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A note on norm inequalities for positive matrices

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ABSTRACT: In this short note, we present some generalizations of a norm inequality due to Huang et al [*J Inequal Appl* **171**, 1–4 (2014)].

KEYWORDS: norm inequalities, polar decomposition, majorization

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INTRODUCTION

Given a real vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we rearrange its components as $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$. For $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \dots, n,$$

then we say that x is weakly majorized by y and denote $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ hold, then we say that x is majorized by y and denote $x \prec y$. If

$$\prod_{i=1}^{k} x_{[i]} \leqslant \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \dots, n,$$

then we write $x \prec_{w\log} y$.

As usual, the set of $n \times n$ complex matrices is denoted by M_n. For $A \in M_n$, we use $s_i(A)$ to present the singular values of A with $s_1(A) \geqslant \cdots \geqslant s_n(A)$. Let $s(A) = (s_1(A), \cdots, s_n(A))$. If $A \in M_n$ is Hermitian, then all eigenvalues of A are real and ordered as $\lambda_1(A) \geqslant \cdots \geqslant \lambda_n(A)$ and set $\lambda(A) = (\lambda_1(A), \cdots, \lambda_n(A))$. Note that $s_i(A) = \lambda_i(|A|)$, where $|A|$ is the modulus of A, i.e. $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the conjugate transpose of A. If A and B are Hermitian matrices and $A–B$ is positive semidefinite, then we say that $A \geqslant B$.

Let A, B be positive semidefinite matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. Bhatia and Kittaneh [1] proved that for any positive integer m,

$$\|A^m + B^m\| \leqslant \|(A+B)^m\|. \tag{1}$$

Huang et al [2] presented a generalization of inequality (1), they proved the following result: For any $A, B \in M_n$ and suppose that p, q be real numbers with $p > 1$ and $1/p + 1/q = 1$. Then for any positive integer m,

$$\left\| A|A|^{m-1} + B|B|^{m-1} \right\| \leqslant \left\| (|A|^m + |B|^m)^{\frac{p}{2}} \right\|^{\frac{1}{p}} \left\| (|A^*|^m + |B^*|^m)^{\frac{q}{2}} \right\|^{\frac{1}{q}}. \tag{2}$$

In order to apply inequality (2) to other fields better, we will consider some extensions of it. In this short note, we present some generalizations of (2).

MAIN RESULTS

Lemma 1 ([3]) *Let $A, B \in M_n$ be positive semidefinite matrices. Then $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant 0$ if and only if $X = A^{\frac{1}{2}} K B^{\frac{1}{2}}$ for some contraction K.*

Lemma 2 ([4]) *Let $A_i \in M_n$ $(i = 1, 2, \dots, k)$. Then*

$$s(A_1 A_2 \cdots A_k) \prec_{w\log} s(A_1)s(A_2)\cdots s(A_k) \quad \textit{and}$$
$$s(A_1 A_2 \cdots A_k) \prec_{w} s(A_1)s(A_2)\cdots s(A_k).$$

Lemma 3 ([4]) *Let $A, B \in M_n$ and $s(A) \prec_w s(B)$. Then*

$$\|A\| \leqslant \|B\|.$$

Lemma 4 ([5]) *Let $A, B \in M_n$ be positive semidefinite matrices. Then, for $p > 1$ and $1/p + 1/q = 1$,*

$$\|AB\| \leqslant \|A^p\|^{\frac{1}{p}} \|B^q\|^{\frac{1}{q}}.$$

Theorem 1 *For any $A, B \in M_n$ and suppose that p, q be real numbers with $p > 1$ and $1/p + 1/q = 1$. Then* *for any positive integer m,*

$$\left\|\sum_{i=1}^{k} A_i|A_i|^{m-1}\right\| \leqslant \left\|\left(\sum_{i=1}^{k}|A_i|^{m-s_i}\right)^{\frac{p}{2}}\right\|^{\frac{1}{p}} \left\|\left(\sum_{i=1}^{k}|A_i^*|^{m+s_i}\right)^{\frac{q}{2}}\right\|^{\frac{1}{q}},$$

where $s_i \in [-1,1]$.

Proof: Let $A_i \in M_n$ $(1 \leqslant i \leqslant k)$ with polar decomposition $A_i = U_i\,|A_i|$. It follows from

$$\begin{bmatrix}|A_i|^{r_i}\\|A_i|^{1-r_i}\end{bmatrix} A_i^{m-1}\begin{bmatrix}|A_i|^{r_i} & |A_i|^{1-r_i}\end{bmatrix} = \begin{bmatrix}|A_i|^{m+2r_i-1} & |A_i|^{m}\\ |A_i|^{m} & |A_i|^{m+1-2r_i}\end{bmatrix} \geqslant 0$$

for $0 \leqslant r_i \leqslant 1$ and $\left|A_i^*\right| = U_i\,|A_i|\,U_i^*$ that

$$\sum_{i=1}^{k}\begin{bmatrix}I & 0\\0 & U_i\end{bmatrix}\begin{bmatrix}|A_i|^{m+2r_i-1} & |A_i|^{m}\\ |A_i|^{m} & |A_i|^{m+1-2r_i}\end{bmatrix}\begin{bmatrix}I & 0\\0 & U_i^*\end{bmatrix} = \begin{bmatrix}\sum_{i=1}^{k}|A_i|^{m+2r_i-1} & \sum_{i=1}^{k}|A_i|^{m}U_i^*\\ \sum_{i=1}^{k}U_i^{m}|A_i| & \sum_{i=1}^{k}\left|A_i^*\right|^{m+1-2r_i}\end{bmatrix} \geqslant 0.$$

Using Lemma 1, we get

$$\sum_{i=1}^{k}|A_i|^{m}U_i^* = \left(\sum_{i=1}^{k}|A_i|^{m+2r_i-1}\right)^{\frac{1}{2}} K \left(\sum_{i=1}^{k}\left|A_i^*\right|^{m+1-2r_i}\right)^{\frac{1}{2}}$$

for contraction K. By Lemma 2, we obtain for $t = 1,2,\dots,n$,

$$\begin{aligned}
&\sum_{j=1}^{t}s_j\left(\sum_{i=1}^{k}U_i\,|A_i|^{m}\right)\\
&= \sum_{j=1}^{t}s_j\left(\sum_{i=1}^{k}|A_i|^{m}U_i^*\right)\\
&= \sum_{j=1}^{t}s_j\left(\left(\sum_{i=1}^{k}|A_i|^{m+2r_i-1}\right)^{\frac{1}{2}} K \left(\sum_{i=1}^{k}\left|A_i^*\right|^{m+1-2r_i}\right)^{\frac{1}{2}}\right)\\
&\leqslant \sum_{j=1}^{t}s_j\left(\sum_{i=1}^{k}|A_i|^{m+2r_i-1}\right)^{\frac{1}{2}} s_j(K) s_j\left(\sum_{i=1}^{k}\left|A_i^*\right|^{m+1-2r_i}\right)^{\frac{1}{2}}\\
&\leqslant \sum_{j=1}^{t}s_j\left(\sum_{i=1}^{k}|A_i|^{m+2r_i-1}\right)^{\frac{1}{2}} s_j\left(\sum_{i=1}^{k}\left|A_i^*\right|^{m+1-2r_i}\right)^{\frac{1}{2}}.
\end{aligned}$$

Let $X = \operatorname{diag} s_j\left(\sum_{i=1}^{k}|A_i|^{m+2r_i-1}\right)^{\frac{1}{2}}$ and $Y = \operatorname{diag} s_j\left(\sum_{i=1}^{k}\left|A_i^*\right|^{m+1-2r_i}\right)^{\frac{1}{2}}$. It is obvious that

$$\sum_{j=1}^{t}s_j\left(\sum_{i=1}^{k}U_i\,|A_i|^{m}\right) \leqslant \sum_{j=1}^{t}s_j(XY). \tag{3}$$

Using Lemma 3, we conclude that inequality (3) is equivalent to

$$\left\|\sum_{i=1}^{k}U_i\,|A_i|^{m}\right\| \leqslant \|XY\|. \tag{4}$$

According to Lemma 4,

$$\|XY\| \leqslant \|X^p\|^{\frac{1}{p}}\,\|Y^q\|^{\frac{1}{q}}. \tag{5}$$

It follows from (4) and (5) that

$$\left\|\sum_{i=1}^{k}U_i\,|A_i|^{m}\right\| \leqslant \left\|\left(\sum_{i=1}^{k}|A_i|^{m+2r_i-1}\right)^{\frac{p}{2}}\right\|^{\frac{1}{p}} \left\|\left(\sum_{i=1}^{k}\left|A_i^*\right|^{m+1-2r_i}\right)^{\frac{q}{2}}\right\|^{\frac{1}{q}}.$$

Let $s_i = 1-2r_i$. Then

$$\left\|\sum_{i=1}^{k}A_i\,|A_i|^{m-1}\right\| \leqslant \left\|\left(\sum_{i=1}^{k}|A_i|^{m-s_i}\right)^{\frac{p}{2}}\right\|^{\frac{1}{p}} \left\|\left(\sum_{i=1}^{k}\left|A_i^*\right|^{m+s_i}\right)^{\frac{q}{2}}\right\|^{\frac{1}{q}}, \tag{6}$$

where $s_i \in [-1,1]$. □

Corollary 1 ([1]) *Let* $A,B \in M_n$, $p > 1$ *and* $1/p + 1/q = 1$. *Then for any positive integer m,*

$$\left\|A|A|^{m-1} + B|B|^{m-1}\right\| \leqslant \left\|(|A|^{m}+|B|^{m})^{\frac{p}{2}}\right\|^{\frac{1}{p}} \left\|(|A^*|^{m}+|B^*|^{m})^{\frac{q}{2}}\right\|^{\frac{1}{q}}.$$

Proof: This follows from inequality (6) by letting $k = 2$, $s_1 = s_2 = 0$ and $A_1 = A$, $A_2 = B$. □

Theorem 2 *For any* $A,B \in M_n$, *positive integer m and* s_1, $s_2 \in [-1,1]$,

$$\begin{aligned}
&\left\|A|A|^{m-1} + B|B|^{m-1}\right\|\\
&\leqslant (2-r)\left\|(|A|^{m}+|B|^{m})^{\frac{p_1}{2}}\right\|^{\frac{1}{p_1}} \left\|(|A^*|^{m}+|B^*|^{m})^{\frac{q_1}{2}}\right\|^{\frac{1}{q_1}}\\
&\quad + (r-1)\left\|(|A|^{m-s_1}+|B|^{m-s_2})^{\frac{p_2}{2}}\right\|^{\frac{1}{p_2}}\\
&\quad \times \left\|(|A^*|^{m+s_1}+|B^*|^{m+s_2})^{\frac{q_2}{2}}\right\|^{\frac{1}{q_2}}
\end{aligned}$$

for p_1, $p_2 > 1$ *and* $1/p_1 + 1/q_1 = 1$, $1/p_2 + 1/q_2 = 1$, $1 \leqslant r \leqslant 2$.

Proof: Let $p = 1/(2-r)$, $q = 1/(r-1)$, then $1/p + 1/q = 1$ and $p, q > 0$. It follows from inequality (2) and inequality (6) that

$$\begin{aligned}
&\left\|A|A|^{m-1} + B|B|^{m-1}\right\| \\
&= \left\|A|A|^{m-1} + B|B|^{m-1}\right\|^{2-r} \left\|A|A|^{m-1} + B|B|^{m-1}\right\|^{r-1} \\
&\leqslant \left(\left\|(|A|^m + |B|^m)^{\frac{p_1}{2}}\right\|^{\frac{1}{p_1}} \left\|(|A^*|^m + |B^*|^m)^{\frac{q_1}{2}}\right\|^{\frac{1}{q_1}} \right)^{2-r} \\
&\quad \times \left(\left\|(|A|^{m-s_1} + |B|^{m-s_2})^{\frac{p_2}{2}}\right\|^{\frac{1}{p_2}} \right. \\
&\quad \left. \times \left\|(|A^*|^{m+s_1} + |B^*|^{m+s_2})^{\frac{q_2}{2}}\right\|^{\frac{1}{q_2}} \right)^{r-1}.
\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}
&\left(\left\|(|A|^m + |B|^m)^{\frac{p_1}{2}}\right\|^{\frac{1}{p_1}} \left\|(|A^*|^m + |B^*|^m)^{\frac{q_1}{2}}\right\|^{\frac{1}{q_1}} \right)^{2-r} \times \\
&\left(\left\|(|A|^{m-s_1} + |B|^{m-s_2})^{\frac{p_2}{2}}\right\|^{\frac{1}{p_2}} \left\|(|A^*|^{m+s_1} + |B^*|^{m+s_2})^{\frac{q_2}{2}}\right\|^{\frac{1}{q_2}} \right)^{r-1} \\
&\leqslant (2-r) \left\|(|A|^m + |B|^m)^{\frac{p_1}{2}}\right\|^{\frac{1}{p_1}} \left\|(|A^*|^m + |B^*|^m)^{\frac{q_1}{2}}\right\|^{\frac{1}{q_1}} \\
&\quad + (r-1) \left\|(|A|^{m-s_1} + |B|^{m-s_2})^{\frac{p_2}{2}}\right\|^{\frac{1}{p_2}} \\
&\quad \times \left\|(|A^*|^{m+s_1} + |B^*|^{m+s_2})^{\frac{q_2}{2}}\right\|^{\frac{1}{q_2}}.
\end{aligned}$$

This completes the proof. □

Remark 1 Putting $r = 1$ in Theorem 2, we get inequality (2). Putting $r = \frac{3}{2}$, $s_1 = s_2 = 0$ in Theorem 2, we get

$$\begin{aligned}
&\left\|A|A|^{m-1} + B|B|^{m-1}\right\| \\
&\leqslant \frac{1}{2} \left\|(|A|^m + |B|^m)^{\frac{p_1}{2}}\right\|^{\frac{1}{p_1}} \left\|(|A^*|^m + |B^*|^m)^{\frac{q_1}{2}}\right\|^{\frac{1}{q_1}} \\
&\quad + \frac{1}{2} \left\|(|A|^{m-s_1} + |B|^{m-s_2})^{\frac{p_2}{2}}\right\|^{\frac{1}{p_2}} \\
&\quad \times \left\|(|A^*|^{m+s_1} + |B^*|^{m+s_2})^{\frac{q_2}{2}}\right\|^{\frac{1}{q_2}} \qquad (7)
\end{aligned}$$

for $p_1, p_2 > 1$ and $1/p_1 + 1/q_1 = 1$, $1/p_2 + 1/q_2 = 1$. Inequality (7) is a generalization of inequality (2). Putting $r = 2$ and $s_1 = s_2 = 0$ in Theorem 2, we get inequality (2). Therefore, Theorem 2 is another generalization of inequality (2).

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