use of edge elements can reduce the number of unknown parameters and the time of computation. Hence, the method can yield good accuracy compared with a method that uses a vector shape function. The result of comparison is shown in the example of a dielectric-loaded waveguide. To verify the usefulness of this method, an example of a circular waveguide partially loaded with ferrite where the applied static magnetic field in the longitudinal direction of propagation is demonstrated. The computational results show that the dispersion curves of this method satisfy those of other numerical methods in the past and have a good accuracy.

Keywords: Finite element method, gyromagnetic waveguide, ferrite

Introduction

The finite element method (FEM) has been widely used to analyze waveguide devices for the last 3 decades, because it is an effective and accurate method that is appropriate for complex structures and material properties. On the other hand, the analytical method is limited to a few simple geometries and materials such as rectangular and circular waveguides. Therefore,

a finite element method for analysis and design of these devices or of new microwave devices has been developed. In the past, research has focused on simulation of these devices. In particular, ferrite loaded waveguides or gyromagnetic waveguides have been investigated by using vector finite element methods. The ferrite loaded waveguide analyzed

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by using the FEM with a quadrilateral element (Dillon *et al.*, 1993) had problems using this element and it is not the typical triangular element. A three-component variational formulation with edge element has been investigated for resonant frequencies of ferrite loaded cavities (Wang and Ida, 1992); however, this approach is not suitable for waveguide analysis with frequencies as the eigenvalues. The solutions of ferrite loaded waveguides where an applied dc magnetic field is parallel or transverse to the direction of propagation has been proposed by using a vector finite element method (Anderson and Cendez, 1995). This method still has a large eigenvalue equation, which leads to lost computational time. Research has developed new structures of microwave devices at high frequencies using the finite element method (Dillon and Gibson, 2000, 2002).

In this paper, we propose the FEM formulation by using a variational expression first proposed by Angkaew *et al.* (1987) with edge elements. Edge elements are known for eliminating spurious modes that are non-physical modes, and also give continuities of tangential electric and magnetic fields between elements. Then, these functions are efficient vector shape functions in the present. This method is presented for the first time by improvement of the FEM (Angkaew *et al.*, 1987). In order to test the efficiency and accuracy of this approach, a dielectric-loaded waveguide is considered. The next examples of waveguides considered are gyromagnetic waveguides in order to find out the dispersion characteristics.

Finite Element Formulation

We consider an anisotropic waveguide with an arbitrary cross section in the x-y plane and the waveguide is assumed to be uniform in the longitudinal z-axis. The kind of anisotropic material in this paper is ferrite, in which the magnetic induction B is not parallel to the magnetic field H . A ferrite with an applied dc magnetic field can be represented with permeability tensor and permittivity scalar. This can be expressed as:

$$[\mu] = \mu_0 \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix} \quad (1)$$

and

$$\varepsilon = \varepsilon_0 \varepsilon_r \quad (2)$$

where μ_0 is the free-space permeability, ε_0 is the free-space permittivity, and ε_r is the relative permittivity of the material. When the applied dc magnetic field H_0 is parallel to the longitudinal z-axis, the elements of tensor in (1) is given by:

$$\mu = 1 + \frac{\gamma^2 H_0 M_s}{(\gamma H_0)^2 - \omega^2}, \quad (3a)$$

$$\kappa = \frac{\omega \gamma M_s}{(\gamma H_0)^2 - \omega^2}, \quad (3b)$$

$$\mu_{zz} = 1, \quad (3c)$$

where ω is the angular frequency, and M_s and γ have their meanings from Anderson and Cendez, (1995). The material whose permeability and permittivity are in the form of Eqns. (1) and (2) is called gyromagnetic. With a time dependence of the form $\exp(j\omega t)$ being implied, the electric field and magnetic field can be given by:

$$E(x, y, z) = (E_t(x, y) + a_z E_z(x, y)) \exp(j\beta z) \quad (4)$$

$$H(x, y, z) = (H_t(x, y) + a_z H_z(x, y)) \exp(j\beta z) \quad (5)$$

where E_t and H_t are the transverse components of the electric and magnetic fields, E_z and H_z are the longitudinal components, a_z is the unit vector in the z direction, and β is the phase constant. The solutions of propagation modes in a waveguide, which satisfy Maxwell's equations and boundary conditions (BC), can be expressed as variational (Angkaew *et al.*, 1987):

$$\beta(E_t, H_t) = \frac{\sum_{\varepsilon} A(E_t, H_t)}{\sum_{\varepsilon} B(E_t, H_t)} \quad (6)$$

where

$$\begin{aligned}
 A(E_t, H_t) = & \iint_e [E_t^* \cdot \omega \epsilon \cdot E_t + H_t^* \cdot \omega \mu_t \cdot H_t \\
 & - \frac{1}{\omega \mu_{zz}} (\nabla_t \times E_t) \cdot (\nabla_t \times E_t)^* \\
 & - \frac{1}{\omega \epsilon} (\nabla_t \times H_t) \cdot (\nabla_t \times H_t)^*] dx dy \quad (7) \\
 B(E_t, H_t) = & \iint_e [a_z \cdot (E_t^* \times H_t - H_t^* \times E_t)] dx dy.
 \end{aligned}$$

Here, \sum_e is a summation of all elements on the cross section of the waveguide, and the asterisk denotes a complex conjugate. Obviously, the above equation is for the transverse components of the electric and magnetic fields, and is suitable for calculating the guided-mode in a waveguide. The advantage of Eqn. (6) is that it gives directly the phase constants.

The trial functions of transverse electric and magnetic fields used in (6) have to satisfy the boundary conditions:

for BC between elements:

$$\begin{aligned}
 a_z \cdot (n \times E_t) \\
 a_z \cdot (n \times H_t)
 \end{aligned}$$

for BC between wall and element:

$$\begin{aligned}
 a_z \cdot (n \times E_t) = 0 \\
 a_z \cdot (n \times H_t) = 0
 \end{aligned}$$

Here n denotes an outward normal unit vector at the boundary condition. These trial functions in each element can be expressed as the term of a vector shape function and unknown parameters:

$$E_t = \sum_{m=1}^M N_m(x, y) \phi_m \quad (8)$$

$$H_t = \sum_{m=1}^3 N_m(x, y) \psi_m \quad (9)$$

where M is a number of unknown parameters in one element, $N_m(x, y)$ is a vector shape function, ϕ_m and ψ_m are unknown parameters of the transverse electric and magnetic fields in one element, respectively.

Substituting (8) and (9) into (6) and (7)

and then taking a surface integral in each element, we obtain:

$$A = \{\xi\}_e^{*T} [P]_e \{\xi\} \quad (10)$$

$$B = \{\xi\}_e^{*T} [Q]_e \{\xi\} \quad (11)$$

where superscript T is the transpose operator, $\{\xi\}$ is the column vector which consists of unknown parameters in each element, $[P]_e$ and $[Q]_e$ are square matrices which are made of permittivity, permeability and a vector shape function for each element. After summing A and B according to (10) and (11) for all elements on the waveguide cross section and imposing boundary conditions, we obtain the following eigenvalue problem that provides a solution for propagation constant β and field distribution in the transverse plane $\{\xi\}$:

$$[Q] \{\xi\} = \frac{1}{\beta} [P] \{\xi\} \quad (12)$$

where $\{\xi\}$ is the column vector that contains unknown parameters of transverse electric and magnetic fields on the waveguide cross section, matrix $[P]$ is the Hermitian matrix, and matrix $[Q]$ is a symmetric matrix.

Vector Shape Function with Edge Elements

Edge elements are vector shape functions that usually are used in the finite element method. These functions have unknown parameters at the edges of triangular elements. From this reason it is convenient to impose boundary conditions between elements or between elements and the waveguide wall. Moreover, there are no spurious solutions of guided modes. Nowadays, there is much research that improves the accuracy of the finite element analysis. One method of doing so is to use higher order vector shape functions such as research from Koshiba *et al.* (1994) and Savage and Peterson (1996) because the lowest order vector shape function is, in general, insufficient. In this section, we will show a constant edge element and

linear edge element, which are vector shape functions.

Constant Edge Element

A constant edge element (Jin, 1993) is the lowest order vector shape function. The essential thing of this shape function is that unknown parameters of transverse electric components or transverse magnetic components are at the edge of the triangular element and are also tangential components. There are no unknown parameters of normal components. As shown in Figure 1, the triangular element consists of 3 unknown parameters of electric field ϕ_{ik} ($k = 1,2,3$) and magnetic field ψ_{ik} ($k = 1,2,3$) respectively, and 3 nodes because of the variational expression in (6), so there are a total of 6 unknown parameters per element. Therefore, this shape function always has continuity of tangential electric fields or magnetic fields between elements, but has discontinuity of normal components between elements. It is given by:

$$\mathbf{N}_1 = (L_1 \nabla L_2 - L_2 \nabla L_1) l_1 \quad (13a)$$

$$\mathbf{N}_2 = (L_2 \nabla L_3 - L_3 \nabla L_2) l_2 \quad (13b)$$

$$\mathbf{N}_3 = (L_3 \nabla L_1 - L_1 \nabla L_3) l_3 \quad (13c)$$

where l_1 , l_2 and l_3 are the length of the side between node 1-2, node 2-3 and node 3-1 respectively, and ∇ is the gradient operator. For the area coordinates L_k ($k = 1,2,3$), they are given by:

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{1}{2A_e} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \quad (14)$$

The area of the element A_e is given by:

$$A_e = 0.5 \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad (15)$$

and coefficients a_k, b_k, c_k are given by:

$$a_k = x_l y_m - x_m y_l \quad (16)$$

$$b_k = y_m - y_l \quad (17)$$

$$c_k = x_m - x_l \quad (18)$$

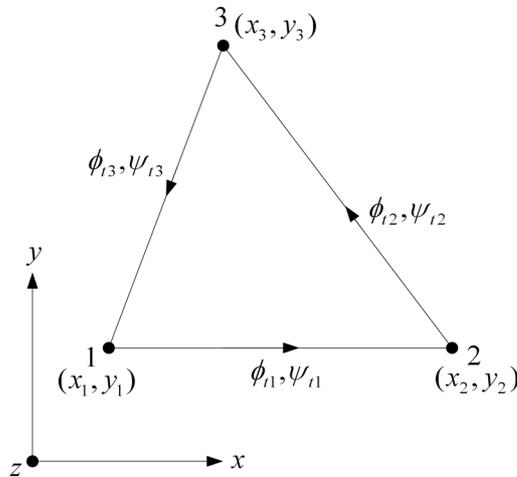


Figure 1. Unknown parameters on each side of a triangular element for a constant edge element

Here, the subscripts k, l, m always progress modulo 3 around the 3 vertices of the triangle, points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) are the Cartesian coordinates of the corner points 1 to 3 of the triangular element. After substituting vector shape function (13) into (8) and (9), one can get the following equations:

$$E_t = \sum_{m=1}^3 N_m(x, y)\phi_m \quad (19)$$

$$H_t = \sum_{m=1}^3 N_m(x, y)\psi_m \quad (20)$$

Linear Edge Element

The linear edge element is a high-order vector shape function and has been developed for the analysis of waveguides and other microwave devices (Koshiba *et al.*, 1994; Savage and Peterson, 1996). Much research has proved that a vector finite element method with this function for waveguide problems provided accurate results of computation. Of the various formulations, a vector finite element method uses either a full vector electric field or a full vector magnetic field, but in this paper we use a different type of vector finite element method as

a variational function (6) with both vector fields.

As shown in Figure 2, the element is composed of 6 unknown parameters of tangential electric field ϕ_{ik} ($k = 1, 2, \dots, 6$) and magnetic field ψ_{ik} ($k = 1, 2, \dots, 6$) respectively, and of 2 normal electric fields ϕ_{nk} ($k = 1, 2$) and magnetic fields ψ_{nk} ($k = 1, 2$) respectively. Therefore, there are 16 unknown parameters in each element. The vector shape functions for tangential components are given by:

$$N_1 = (L_1 \nabla L_2) l_1 \quad (21)$$

$$N_2 = (L_2 \nabla L_3) l_2 \quad (22)$$

$$N_3 = (L_3 \nabla L_1) l_3 \quad (23)$$

$$N_4 = (-L_2 \nabla L_1) l_1 \quad (24)$$

$$N_5 = (-L_3 \nabla L_2) l_2 \quad (25)$$

$$N_6 = (-L_1 \nabla L_3) l_3, \quad (26)$$

and for normal components are given by:

$$g_1 = (-L_2 L_3 \nabla L_1) l_2 \quad (27)$$

$$g_2 = (-L_2 L_1 \nabla L_3) l_1, \quad (28)$$

when substituting vector shape functions (21) to (28) into (8) and (9), one can get the following

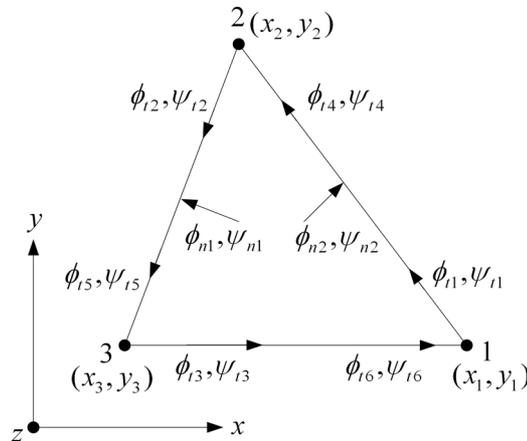


Figure 2. Unknown parameters on each side of a triangular element for a linear edge element

equations:

$$\mathbf{E}_t = \sum_{m=1}^6 \mathbf{N}_m(x, y) \phi_{tm} + \mathbf{g}_1(x, y) \phi_{n1} + \mathbf{g}_2(x, y) \phi_{n2} \quad (29)$$

$$\mathbf{H}_t = \sum_{m=1}^6 \mathbf{N}_m(x, y) \psi_{tm} + \mathbf{g}_1(x, y) \psi_{n1} + \mathbf{g}_2(x, y) \psi_{n2} \quad (30)$$

Numerical Results and Discussion

The Accuracy of FEM with Edge Element

In this section, numerical results of waveguide problems based on a finite element method with constant edge element and linear edge element are presented. This approach is applied to find the propagation constants and the corresponding field distribution of waveguide modes. In order to verify the accuracy of this approach, the well-known half-filled dielectric load waveguide as shown in Figure 3 is considered. Such a waveguide was widely used to test for accuracy, such as the research from Koshiba *et al.* (1994). The cross section of the waveguide has a size $a \times 2a$ bounded by a perfect electric conductor, and half of the waveguide is filled with a dielectric whose relative permittivity is 2.25. The boundary of the waveguide is assumed to be a perfect electric conductor. So the necessary boundary condition is that the tangential components of the electric field are zero. All quantities related to

length are normalized with the free space wave number, k_0 . In order to compare our results with previously published literature (Angkaew, 1989) and test the accuracy, Table 1 shows the results of computing β / k_0 at $k_0 a = 4.0$ in the first 4 modes of this waveguide. There is a matrix size of about 384 and eigenvalues of Eqn. (12) correspond with phase constant β . The first 4 modes of the guide are LSE_{10} , LSM_{11} , LSE_{11} , and LSE_{20} , respectively. In this case, our results agree with analytical solutions and have a percentage of error less than 0.5, if a linear edge element is applied. Moreover, the propagation constants computed by this approach do not show spurious modes. The constant edge element applied in this finite element method provides a high percentage of error of 3 vector shape functions because it has only 6 unknowns per element and is the lowest order shape function. The more unknowns that are used in each element, the less the percentage of error one obtains, but there are more complicated vector shape functions. Figure 4 shows the percentage of error for computing β of the fundamental LSE_{10} mode in this waveguide, where unknowns here are the number of degrees of freedom and they range from 50 to 800. From this Figure, it is confirmed that a finite element method with a linear edge element can provide more accurate results than a finite element method with a constant edge element.

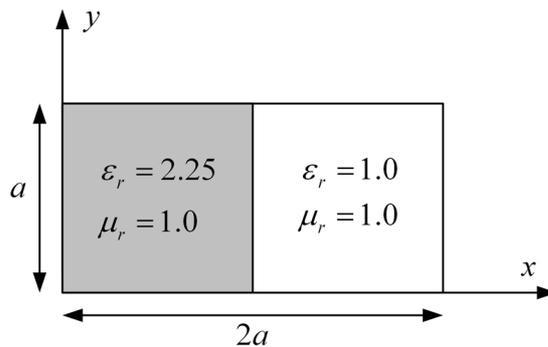


Figure 3. Cross section of a dielectric-loaded waveguide

Table 1. Comparison of the analytical solution of β / k_0 at $k_0 a = 4.0$ in the first 4 modes of a dielectric-loaded waveguide with numerical solution by using the finite element method

Modes	Analytical solution	Numerical solution by vector finite element method			
		Linear vector shape function* (12 unknowns)	Quadratic vector shape function* (24 unknowns)	Constant edge element (6 unknowns)	Linear edge element (16 unknowns)
LSE_{10} % error	1.35913	1.35508 0.30%	1.35903 0.0074%	1.37467 1.143%	1.36014 0.0743%
LSM_{11} % error	1.22746	1.21560 0.97 %	1.22691 0.045%	1.26171 2.790%	1.22990 0.19%
LSE_{11} % error	1.10923	1.08726 2.0%	1.10792 0.12%	1.14680 3.387%	1.11280 0.321%
LSE_{20} % error	0.92412	0.89710 2.9%	0.92159 0.27%	0.95874 3.746%	0.92730 0.344%

* : Ph.D. dissertation, Angkaew, (1989)

An Example of Anisotropic Waveguide

The validity of this method for the case of anisotropic waveguides has been investigated in the gyromagnetic waveguide. This waveguide is filled with permeability in terms of tensor form (1) and permittivity in term of scalar form (2). Three numerical examples of gyromagnetic waveguides have been carried out.

Figure 5 shows the cross section of a circular gyromagnetic waveguide bounded by either an electric wall or a magnetic wall and the radius is a. The electric or magnetic wall always causes different dispersion characteristics. By subdividing a cross section of the waveguide into 208 triangles for the FEM with constant edge element and 75 triangles for the FEM with linear edge element, and imposing boundary conditions between a wall and elements, we can get 304 and 360 degrees of freedom for the FEM with constant edge element and linear edge element, respectively. In order to compare our results with a previously published finite element approach that was extensively studied

by Gibson and Helszajn (1989), the geometry and parameters are chosen to be the same as those of that research: $\mu = 1.0$, $\epsilon_r = 15.0$, $k_0 a = 0.6$. Moreover, each term of the permeability tensor (1) is normalized by κ / μ since this is convenient to find out solutions.

Figure 6(a-b) illustrates the dispersion characteristics of the fundamental mode for the waveguide shown in Figure 5. As seen from the graph, our results agree exactly with the results of the research of Gibson and Helszajn (1989). The fundamental mode of this waveguide with an electric wall, assigned to HE_{11} , is not the same as TE or TM of a hollow circular waveguide and it is called a hybrid mode because both electric and magnetic fields in the longitudinal axis of the mode are not zero. This mode also has a split mode which is the HE_{11}^+ and HE_{11}^- , where plus and minus symbols show the right-hand circular polarization and the left-hand circular polarization, respectively. Furthermore, they can be represented as wave propagating in $+z$ and $-z$ directions, respectively. The solution in

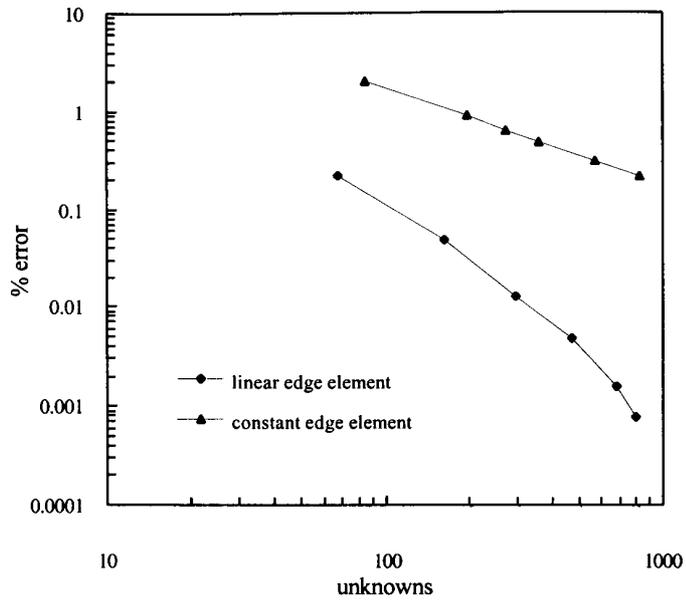


Figure 4. Percentage of error for the fundamental LSE_{10} mode in a dielectric-loaded waveguide

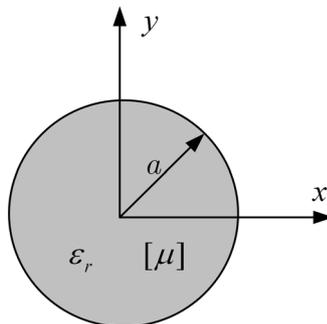
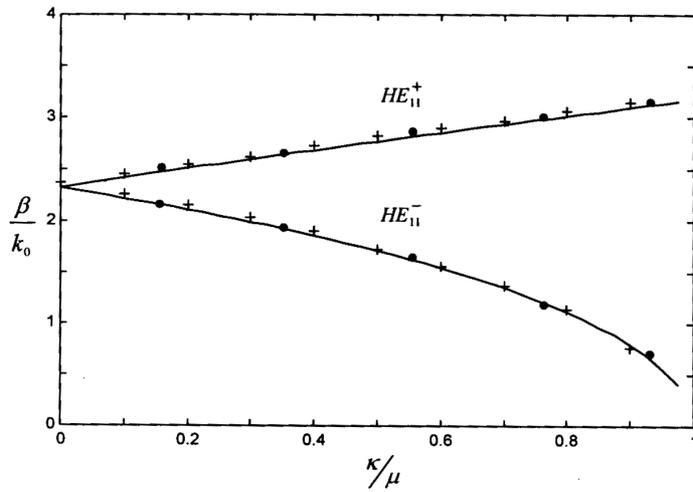
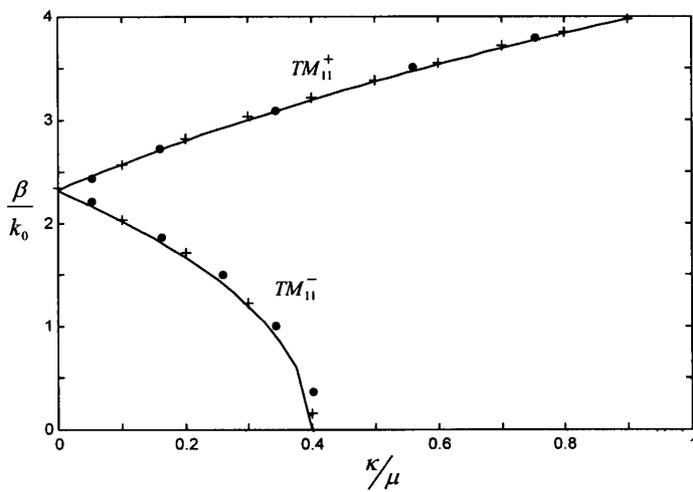


Figure 5. Cross section of a circular gyromagnetic waveguide



(a) waveguide bounded with an electric wall



(b) waveguide bounded with a magnetic wall

Figure 6. Dispersion characteristics of fundamental mode for the waveguide shown in Figure 5 (Solid line for FEM with constant edge element, dot for FEM with linear edge element, and plus for FEM by Gibson and Helszajn (1989))

Figure 6(b) provides a split mode as well, which consists of TM_{11}^+ and TM_{11}^- . It should be noted that the difference between the solutions in the 2 Figures is the existence of a split mode along k/m .

The next numerical example, an elliptical gyromagnetic waveguide is shown in Figure 7. The subdivision of the cross section of the waveguide and the imposition of the boundary conditions are similar to those of the previous waveguide. As mentioned above, the geometry and parameters are selected to be the same as those of the research of Gibson and Helszajn (1989): $\mu = 1.0$, $\epsilon_r = 10.0$ and $k_0a = 0.9696$ for an electric wall case, and $\mu = 1.0$, $\epsilon_r = 15.0$ and $k_0a = 0.60$ for a magnetic wall case. The eccentricity of a waveguide with electric and magnetic wall case is equal to 0.648 and 0.4, respectively. Figure 8(a-b) presents dispersion characteristics of a fundamental mode for the waveguide shown in Figure 7. In Figure 8 our results are very close to the results by Gibson and Helszajn (1989) as well, and there are no spurious modes.

To demonstrate the validity and usefulness of the method, an inhomogenous gyromagnetic waveguide shown in Figure 9 is taken into consideration. The material filled in a waveguide consists of gyromagnetic and dielectric properties. The dimension and parameters of a waveguide are chosen to allow the comparison with the results of the research of Dillon *et al.* (1993): $\epsilon_1 = 1.0$, $\epsilon_r = 10.0$, $\mu = 1.0$, and $k_0a = 2.5133$.

Figure 10 illustrates the dispersion characteristics as a function of b/a for the fundamental mode, HE_{11}^+ and HE_{11}^- , for 2 values of k/m . As seen from a graph, the computed results of the propagation constants based on this approach are very close to results of the method of Dillon *et al.* (1993). For $\kappa/\mu = 0$, this means that there is not an applied dc magnetic field. This leads the permeability tensor to scalar form, and only the HE_{11}^+ mode occurs on the dispersion characteristics. For $\kappa/\mu = 0.75$, a split mode, which is HE_{11}^+ and HE_{11}^- , occurs and has circular polarizations. Furthermore, the ratio of radius in the graph has an effect on the propagation modes.

Conclusions

In this paper, a finite element method with edge element for the analysis of anisotropic and inhomogenous waveguides is presented to investigate the propagation constants. The formulation is in terms of a variational expression that uses transverse electric and magnetic components and was first introduced by Angkaew (1989). The advantage of this variational function is that the matrices involved in the final eigensystem are expressed as propagation constants. The trial functions for the transverse electric and magnetic components in each element can be expressed as the terms of edge element and unknown parameters. The vector shape functions used here are in the form of constant edge and linear edge elements.

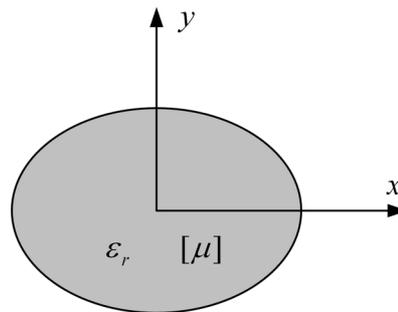
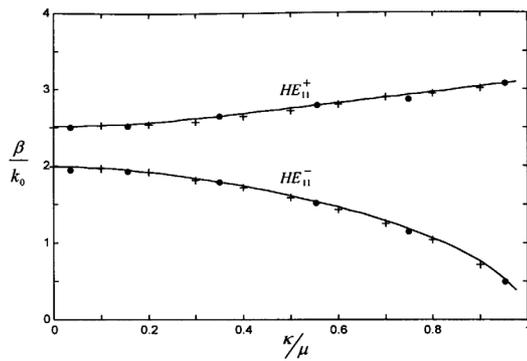


Figure 7. Cross section of an elliptical gyromagnetic waveguide

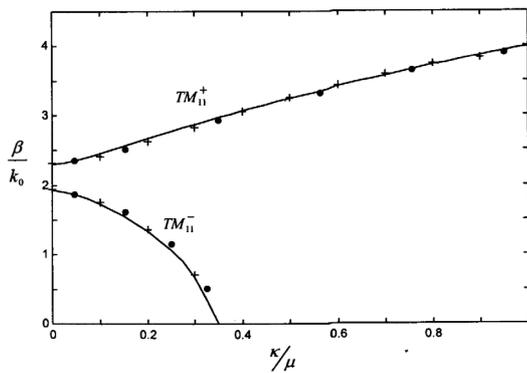
Numerical results for a dielectric-loaded waveguide case are in good agreement with those of analytic data and previous literature. It can be noted that this method provides accuracy and efficiency of computing the propagation constants in waveguides. For the other cases, the computed results for a gyromagnetic waveguide are close to results published before and there are no spurious modes.

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(a) waveguide bounded with an electric wall and $k_0a = 0.9696$



(b) waveguide bounded with a magnetic wall and $k_0a = 0.60$

Figure 8. Dispersion characteristics of fundamental mode for the waveguide shown in Figure 7 (Solid line for FEM with constant edge element, dot for FEM with linear edge element, plus for FEM by Gibson and Helszajn (1989))

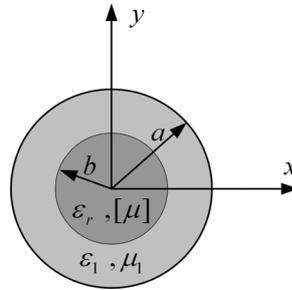


Figure 9. Cross section of an inhomogeneous gyromagnetic waveguide

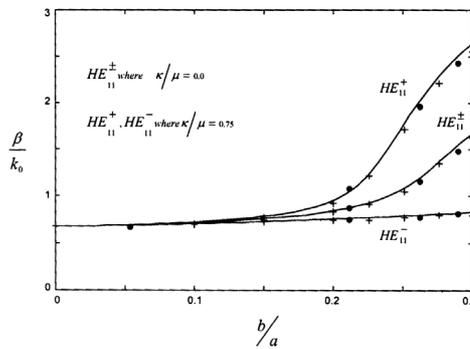


Figure 10. Dispersion characteristics of fundamental mode for the waveguide shown in Figure 9 (Solid line for FEM with constant edge element, dot for FEM with linear edge element, plus for FEM by Dillon *et al.* (1993))

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