SOLUTIONS FOR ONE CLASS OF NONLINEAR FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Some solutions for one class of nonlinear fourth-order partial differential equations

$$u_{tt}=\left(\kappa u+\gamma u^2\right)_{xx}+\nu u u_{xxxx}+\mu u_{xxtt}+\alpha u_x u_{xxx}+\beta u^2{}_{xx}$$

where α, β, γ, μ, ν and κ are arbitrary constants are presented in the paper. This equation may be thought of as a fourth-order analogue of a generalization of the Camassa-Holm equation, in which there has been considerable interest recently. Furthermore, this equation is a Boussinesq-type equation which arises as a model of vibrations of harmonic mass-spring chain. The idea of travelling wave solutions and linearization criteria for fourth-order ordinary differential equations by point transformations is applied to this problem.

Keywords: Linearization problem, point transformation, nonlinear ordinary differential equation, travelling wave solution

Introduction

Almost all important governing equations in physics take the form of nonlinear differential equations, and, in general, are very difficult to solve explicitly. While solving problems related to nonlinear ordinary differential equations it is often expedient to simplify equations by a suitable change of variables.

Many methods of solving differential equations use a change of variables that transforms a given differential equation into another equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there is the problem of transforming a given differential equation into a linear equation. This problem, is called a linearization problem. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows constructing an exact solution of the original equation.

One of the most interesting nonlinear problems but also difficult to solve is the

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problem of nonlinear fourth-order partial differential equations

$$u_{tt}=\left(\kappa u+\gamma u^2\right)_{xx}+\nu u u_{xxxx}+\mu u_{xxtt}+\alpha u_x u_{xxx}+\beta u^2{}_{xx} \quad (1)$$

where α, β, γ, μ, ν and κ are arbitrary constants. The main difficulty in solving this problem comes from the terms of nonlinear partial differential equations and the large number of order. Because of this difficulty, there have been only a few attempts to solve this problem.

In 1999, Clarkson and Priestley (Clarkson and Priestly, 1999) studied symmetry reductions of (1). An interesting aspect of their results is that the class of reductions given by the nonclassical method, which are not obtainable using the classical Lie method, were much more plentiful and richer than the analogous results given in (Clarkson *et al.*, 1997).

In 2008, Ibragimov, Meleshko and Suksern (Ibragimov *et al.*, 2008) found the explicit form of the criteria for linearization of fourth-order ordinary differential equations by point transformations. Moreover, the procedure for the construction of the linearizing transformation was presented. So that the idea for solving the problem of nonlinear fourth-order partial differential Equations (1) was happened.

The way to solve this problem is organized as follows. Firstly, substituting the form of travelling wave solutions into the nonlinear fourth-order partial differential equations. Then, applying the criteria for linearization in (Ibragimov *et al.*, 2008). After that, finding the exactsolutions of linear equations.

Linearization Criteria for Fourth-Order Ordinary Differential Equations by Point Transformations

The important tool for this research is the linearization criteria for fourth-order ordinary differential equations by point transformations. Based on (Ibragimov *et al.*, 2008), we have the following theorems.

Theorem 1 Any fourth-order ordinary differential Equation $y^{(4)}=F\left(x,y,y',y'',y'''\right)$ can be reduced by a point transformation can be reduced by a point transformation

$$t=\varphi\left(x,y\right),\ u=\psi\left(x,y\right), \quad (2)$$

to the linear equation

$$u^{(4)}+\alpha\left(t\right)u'+\beta\left(t\right)u=0, \quad (2)$$

where t and u are the independent and dependent variables, respectively, if it belongs to the class of equations

$$y^{(4)}+\left(A_1y'+A_0\right)y'''+B_0y''^2+\left(C_2y'^2+C_1y'+C_0\right)y''+D_4y'^4+D_3y'^3+D_2y'^2+D_1y'+D_0=0, \quad (4)$$

or

$$y^{(4)}+\frac{1}{y'+r}(-10\,y''+F_2y'^2+F_1y'+F_0)y'''+\frac{1}{(y'+r)^2}[15y''^3+(H_2y'^2+H_1y'+H_0)y''^2+(J_4y'^4+J_3y'^3+J_2y'^2+J_1y'+J_0)y''+K_7y'^7+K_6y'^6+K_5y'^5+K_4y'^4+K_3y'^3+K_2y'^2+K_1y'+K_0]=0, \quad (5)$$

where
$A_j=A_j(x,y)$, $B_j=B_j(x,y)$, $C_j=C_j(x,y)$, $D_j=D_j(x,y)$, $r=r(x,y)$, $F_j=F_j(x,y)$, $H_j=H_j(x,y)$, $J_j=J_j(x,y)$ and $K_j=K_j(x,y)$ are arbitrary functions of x, y.

Since this research deals with the first class, let us emphasize the first class for other theorems that we need to use.

Theorem 2 Equation (4) is linearizable if and only if its coefficients obey the following conditions:

$$A_{0y}-A_{1x}=0, \quad (6)$$

$$4B_0-3A_1=0, \quad (7)$$

$$12A_{1y}+3A_1^2-8C_2=0, \quad (8)$$

$$12A_{1x} + 3A_0A_1 - 4C_1 = 0, \quad (9)$$

$$32C_{0y} + 12A_{0x}A_1 - 16C_{1x} + 3A_0^2A_1 - 4A_0C_1 = 0, \quad (10)$$

$$4C_{2y} + A_1C_2 - 24D_4 = 0, \quad (11)$$

$$4C_{1y} + A_1C_1 - 12D_3 = 0, \quad (12)$$

$$16C_{1x} - 12A_{0x}A_1 - 3A_0^2A_1 + 4A_0C_1 + 8A_1C_0 - 32D_2 = 0, \quad (13)$$

$$192D_{2x} + 36A_{0x}A_0A_1 - 48A_{0x}C_1 - 48C_{0x}A_1 + 288D_{1y} + 9A_0^3A_1 - 12A_0^2C_1 - 36A_0A_1C_0 - 48A_0D_2 - 32C_0C_1 = 0, \quad (14)$$

$$384D_{1xy} + [3((3A_0A_1 - 4C_1)A_0^2 + 16(2A_1D_1 + C_0C_1) - 16(A_1C_0 - D_2)A_0)A_0 - 32(4(C_1D_1 - 2C_2D_0 + C_0D_2) + (3A_1D_0 - C_0^2)A_1) - 96D_{1y}A_0 + 384D_{0y}A_1 + 1536D_{0yy} - 16(3A_0A_1 - 4C_1)C_{0x} + 12((3A_0A_1 - 4C_1)A_0 - 4(A_1C_0 - 4D_2))A_{0x}] = 0 \quad (15)$$

Theorem 3 Provided that the conditions (6)-(15) are satisfied, the linearizing transformation (2) is defined by a fourth-order ordinary differential equation for the function $\varphi(x)$, namely by the Riccati equation

$$40\frac{d\chi}{dx} - 20\chi^2 = 8C_0 - 3A_0^2 - 12A_{0x} \quad (16)$$

for

$$\chi = \frac{\varphi_{xx}}{\varphi_x}, \quad (17)$$

and by the following integrable system of partial differential equations for the function $\psi(x, y)$

$$4\psi_{yy} = \psi_yA_1, \quad 4\psi_{xy} = \psi_y(A_0 + 6\chi), \quad (18)$$

and

$$1600\psi_{xxxx} = 9600\psi_{xxx}\chi + 160\psi_{xx}(-12A_{0x} - 3A_0^2 - 90\chi^2 + 8C_0) + 40\psi_x(12A_{0x}A_0 + 72A_{0x}\chi - 16C_{0x} + 3A_0^3 + 18A_0^2\chi - 12A_0C_0 + 120\chi^3 - 48\chi C_0 + 24D_1 - 8\Omega) + \psi(144A_{0x}^2 + 72A_{0x}A_0^2 - 352A_{0x}C_0 - 160C_{0xx} - 80C_{0x}A_0 - 1600D_{0y} + 640D_{1x} - 80\Omega_x + 9A_0^4 - 88A_0^2C_0 + 160A_0D_1 + 30A_0\Omega - 400A_1D_0 + 300\chi\Omega + 144C_0^2) + 1600\psi_yD_0, \quad (19)$$

where χ is given by Equation (17) and Ω is the following expression

$$\Omega = A_0^3 - 4A_0C_0 + 8D_1 - 8C_{0x} + 6A_{0x}A_0 + 4A_{0xx}. \quad (20)$$

Finally, the coefficients α and β of the resulting linear Equation (3) are

$$\alpha = \frac{\Omega}{\varphi_x^3}, \quad (21)$$

$$\beta = (1600\varphi_x^4)^{-1}(-144A_{0x}^2 - 72A_{0x}A_0^2 + 352A_{0x}C_0 + 160C_{0xx} + 80C_{0x}A_0 + 1600D_{0y} - 640D_{1x} + 80\Omega_x - 9A_0^4 + 88A_0^2C_0 - 160A_0D_1 - 30A_0\Omega + 400A_1D_0 - 300\chi\Omega - 144C_0^2). \quad (22)$$

Method and Result

Let us consider the nonlinear fourth-order partial differential Equation (1)

$$u_{tt} = (\kappa u + \gamma u^2)_{xx} + \nu u u_{xxxx} + \mu u_{xxtt} + \alpha u_x u_{xxx} + \beta u_{xx}^2$$

where α, β, γ, μ, ν and κ are arbitrary constants.

Of particular interest among solutions of Equation (1) are traveling wave solutions:

$$u(x, t) = H(x-Dt),$$

where D is a constant phase velocity and the argument x-Dt is a phase of the wave. Substituting the representation of a solution into Equation (1), one finds

$$\left(\nu H+\mu D^2\right)H^{(4)}+\alpha H'H'''+\beta H''^2+$$
$$\left(2\gamma H+\kappa-D^2\right)H''+2\gamma H'^2=0. \quad (23)$$

This is an equation of the form (4) with coefficients

$$A_1=\frac{\alpha}{\nu H+\mu D^2},\quad A_0=0,\quad B_0=\frac{\beta}{\nu H+\mu D^2},$$
$$C_2=C_1=0,\quad C_0=\frac{2\gamma H+\kappa-D^2}{\nu H+\mu D^2},$$
$$D_4=D_3=0,\quad D_2=\frac{2\gamma}{\nu H+\mu D^2},\quad D_1=D_0=0.$$

Substituting these coefficients into the linearization conditions (6)-(15), one obtains the following results.

Case 1 : $\nu=0$

• If $\nu = 0$, then Equation (23) is linearizable if and only if

$$\alpha=\beta=\gamma=0.$$

The solution of Equation (23) is

$$u(x,t)=C_1\sin\sqrt{\frac{\kappa-D^2}{\mu D^2}}(x-Dt)+$$
$$C_2\cos\sqrt{\frac{\kappa-D^2}{\mu D^2}}(x-Dt), \quad (24)$$

where C_1 and C_2 are arbitrary constants.

Case 2 : $\nu \neq 0$

• If $\nu\neq0, \gamma=0$, and $\beta=0$, then Equation (23) is linearizable if and only if

$$\alpha = 0, \kappa = D^2.$$

Next, to find the linear form and the solutions of Equation (23). From theorem 3, we found that Equation (23) is mapped by the transformations

$$\varphi = x - Dt, \psi = H$$

to the linear Equation

$$\psi^{(4)} = 0.$$

Hence, the solution of Equation (1) is

$$u(x,t)=C_0+C_1(x-Dt)+C_2(x-Dt)^2$$
$$+C_3(x-Dt)^3, \quad (25)$$

where C_0, C_1, C_2 and C_3 are arbitrary constants.

• If $\nu \neq 0$, $\gamma = 0$, and $\beta = 3\nu$, then Equation (23) is linearizable if and only if

$$\alpha = 4\nu, \kappa = D^2.$$

By using theorem 3, we obtained that Equation (23) is mapped by the transformations

$$\varphi = x - Dt,\ \psi = \frac{H^2}{2}+\frac{D^2\mu}{\nu}H$$

to the linear equation

$$\psi^{(4)} = 0.$$

So that we obtain the implicit solution of Equation (1) in the form

$$\frac{u^2}{2}+\frac{D^2\mu}{\nu}u=C_0+C_1(x-Dt)+$$
$$C_2(x-Dt)^2+C_3(x-Dt)^3, \quad (26)$$

where C_0, C_1, C_2 and C_3 are arbitrary constants.

Discussion and Conclusions

In the present work, we found some solutions

for nonlinear fourth-order partial differential Equation (1), which are in the form of Equations (24), (25), and (26), respectively. An interesting aspect of the results in this paper is that the class of exact solutions of the original nonlinear problems are found.

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