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# Characterization of a group-norm by maximum functional equation and stability results

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**ABSTRACT**: Let $(G, \|\cdot\|, +)$ be a normed group, where $\|\cdot\|: G \to \mathbb{R}$. We study the equation

$$\max\{\|x+y\|, \|x-y\|\} = \|x\| + \|y\| \qquad \text{for all } x, y \in G.$$

Without a commutativity assumption of the normed group $G$, we analyze the stability results and characterization of a group-norm by the given equation.

**KEYWORDS**: normed group, discretely normed abelian group, Tabor weakly commutative, stability

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## INTRODUCTION

Simon et al [4] gave the characterization $f(g) = |\eta(g)|$ for an additive function $\eta: G \to \mathbb{R}$ such that $\eta(g_1 + g_2) = \eta(g_1) + \eta(g_2)$, which fulfills the equations

$$\max\{f(g_1 - g_2), f(g_1 + g_2)\} = f(g_1) + f(g_2), \quad (1)$$

$$\min\{f(g_1 - g_2), f(g_1 + g_2)\} = |f(g_1) - f(g_2)| \quad (2)$$

for all $g_1, g_2 \in G$, assuming that the domain of $f$ is an abelian group $G$. However, according to the stability results of Przebieracz [9], (2) is stable, and she presented a general theorem that proves the stability of (2), where $f: \mathbb{R} \to \mathbb{R}$ is considered as a continuous function of real variables.

Gilanyi et al [5] took into account the generalized version of (1), that is

$$\max\{f((g_1 g_2)g_2), f(g_1)\} = f(g_1 g_2) + f(g_2) \quad (3)$$

for all $g_1, g_2 \in G$, and demonstrated its stability for the real-valued function $f: G \to \mathbb{R}$ under the assumption of left identity, where $G$ is considered as a square-symmetric groupoid. Consequently, Volkmann [15] gave the generalization of (1) under the condition that $f(g_1 g_2 g_3) = f(g_1 g_3 g_2)$ holds for all $g_1, g_2, g_3 \in G$. Stability results in connection with generalization of (1) can be found in [12], while a generalized version of (1) without commutativity condition can be seen in [14].

Furthermore, Redheffer in a joint paper with Volkmann [10] gave the solution to a Pexiderized version of (1),

$$f(x) + g(y) = \max\{h(x-y), h(x+y)\} \quad (4)$$

for all $x, y \in G$, where $f$, $g$ and $h$ are mappings from an abelian group $(G, +)$ to $\mathbb{R}$.

Since group-norms play an important role in establishing relation between norms and group structures; therefore, in the next section, it will be shown that the proposed equation

$$\|x\| + \|y\| = \max\{\|x+y\|, \|x-y\|\} \quad (5)$$

for all $x, y \in G$, characterizes the group-norm. Therefore, we established a reliable relation between normed-groups and functional equation (5) through the characterization of a group-norm function from a normed group $(G, \|\cdot\|, +)$ to $\mathbb{R}_{\geqslant 0}$ defined by $\|x\| := |x|$ for all $x \in G$. A presentation of our proposed definition in the form of group-norm equivalent to (5) was investigated in [5] as

$$\max\{\|(g_1 g_2)g_2\|, \|g_1\|\} = \|g_1 g_2\| + \|g_2\| \quad (6)$$

for all $g_1, g_2 \in G$.

The last section is devoted to the stability results of (5), where $(G, \|\cdot\|, +)$ is a normed group. Moreover, we will analyze the stability of (5) for a real-valued function defined on a normed group $G$. As a consequence of our main stability theorem of (5), we obtain the stability results of (5) on a Tabor weakly commutative group.

## ANALYSIS OF (5)

Throughout this article, our normed group $G$ will in general be $(G, \|\cdot\|, +)$, and 0 is considered to be the neutral element unless otherwise stated.

**Definition 1** Let $(G, +)$ be a group with the neutral element 0, then we say its norm $\|\cdot\| : G \to [0, \infty)$ is called a group-norm if, for any $a, b \in G$, it fulfills the following properties:

(i) $\|a+b\| \leqslant \|a\| + \|b\|$;

(ii) $\|a\| \geqslant 0$, with $\|a\| = 0$ if $a = 0$;

(iii) $\|-a\| = \|a\|$.

If (i) and (ii) are satisfied, then the norm $\|\cdot\|$ is known as a pre-norm; if only (i) holds, then we say the norm $\|\cdot\|$ is a semi-norm. For instance, see [2, 13]. A normed group is denoted by $(G, \|\cdot\|, +)$ where $\|\cdot\|$ is a group-norm and $(G, +)$ is a group.

**Theorem 1** *Suppose that $(G, +)$ is a group. A mapping $\|\cdot\| : G \to \mathbb{R}$ satisfies (5) if and only if $(G, \|\cdot\|, +)$ is a normed group.*

*Proof*: Suppose that (5) holds. Setting $x = 0$, we can compute $\|0\| + \|0\| = \max\{\|0\|, \|0\|\} = \|0\|$, which implies that $\|0\| = 0$.

Also given condition yields that $\|x\| + \|x\| = \max\{\|x+x\|, \|x-x\|\} = \max\{\|2x\|, \|0\|\} \geqslant 0$, which gives that $2\|x\| \geqslant 0$, so $\|x\| \geqslant 0$. Since $\|x\| \geqslant 0$ and $\|x\| = 0$ whenever $x = 0$, then we can see $\|2x\| \geqslant 0 = \|0\|$, so we have $\|2x\| \geqslant \|x - x\|$, then

$$2\|x\| = \|x\| + \|x\| = \max\{\|x-x\|, \|2x\|\} = \|2x\|,$$

therefore $2\|x\| = \|2x\|$.

Moreover, setting $y = -x$ in (5) implies that $\|x\| + \|-x\| = \max\{\|x+x\|, \|x-x\|\} = \max\{\|0\|, \|2x\|\} = \|2x\| = 2\|x\|$, which gives that $\|-x\| = \|x\|$. Furthermore, we can observe from (5) that $\|x-y\| \leqslant \|x\| + \|y\|$ or $\|x+y\| \leqslant \|x\| + \|y\|$; hence, in either case, the triangle inequality holds. Hence $(G, \|\cdot\|, +)$ is a normed group. Conversely, let $G$ be a normed group defined by the group-norm $\|\cdot\|$. Obviously, $\|x\| + \|y\| = \max\{\|x-y\|, \|x+y\|\}$ for any $x, y \in G$. □

**Corollary 1** *For a normed group $(G, \|\cdot\|, +)$, a group-norm $\|\cdot\| : G \to \mathbb{R}$ fulfilling (5) is a conjugation and abelian group-norm.*

*Proof*: Let $x, y \in G$, then the proof of conjugation group-norm consists of the following simple computation:

$$\begin{aligned}
\|y\| + \|-y+x+y\| \\
&= \max\{\|y-y+x+y\|, \|y-y-x+y\|\} \\
&= \max\{\|x+y\|, \|-x+y\|\} \\
&= \max\{\|-y-x\|, \|-y+x\|\|\} \\
&= \max\{\|-y+x\|, \|-y-x\|\} \\
&= \|-y\| + \|x\| \\
&= \|y\| + \|x\|,
\end{aligned}$$

Therefore, $\|-y+x+y\| = \|x\|$. Writing $y+x$ instead of $x$, we can obtain that

$$\|y+x\| = \|-y+y+x+y\| = \|x+y\|,$$

which implies that $\|x+y\| = \|y+x\|$ for any $x, y \in G$. Thus, the group-norm is abelian. □

**Theorem 2** *Let $(G, \|\cdot\|, +)$ be a normed group then $\min\{\|x-y\|, \|x+y\|\} \leqslant |\,\|x\| - \|y\|\,|$ holds for all $x, y \in G$.*

*Proof*: Since $(G, \|\cdot\|, +)$ is a normed group, a group-norm $\|\cdot\| : G \to \mathbb{R}$ satisfies (5). Making use of the conjugation group-norm, we first compute that

$$\begin{aligned}
2\|x\| &= \|x\| + \|x\| \\
&= \|x\| + \|-y+x+y\| \\
&= \max\{\|x-y+x+y\|, \|x-y-x+y\|\} \\
&\geqslant \|x-y+x+y\|.
\end{aligned}$$

Then we can obtain the required result by the following simple calculation:

$$\begin{aligned}
&\max\{\|x-y\|, \|x+y\|\} + \min\{\|x-y\|, \|x+y\|\} \\
&\qquad = \|x-y\| + \|x+y\| \\
&\|x\| + \|y\| + \min\{\|x-y\|, \|x+y\|\} = \|x-y\| + \|x+y\| \\
&\min\{\|x-y\|, \|x+y\|\} = \|x-y\| + \|x+y\| - \|x\| - \|y\| \\
&\quad = \max\{\|x-y+x+y\|, \|x-y-y-x\|\} - \|x\| - \|y\| \\
&\quad = \max\{\|x-y+x+y\|, \|x-2y-x\|\} - \|x\| - \|y\| \\
&\quad \leqslant \max\{2\|x\|, \|-2y\|\} - \|x\| - \|y\| \\
&\quad = \max\{2\|x\|, 2\|y\|\} - \|x\| - \|y\| \\
&\quad = 2\max\{\|x\|, \|y\|\} - \|x\| - \|y\| \leqslant |\,\|x\| - \|y\|\,|.
\end{aligned}$$

□

By adding a certain condition for Theorem 2, we can extend the proof to remove the inequality. The following definition will play a key role in the proof of Theorem 3.

**Definition 2** ([7]) For a normed group $(G,\|\cdot\|,+)$, we say a mapping $\|\cdot\|\colon G\to\mathbb{R}$ satisfies condition (C) if

$$\|u+z+v\|=\|u+v+z\| \qquad \text{for any } u,z,v\in G.$$

Obviously, any normed group $(G,\|\cdot\|,+)$ fulfills the proposed condition (C) whenever the group $G$ is abelian.

**Theorem 3** *Suppose that $(G,+)$ is a group and a mapping $\|\cdot\|\colon G\to\mathbb{R}$ fulfills the given condition (C), then $(G,\|\cdot\|,+)$ is a normed group if and only if the group-norm is 2-homogeneous, and also*

$$\min\{\|x-y\|,\|x+y\|\}=|\,\|x\|-\|y\|\,| \tag{7}$$

*holds for all $x,y\in G$.*

*Proof*: If $(G,\|\cdot\|,+)$ is a normed group, then it is obvious that $\|0\|=0$ holds and also $\|x\|\geqslant 0$ for any $x\in G$. Also we can obtain that $\|x\|+\|y\|=\max\{\|x-y\|,\|x+y\|\}$; therefore, setting $x=y=0$, we can compute that $\|2x\|=2\|x\|$, i.e., group-norm is 2-homogeneous. The following simple computation gives the proof of (7):

$$\begin{aligned}
&\min\{\|x-y\|,\|x+y\|\}+\|x\|+\|y\|\\
&\quad=\min\{\|x-y\|,\|x+y\|\}+\max\{\|x-y\|,\|x+y\|\}\\
&\quad=\|x-y\|+\|x+y\|\\
&\quad=\max\{\|x+y+x-y\|,\|x+y+y-x\|\}\\
&\quad=\max\{\|x+y+x-y\|,\|x+2y-x\|\}\\
&\quad=\max\{\|2x\|,\|2y\|\}\\
&\min\{\|x-y\|,\|x+y\|\}=\max\{2\|x\|,2\|y\|\}-\|x\|-\|y\|\\
&\min\{\|x-y\|,\|x+y\|\}=|\,\|x\|-\|y\|\,|.
\end{aligned}$$

Conversely, assume that a group-norm fulfills (7) and also is 2-homogeneous. Then we can see that

$$\max\{\|x-y\|,\|x+y\|\}-\min\{\|x-y\|,\|x+y\|\}=|\,\|x+y\|-\|x-y\|\,|.$$

Applying (7) in the following computation, we obtain that

$$\begin{aligned}
&\max\{\|x-y\|,\|x+y\|\}-|\,\|x\|-\|y\|\,|=|\,\|x+y\|-\|x-y\|\,|\\
&\quad=\min\{\|x+y+x-y\|,\|x+y+y-x\|\}\\
&\quad=\min\{\|2x\|,\|x+2y-x\|\}\\
&\quad=\min\{\|2x\|,\|2y\|\}\\
&\max\{\|x-y\|,\|x+y\|\}=2\min\{\|x\|,\|y\|\}+|\,\|x\|-\|y\|\,|\\
&\max\{\|x-y\|,\|x+y\|\}=\|x\|+\|y\|,
\end{aligned}$$

which implies that $(G,\|\cdot\|,+)$ is a normed group. □

**Corollary 2** *If $(G,+)$ is a group and a mapping $\|\cdot\|\colon G\to\mathbb{R}$ fulfills the proposed condition (C), then $(G,\|\cdot\|,+)$ is a normed group if and only if*

$$\|x\|+\|y\|=\|x-y\|+\|x+y\|-|\,\|x\|-\|y\|\,|, \tag{8}$$

*for any $x,y\in G$, and also $\|0\|=0$ holds.*

*Proof*: Assume that (8) holds and also $\|0\|=0$. Then (8) yields that

$$2\max\{\|x\|,\|y\|\}=\|x-y\|+\|x+y\|,\quad x,y\in G. \tag{9}$$

Since $\|0\|=0$, replacing $y$ with $x$, we can easily compute that $\|2x\|=2\|x\|$. Then replacing $x$ with $x+y$ and $y$ with $x-y$ in (9), we have

$$\begin{aligned}
2\max\{\,\|x-y\|,\|x+y\|\,\}&=\{\|2x\|+\|2y\|\}\\
&=\{2\|x\|+2\|y\|\}\\
\max\{\,\|x-y\|,\|x+y\|\,\}&=\|x\|+\|y\|.
\end{aligned}$$

Conversely, suppose $(G,\|\cdot\|,+)$ is a normed group, then, by Theorem 3, we can determine that $\|0\|=0$, and (8) also holds. □

## STABILITY OF (5)

To analyze the stability results of (5) involving variables $x$ and $y$, first replacing $y$ with $x$ in (5), we will find the stability result of (5) in a single variable $x$ in the following theorem.

**Theorem 4** *Suppose that $(G,+)$ is a group and for some $\delta\geqslant 0$ a mapping $\|\cdot\|^*\colon G\to\mathbb{R}$ satisfies*

$$|\max\{\|2x\|^*,\|0\|^*\}-2\|x\|^*|\leqslant\delta,\quad x\in G, \tag{10}$$

*then, we can obtain a group-norm $\|\cdot\|\colon G\to\mathbb{R}$ of*

$$\max\{\|2x\|,\|0\|\}=2\|x\|,\quad x\in G, \tag{11}$$

*such that*

$$-3\delta\leqslant\|x\|-\|x\|^*\leqslant\delta. \tag{12}$$

*Also, the group-norm $\|\cdot\|$ can be written as*

$$\|x\|=\lim_{n\to\infty}\frac{1}{2^n}\|2^n x\|^*,\quad x\in G. \tag{13}$$

*By (11), $\|\cdot\|$ is uniquely determined, and by (12), $\|\cdot\|-\|\cdot\|^*$ is also bounded.*

*Proof*: First, setting $x=0$ in (10) implies that $|\max\{\|0\|^*,\|0\|^*\}-2\|0\|^*|\leqslant\delta$, therefore $|\|0\|^*|\leqslant\delta$. Additionally, (10) also gives that

$$\begin{aligned}
&-\delta+2\|x\|^*\leqslant\max\{\|2x\|^*,\|0\|^*\}\leqslant\delta+2\|x\|^*\\
&-\delta\leqslant\|0\|^*\leqslant\delta+2\|x\|^*\\
&-\delta\leqslant\|x\|^*, \qquad\qquad \text{for all } x\in G.
\end{aligned} \tag{14}$$

Replacing $x$ with $2x$ gives that $-\delta \leqslant \|2x\|^*$, so we can obtain

$$\|0\|^* \leqslant \delta = 2\delta - \delta \leqslant 2\delta + \|2x\|^*$$
$$\|0\|^* \leqslant 2\delta + \|2x\|^* \qquad \text{for all } x \in G. \tag{15}$$

By (10) and (15), we can compute

$$2\|x\|^* \leqslant \delta + \max\{\|2x\|^*, \|0\|^*\}$$
$$2\|x\|^* \leqslant \delta + \|0\|^* \leqslant 3\delta + \|2x\|^*$$
$$\text{or} \quad 2\|x\|^* \leqslant \delta + \|2x\|^* \leqslant 3\delta + \|2x\|^*.$$

Joining both cases, we determine that

$$-3\delta \leqslant \|2x\|^* - 2\|x\|^*. \tag{16}$$

By (10), we can see that $\max\{\|2x\|^*, \|0\|^*\} \leqslant \delta + 2\|x\|^*$, which is possible whenever

$$\|2x\|^* - 2\|x\|^* \leqslant \delta. \tag{17}$$

Inequalities (16) and (17) imply that

$$-3\delta \leqslant \|2x\|^* - 2\|x\|^* \leqslant \delta \qquad \text{for all } x \in G. \tag{18}$$

By (18), it can be observed that the mapping $\|\cdot\|\colon G \to \mathbb{R}$ given in (13) exists and $\|\cdot\|$ fulfills

$$\|2x\| = 2\|x\| \qquad \text{for all } x \in G. \tag{19}$$

Moreover, $\|\cdot\|$ satisfies (12). Also, replacing $x$ with $2^n x$ in (14) and dividing by $2^n$, then taking the limit $n \to \infty$ while utilizing (13), we can get that $\|x\| \geqslant 0$ for all $x \in G$. Therefore, we get (11) from (19). In view of (11) and using fact that $\|2x\| = 2\|x\| \geqslant 0$ for every $x \in G$, we can see the uniqueness of $\|\cdot\|$. □

Stability results of (5) involving two variables can be easily shown with the help of Theorem 4 as follows.

**Theorem 5** *Suppose that $(G,+)$ is a group and a mapping $\|\cdot\|^*\colon G \to \mathbb{R}$ satisfies*

$$|\max\{\|x-y\|^*, \|x+y\|^*\} - \|x\|^* - \|y\|^*| \leqslant \delta, \tag{20}$$

*for all $x, y \in G$ and for some $\delta \geqslant 0$. Then, there exists a unique group-norm $\|\cdot\|\colon G \to \mathbb{R}$ of*

$$\|x\| + \|y\| = \max\{\|x-y\|, \|x+y\|\}, \quad x, y \in G \tag{21}$$

*such that*

$$-3\delta \leqslant \|x\| - \|x\|^* \leqslant \delta, \qquad x \in G. \tag{22}$$

*Also,*

$$\|x\| = \lim_{n\to\infty} \frac{1}{2^n} \|2^n x\|^*, \qquad x \in G. \tag{23}$$

*Proof*: By Theorem 4, we can show the required stability results. First, replacing $y$ with $x$ in (20), and by Theorem 4 we can get a mapping $\|\cdot\|\colon G \to \mathbb{R}$. Moreover, we show that this mapping $\|\cdot\|$ fulfills (21). By replacing $x$ with $2^n x$ and $y$ with $2^n y$ in (20) and dividing by $2^n$, then applying the limit $n \to \infty$ and also utilizing (23), we get the required result in the form of (21). □

**Theorem 6** *Let $(G,+)$ be a group. Assume that a mapping $\|\cdot\|^*\colon G \to \mathbb{R}$ satisfies (20), then it fulfills the proposed condition*

$$\lim_{n\to\infty} \frac{1}{2^n}\Big[\max\{\|2^n x - 2^n y\|^*, \|2^n x + 2^n y\|^*\} - \max\{\|2^n[x-y]\|^*, \|2^n[x+y]\|^*\}\Big] = 0 \tag{24}$$

*if and only if $(G, \|\cdot\|, +)$ is a normed group.*

*Proof*: Suppose that $(G, \|\cdot\|, +)$ is a normed group, then (21) holds. Taking any elements $x, y \in G$, we have

$$\begin{aligned}
&\Big|\max\{\|2^n x - 2^n y\|^*, \|2^n x + 2^n y\|^*\} - \max\{\|2^n[x-y]\|^*, \|2^n[x+y]\|^*\}\Big| \\
&\leqslant |\max\{\|2^n x - 2^n y\|^*, \|2^n x + 2^n y\|^*\} - \|2^n x\|^* - \|2^n y\|^*| \\
&\quad + |\max\{\|2^n[x-y]\|^*, \|2^n[x+y]\|^*\} - \|2^n x\|^* - \|2^n y\|^*| \\
&\leqslant \delta + |\max\{\|2^n[x-y]\|^*, \|2^n[x+y]\|^*\} - \|2^n x\|^* - \|2^n y\|^*|.
\end{aligned}$$

To obtain the required statement, first dividing both sides by $2^n$, taking the limit $n \to \infty$, and using the proposed condition (21), we can see that

$$\lim_{n\to\infty} \frac{1}{2^n}\Big[\max\{\|2^n x - 2^n y\|^*, \|2^n x + 2^n y\|^*\} - \max\{\|2^n[x-y]\|^*, \|2^n[x+y]\|^*\}\Big] = 0.$$

Conversely, suppose that condition (24) holds. Replacing $x$ with $2^n x$ and $y$ with $2^n y$ in (20), then taking the limit $n \to \infty$ after dividing by $2^n$, we have

$$\lim_{n\to\infty} \frac{1}{2^n} \max\{\|2^n x - 2^n y\|^*, \|2^n x + 2^n y\|^*\} = \|x\| + \|y\|,$$

$$\lim_{n\to\infty} \frac{1}{2^n} \max\{\|2^n[x-y]\|^*, \|2^n[x+y]\|^*\} = \max\{\|x-y\|, \|x+y\|\}.$$

By condition (24), we can compute $\|x\| + \|y\| = \max\{\|x-y\|, \|x+y\|\}$.

Also, the proposed condition (24) associated with function $\|\cdot\|^*$ is not directly related to $(G,+)$, but some valuable properties about $G$ can be observed. We have given below a modified condition that is equivalent to the proposed condition (24). Consider a subsequence $m(n)$ of $\mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{2^{m(n)}}\Big[\max\{\|2^{m(n)}x-2^{m(n)}y\|^*, \|2^{m(n)}x+2^{m(n)}y\|^*\} - \max\{\|2^{m(n)}[x-y]\|^*, \|2^{m(n)}[x+y]\|^*\}\Big]=0,$$

which implies the mapping $\|\cdot\|^*$. Moreover, it holds because both of the limits

$$\lim_{n\to\infty} \frac{1}{2^{m(n)}}\max\{\|2^{m(n)}x-2^{m(n)}y\|^*, \|2^{m(n)}x+2^{m(n)}y\|^*\}$$

and

$$\lim_{n\to\infty} \frac{1}{2^{m(n)}}\max\{\|2^{m(n)}[x-y]\|^*, \|2^{m(n)}[x+y]\|^*\}$$

exist and are finite. □

**Corollary 3** *Assume that a mapping* $\|\cdot\|^*\colon G\to\mathbb{R}$ *satisfies* (11). *Then it fulfills the proposed condition*

$$\lim_{n\to\infty}\frac{1}{2^n}\Big[\max\{\|2^nx+2^ny\|^*, \|2^nx-2^ny\|\} - \|2^n[x+y]\|^*\Big]=0 \quad \textit{for all } x,y\in G, \quad (25)$$

*if and only if* $\|x\|+\|y\|=\|x+y\|$ *holds for every* $x,y\in G$.

**Remark 1** The proposed condition (24) satisfies when $G$ is an $n$-abelian group (a group $G$ is known as an $n$-abelian group if condition $n(u+v)=nu+nv$ holds for every integer $n$ and for every $u,v\in G$, for instance, see [1,3]).

**Remark 2** Proposed condition (24) also holds when $G$ is related to the class of groups $C_n$ for every natural number $n$ belongs to $\mathbb{N}$ ($C_n$ is a notation for the class of groups, which fulfills the condition $nv+nu=nu+nv$ for every $n\in\mathbb{N}$ and $u,v\in G$).

**Remark 3** If a group-norm $\|\cdot\|$ is abelian, then the proposed condition (24) is also true.

**Theorem 7** *Assume that* (5) *is stable on a normed group* $(G,\|\cdot\|,+)$. *Then a free abelian group* $H$ *can be embedded into* $G$.

*Proof*: Let $\|\cdot\|\colon G\to\mathbb{R}$ and for some $\delta\geqslant 0$, we have

$$|\max\{\|x-y\|,\|x+y\|\}-\|x\|-\|y\||\leqslant\delta, \quad x,y\in G. \quad (26)$$

$H$ is a torsion-free group because $H$ is a free abelian group. By using the concept of HNN-extensions, for instance, see [6,8]. Any torsion-free group $H$ can be embedded into $G$, if for every $h\in H$, there exists an element $g\in G$ such that $g+h-g=2h$. If $H$ is embedding into $G$, then (26) implies that

$$|\max\{\|h+g\|,\|h-g\|\}-\|h\|-\|g\||\leqslant\delta, \quad h,g\in G. \quad (27)$$

It will be shown that $\|\cdot\|$ is bounded. The proof is obvious when $\delta=0$, so assume that $\delta>0$. More explicitly, we show that $\|h\|<2\delta$ for every $h\in G$. By (27), we have two possible values that either $|\|h-g\|-\|h\|-\|g\||\leqslant\delta$ or $|\|h+g\|-\|h\|-\|g\||\leqslant\delta$. Considering the first possibility and setting $h=g=0$, we can see that $\|0\|\leqslant\delta$. Setting $g=h$, we can conclude that

$$\begin{aligned} |\|0\|-2\|h\||&\leqslant\delta\\ |2\|h\||-|\|0\||&\leqslant\delta\\ 2\|h\|&\leqslant\delta+\|0\|\\ 2\|h\|&\leqslant\delta+\delta\\ \|h\|&\leqslant\delta. \end{aligned}$$

For the second possibility, we can obtain

$$|\|h+g\|-\|h\|-\|g\||\leqslant\delta. \quad (28)$$

On the contrary, assume that $\|h\|\geqslant 2\delta$ for some $h\in G$. Setting $g=h$ in (28), we have

$$\begin{aligned} |\|2h\|-2\|h\||&\leqslant\delta\\ |2\|h\|-\|2h\||&\leqslant\delta\\ 2\|h\|-\delta&\leqslant\|2h\|\\ 3\delta&\leqslant\|2h\|. \end{aligned}$$

Again, setting $g=2h$ in (28) we have

$$\begin{aligned} |\|3h\|-\|h\|-\|2h\||&\leqslant\delta\\ |\|h\|+\|2h\|-\|3h\||&\leqslant\delta\\ |\|h\||+|\|2h\||&\leqslant\delta+\|3h\|\\ 5\delta-\delta&\leqslant\|3h\|\\ 4\delta&\leqslant\|3h\|. \end{aligned}$$

Repeating this process for $g=3h$, we can conclude $\|4h\|\geqslant 4\delta$. Continuing the process, we can determine

$$(m+1)\delta\leqslant\|mh\|,$$

where $m=1,2,\dots$, so we can see that $\|mh\|$ is unbounded when the value of $m$ varies.

Also, let $g\in G$ such that $2h=g+h-g$. Then $2mh=g+mh-g$ for any integer $m>0$. Moreover, for any $m$, setting $g=mh$ and $h=mh$ in (28), we have

$$\begin{aligned} |\|2mh\|-2\|mh\||&\leqslant\delta\\ |\|g+mh-g\|-2\|mh\||&\leqslant\delta. \end{aligned} \quad (29)$$

Moreover, (28) follows that

$$\big|\,\|g+mh-g\|-\|g\|-\|mh-g\|\,\big|\leqslant\delta\quad\text{and}$$
$$\big|\,\|mh-g\|-\|mh\|-\|-g\|\,\big|\leqslant\delta;$$

thus, we can get

$$\begin{aligned}\big|\,\|g+mh-g\|&-\|g\|-\|mh\|-\|-g\|\,\big|\\ &\leqslant\big|\,\|g+mh-g\|-\|g\|-\|mh-g\|\,\big|\\ &\quad+\big|\,\|mh-g\|-\|mh\|-\|-g\|\,\big|\\ &\leqslant 2\delta.\end{aligned}\tag{30}$$

From (29) and (30) we have

$$\begin{aligned}&\big|\,\|g+mh-g\|-2\|mh\|+\|mh\|\,\big|\leqslant 2\delta+\|g\|+\|-g\|\\ &\quad\|mh\|\leqslant 2\delta+\|g\|+\|-g\|+\big|\,\|g+mh-g\|-2\|mh\|\,\big|\\ &\quad\|mh\|\leqslant 5\delta,\end{aligned}$$

for $m=1,2,\ldots$, which is a contradiction. This completes the proof because the group-norm $\|\cdot\|$ is bounded. □

**Corollary 4** *Assume that* (5) *is stable on a normed group* $(G,\|\cdot\|,+)$*, and* $H$ *is a discretely normed abelian group. Then* $H$ *is embedding into* $G$*.*

*Proof*: Since $H$ is a discretely normed abelian group, then $H$ is a free group, for instance, see [11]; consequently $H$ is embedding into $G$. □

When we analyzed condition (24), it is noticed that for the stability of (5), the given condition (24) is necessary and sufficient. This condition leads to the following definition.

**Definition 3** ([16]) A group $(G,+)$ is called weakly commutative if for any $a,b\in G$, there exists $n=n(a,b)\geqslant 2$ such that $2^n(a+b)=2^na+2^nb$.

When we consider Theorem 6 and Definition 3 about Tabor weakly commutativity, then it gives the following theorem.

**Theorem 8** *Let* $(G,\|\cdot\|^*,+)$ *be a Tabor weakly commutative, then the group-norm* $\|\cdot\|$ *satisfies* (5).

*Proof*: From Theorem 6, it can be seen that condition (24) satisfies when $G$ is weakly commutative. To prove the second condition presented in Theorem 6, we need to construct a sequence $\{m(n)\}$, which holds the second condition. For this purpose, assume that $m_1=n(x,y)$ for fixed $x,y\in G$. Considering the pair $(2^{m_1}x,2^{m_1}y)$, by our assumption, there exists $n(2^{m_1}x,2^{m_1}y)$ such that

$$\begin{aligned}&2^{n(2^{m_1}x,2^{m_1}y)}(2^{m_1}x+2^{m_1}y)\\ &\qquad=2^{n(2^{m_1}x,2^{m_1}y)}(2^{m_1}x)+2^{n(2^{m_1}x,2^{m_1}y)}(2^{m_1}y).\end{aligned}$$

Since $2^{m_1}(x+y)=2^{m_1}x+2^{m_1}y$, so we get that

$$\begin{aligned}&2^{m_1+n(2^{m_1}x,2^{m_1}y)}(x+y)\\ &\qquad=2^{m_1+n(2^{m_1}x,2^{m_1}y)}x+2^{m_1+n(2^{m_1}x,2^{m_1}y)}y.\end{aligned}$$

Again, assume that $m_1+n(2^{m_1}x,2^{m_1}y)=m_2$; therefore, we have $2^{m_2}(x+y)=2^{m_2}x+2^{m_2}y$. By mathematical induction, it leads to the required sequence $\{m_n\}$. □

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