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γ-total dominating graphs of paths and cycles

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ABSTRACT: A total dominating set for a graph $G=(V(G),E(G))$ is a subset D of $V(G)$ such that every vertex in $V(G)$ is adjacent to some vertex in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of cardinality $\gamma_t(G)$ is called a γ-total dominating set. Let TD_γ be the set of all γ-total dominating sets in G. We define the γ-total dominating graph of G, denoted by $TD_\gamma(G)$, to be the graph whose vertex set is TD_γ, and two γ-total dominating sets D_1 and D_2 from TD_γ are adjacent in $TD_\gamma(G)$ if $D_1 = D_2\backslash\{u\}\cup\{v\}$ for some $u\in D_2$ and $v\notin D_2$. In this paper, we present γ-total dominating graphs of paths and cycles.

KEYWORDS: total dominating set, total dominating subset, total domination number

MSC2010: 05C40

INTRODUCTION

Let $G=(V(G),E(G))$ be a graph where $V(G)$ and $E(G)$ are the set of vertices and the set of edges of G, respectively. A set $D\subseteq V(G)$ is called a *dominating set* if every vertex in $V(G)\backslash D$ is adjacent to some vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a γ-*dominating set* (or γ-set). For basic concepts and notation in domination, see Refs. 1, 2.

Let G be a graph and D_γ the set of all γ-dominating sets. Lakshmanan and Vijayakumar[3] introduced a *gamma graph* $\gamma.G$ of G. The vertex set of $\gamma.G$ is D_γ, and two γ-dominating sets D_1 and D_2 from D_γ are adjacent in $\gamma(G)$ if $D_1=D_2\backslash\{u\}\cup\{v\}$ for some $u,v\in V(G)$. They provided the relationship between the clique number and independence of a graph and its gamma graph. Fricke et al[4] also defined a gamma graph $G(\gamma)$ with a different meaning. The only difference is that two γ-dominating sets D_1 and D_2 from D_γ are adjacent in $G(\gamma)$ if $D_1=D_2\backslash\{u\}\cup\{v\}$ for some adjacent vertices u and v. Note that $G(\gamma)$ is a subgraph of $\gamma.G$ with the same vertex set.

In Ref. 5, Haas and Seyffarth defined a k-*dominating graph* of a graph G, denoted by $D_k(G)$. Its vertex set contains all dominating sets D such that $|D|\leqslant k$, and two such dominating sets are adjacent in $D_k(G)$ if one can be obtained from the other by either adding or deleting a single vertex. The authors gave some conditions for connectivity of $D_k(G)$.

Kulli and Janakiram[6] introduced a *minimal dominating graph* of a graph G, denoted by $MD(G)$, which is the graph whose vertices are minimal dominating sets, and two minimal dominating sets are adjacent in $MD(G)$ if they have at least one vertex in common. They characterized connected minimal dominating graphs.

In Ref. 7, Kulli and Janakiram introduced a *common minimal dominating graph* of a graph G, denoted by $CD(G)$. It has the same vertex set as G, and two vertices are adjacent in $CD(G)$ if there is a minimal dominating set in G which contains them. The authors characterized connected common minimal dominating graphs. They also gave characterization of a graph G for which $CD(G)$ is isomorphic to the complement of G.

A *common minimal total dominating graph* of a graph G, denoted by $CD_t(G)$, is the graph with the same vertex set as G, and two vertices are adjacent in $CD_t(G)$ if there is a minimal total dominating set in G which contains them. This concept was introduced in Ref. 8.

A set D of vertices in a graph G is called a *total dominating set* if every vertex of G is adjacent to some vertex in D. Total dominating sets were introduced by Cockayne et al[9]. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum

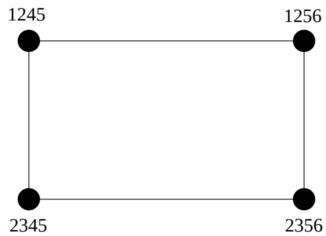

Fig. 1 The γ-total dominating graph of a path with 6 vertices. In this and later figures we write $abcd$ instead of $\{v_a, v_b, v_c, v_d\}$.

cardinality of a total dominating set of G. A total dominating set of cardinality $\gamma_t(G)$ is called a *γ-total dominating set*. Let TD_γ be the set of all γ-total dominating sets in G. The *γ-total dominating graph* of G, denoted by $TD_\gamma(G)$, is the graph whose vertex set is TD_γ, and two γ-total dominating sets D_1 and D_2 from TD_γ are adjacent in $TD_\gamma(G)$ if $D_1 = D_2 \backslash \{u\} \cup \{v\}$ for some $u \in D_2$ and $v \notin D_2$. For instance, the γ-total dominating graph of the path $v_1v_2v_3v_4v_5v_6$ is shown in Fig. 1.

PRELIMINARY RESULTS

Let D be a total dominating set of a graph G, S a subset of D, V' the set of vertices in G which are dominated by the vertices in S, and G' the subgraph of G induced by V'. Then S is called a *total dominating subset* of D if S is a total dominating set of G'.

We first consider the relation between the number of vertices in S and the number of vertices in G dominated by the vertices in S when G is a path or a cycle. We have that any 2 consecutive vertices in G can dominate at most 4 vertices, and 3 consecutive vertices in G can dominate at most 5 vertices, so we easily obtain the following lemma.

Lemma 1 *Let G be a path or cycle with n vertices, D a total dominating set of G, and S a total dominating subset of D of size k. If k is even, then S can dominate at most $2k$ vertices of G; otherwise, S can dominate at most $2k-1$ vertices of G.*

Lemma 2 *Let G be a graph. If v is a support vertex (the vertex adjacent to a vertex of degree one) of G, then v has to be in every total dominating set of G.*

The γ-total domination numbers of paths and cycles were established by Henning[10], as shown in the following theorem.

Theorem 1 *For $n \geqslant 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor \frac{1}{2}n \rfloor + \lceil \frac{1}{4}n \rceil - \lfloor \frac{1}{4}n \rfloor$.*

TOTAL DOMINATING GRAPH OF PATHS

In this section, we consider γ-total dominating graphs of paths. We always let $P_n = v_1v_2 \dots v_n$ be a path with n vertices. If $n = 1$, we have that $TD_\gamma(P_1)$ is the empty graph since P_1 has no γ-total dominating sets. For $n \geqslant 2$, we obtain the following theorems.

Theorem 2 *Let $k \geqslant 1$ be an integer. Then $TD_\gamma(P_{4k}) \cong K_1$.*

Proof: We first show that each γ-total dominating set of P_{4k} cannot contain three or more consecutive vertices of P_{4k}. Suppose for a contradiction that there is a γ-total dominating set D containing three or more consecutive vertices of P_{4k}. Let l be the largest number of these consecutive vertices, so $l \geqslant 3$. Let S be the set obtained from D by removing these l vertices. Then S is a total dominating subset of D. Note that $|D| = 2k$ by Theorem 1. Since these l vertices dominate at most $l+2$ vertices of P_{4k}, the other $2k-l$ vertices in D must dominate at least $4k-(l+2) = 4k-l-2$ vertices of P_{4k}. By Lemma 1, the $2k-l$ vertices in S can dominate at most $4k-2l$ vertices of P_{4k}, which is less than $4k-l-2$ since $l \geqslant 3$. This is a contradiction. Thus every γ-total dominating set must contain k groups of two consecutive vertices. Hence there is only one γ-total dominating set, which is $\{v_2, v_3, v_6, v_7, \dots, v_{4k-2}, v_{4k-1}\}$. □

Theorem 3 *Let $k \geqslant 1$ be an integer. Then $TD_\gamma(P_{4k+1}) \cong P_k$.*

Proof: We prove by induction on k.

Base step. There is only one γ-total dominating sets of P_5, which is $\{v_2, v_3, v_4\}$. Hence $TD_\gamma(P_5) \cong P_1$. Furthermore, there are two γ-total dominating sets of P_9, which are $\{v_2, v_3, v_4, v_7, v_8\}$ and $\{v_2, v_3, v_6, v_7, v_8\}$. Hence $TD_\gamma(P_9) \cong P_2$.

Induction step. Let $k \geqslant 2$. Suppose that $TD_\gamma(P_{4k+1}) \cong P_k$. Without loss of generality, we may assume that $TD_\gamma(P_{4k+1}) = D_1D_2 \dots D_k$, where $D_1 = \{v_2, v_3, v_4, v_7, v_8, \dots, v_{4k-5}, v_{4k-4}, v_{4k-1}, v_{4k}\}$ and for each $l = 2, 3, \dots, k$, $D_l = D_1 \backslash \{v_{4i} \mid i = 1, 2, \dots, l-1\} \cup \{v_{4i+2} \mid i = 1, 2, \dots, l-1\}$. We next show that $TD_\gamma(P_{4k+3}) \cong P_{k+1}$. For each $l = 1, 2, \dots, k$, let $D'_l = D_l \cup \{v_{4k+3}, v_{4k+4}\}$ and $D'_{k+1} = D_k \backslash \{v_{4k}\} \cup \{v_{4k+2}, v_{4k+3}, v_{4k+4}\}$. Hence D'_l is a γ-total dominating set of P_{4k+5} for all $l = 1, 2, \dots, k+1$. Furthermore, $D'_1D'_2 \dots D'_{k+1}$ forms a path with $k+1$

vertices in $TD_\gamma(P_{4k+5})$. We claim that there is no other γ-total dominating set of P_{4k+5} apart from $D'_1, D'_2, \dots, D'_{k+1}$. Suppose for a contradiction that there is another γ-total dominating set D' of P_{4k+5}, which is different from these total dominating sets. By Theorem 1, $\gamma_t(P_{4k}) = 2k$, $\gamma_t(P_{4k+1}) = 2k+1$ and $\gamma_t(P_{4k+5}) = 2k+3$, so $|D'| = 2k+3$. Furthermore, $|D' \cap \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}| \geqslant 2$ and $v_{4k+4} \in D'$ by Lemma 2. We consider the following 3 cases.

Case 1: $|D' \cap \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}| = 2$.

Subcase 1.1: $v_{4k+3}, v_{4k+4} \in D'$, but $v_{4k+2}, v_{4k+5} \notin D'$. Hence $D' \backslash \{v_{4k+3}, v_{4k+4}\}$ is a γ-total dominating set of P_{4k+1}. Thus $D' \backslash \{v_{4k+3}, v_{4k+4}\} = D_l$ for some $l = 1, 2, \dots k$. Hence $D' = D_l \cup \{v_{4k+3}, v_{4k+4}\} = D'_l$, a contradiction.

Subcase 1.2: $v_{4k+4}, v_{4k+5} \in D'$, but $v_{4k+2}, v_{4k+3} \notin D'$. Thus $D' \backslash \{v_{4k+4}, v_{4k+5}\}$ is a total dominating subset of D'. Since v_{4k+4} and v_{4k+5} dominate 3 vertices, the other $2k+1$ vertices in D' must dominate at least $4k+2$ vertices. By Lemma 1, these $2k+1$ vertices in D' can dominate at most $4k+1$ vertices. This is a contradiction.

Case 2: $|D' \cap \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}| = 3$.

Subcase 2.1: $v_{4k+2}, v_{4k+3}, v_{4k+4} \in D'$, but $v_{4k+5} \notin D'$. Suppose for a contradiction that $v_{4k+1} \in D'$. Then $v_{4k} \notin D'$ (otherwise, D' is not minimal). Thus $D' \backslash \{v_{4k+1}, v_{4k+2}, v_{4k+3}, v_{4k+4}\}$ is a total dominating subset of D'. Since v_{4k+1}, v_{4k+2}, v_{4k+3} and v_{4k+4} dominate 6 vertices, the other $2k-1$ vertices in D' must dominate at least $4k-1$ vertices. This contradicts Lemma 1. Hence $v_{4k+1} \notin D'$. Since D' is a γ-total dominating set of P_{4k+5}, $D' \backslash \{v_{4k+2}, v_{4k+3}, v_{4k+4}\}$ is a γ-total dominating set of P_{4k}. By Theorem 2, $\{v_2, v_3, v_6, v_7, \dots, v_{4k-2}, v_{4k-1}\}$ is the only γ-total dominating set of P_{4k}. Thus $D' \backslash \{v_{4k+2}, v_{4k+3}, v_{4k+4}\} = \{v_2, v_3, v_6, v_7, \dots, v_{4k-2}, v_{4k-1}\} = D_k \backslash \{v_{4k}\}$. Hence $D' = D_k \backslash \{v_{4k}\} \cup \{v_{4k+2}, v_{4k+3}, v_{4k+4}\} = D'_{k+1}$, a contradiction.

Subcase 2.2: $v_{4k+2}, v_{4k+4}, v_{4k+5} \in D'$, but $v_{4k+3} \notin D'$. Then $v_{4k+1} \in D'$. We next have that $v_{4k} \notin D'$ (otherwise, D' is not minimal). Thus $D' \backslash \{v_{4k+1}, v_{4k+2}, v_{4k+4}, v_{4k+5}\}$ is a total dominating subset of D'. Similarly, we then obtain a contradiction to Lemma 1, so this case is impossible.

Subcase 2.3: $v_{4k+3}, v_{4k+4}, v_{4k+5} \in D'$, but $v_{4k+2} \notin D'$. This case is impossible since D' is not minimal.

Case 3: $|D' \cap \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}| = 4$. This case is impossible since D' is not minimal. □

Theorem 4 *Let $k \geqslant 0$ be an integer. Then* $TD_\gamma(P_{4k+2}) \cong P_{k+1} \square P_{k+1}$.

Proof: We prove by induction on k. For $k = 0$, there is only one γ-total dominating set of P_2, so $TD_\gamma(P_2) \cong K_1 \cong P_1 \square P_1$. For $k = 1$, the graph $TD_\gamma(P_6)$ is shown in Fig. 1.

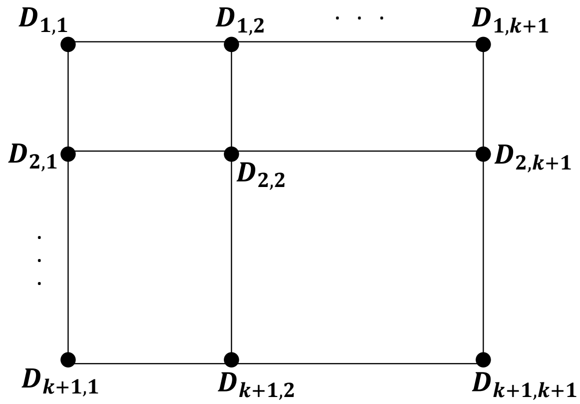

Fig. 2 The γ-total dominating graph of a path with $4k+2$ vertices.

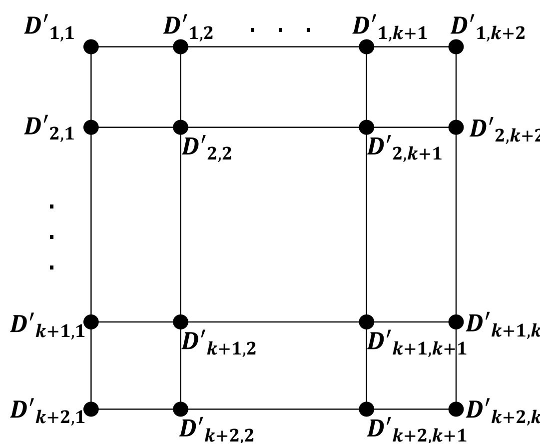

Fig. 3 The γ-total dominating graph of a path with $4k+6$ vertices.

Let $k \geqslant 1$. Suppose that $TD_\gamma(P_{4k+2}) \cong P_{k+1} \square P_{k+1}$. Without loss of generality, we may assume that $TD_\gamma(P_{4k+2})$ is the graph shown in Fig. 2, whose vertices are $D_{i,j} = O_i \cup E_j$ for all integers $1 \leqslant i, j \leqslant k+1$, where $O_1 = \{v_{4i+1} \mid i = 0, 1, \dots, k\}$, $E_1 = \{v_2\} \cup \{v_{4i} \mid i = 1, 2, \dots, k\}$, and for each $l = 1, 2, \dots, k$, $O_{l+1} = \{v_{4i+3} \mid i = 0, 1, \dots, l-1\} \cup \{v_{4i+1} \mid i = l, l+1, \dots, k\}$ and $E_{l+1} = \{v_{4i+2} \mid i = 0, 1, \dots, l\} \cup \{v_{4i} \mid i = l, l+1, \dots, k\}$. It is easy to check that $v_{4k-1} \in O_i$ if and only if $i = k+1$, and $v_{4k+2} \in E_j$ if and only if $j = k+1$.

We next show that $TD_\gamma(P_{4k+6}) \cong P_{k+2} \square P_{k+2}$. For each $i, j = 1, 2, \dots, k+1$, let $D'_{i,j} = D_{i,j} \cup \{v_{4k+4}, v_{4k+5}\}$. For each $i = 1, 2, \dots, k+1$, let $D'_{i,k+2} =$

$D_{i,k+1} \cup \{v_{4k+5}, v_{4k+6}\}$. For each $j = 1, 2, \dots, k+1$, let $D'_{k+2,j} = D_{k+1,j} \backslash \{v_{4k+1}\} \cup \{v_{4k+3}, v_{4k+4}, v_{4k+5}\}$, and $D'_{k+2,k+2} = D_{k+1,k+1} \backslash \{v_{4k+1}\} \cup \{v_{4k+3}, v_{4k+5}, v_{4k+6}\}$. Then $D'_{i,j}$ is a γ-total dominating set of P_{4k+6} for all $i, j = 1, 2, \dots, k+2$. Furthermore, these $D'_{i,j}$'s form the graph $P_{k+2} \Box P_{k+2}$ in $TD_\gamma(P_{4k+6})$ (Fig. 3). Suppose for a contradiction that there is another γ-total dominating set D' of P_{4k+6}, which is different from these γ-total dominating sets. Note that $|D'| = 2k+4$, $|D' \cap \{v_{4k+3}, v_{4k+4}, v_{4k+5}, v_{4k+6}\}| \geqslant 2$, and $v_{4k+5} \in D'$. We consider the following 3 cases.

Case 1: $|D' \cap \{v_{4k+3}, v_{4k+4}, v_{4k+5}, v_{4k+6}\}| = 2$.

Subcase 1.1: $v_{4k+4}, v_{4k+5} \in D'$, but $v_{4k+3}, v_{4k+6} \notin D'$. Hence $D' \backslash \{v_{4k+4}, v_{4k+5}\}$ is a γ-total dominating set of P_{4k+2}. Thus $D' \backslash \{v_{4k+4}, v_{4k+5}\} = D_{i,j}$ for some integers $1 \leqslant i, j \leqslant k+1$. Hence $D' = D_{i,j} \cup \{v_{4k+4}, v_{4k+5}\} = D'_{i,j}$, a contradiction.

Subcase 1.2: $v_{4k+5}, v_{4k+6} \in D'$, but $v_{4k+3}, v_{4k+4} \notin D'$. Then $v_{4k+2} \in D'$. Thus $D' \backslash \{v_{4k+5}, v_{4k+6}\}$ is a γ-total dominating set of P_{4k+2} containing v_{4k+2}. Since v_{4k+2} is in only E_{k+1}, $D' \backslash \{v_{4k+5}, v_{4k+6}\} = D_{i,k+1}$ for some $i \in \{1, 2, \dots, k+1\}$. Hence $D' = D_{i,k+1} \cup \{v_{4k+5}, v_{4k+6}\} = D'_{i,k+2}$, a contradiction.

Case 2: $|D' \cap \{v_{4k+3}, v_{4k+4}, v_{4k+5}, v_{4k+6}\}| = 3$.

Subcase 2.1: $v_{4k+3}, v_{4k+4}, v_{4k+5} \in D'$, but $v_{4k+6} \notin D'$.

Subcase 2.1.1: $v_{4k+2} \in D'$. Clearly, $v_{4k+1} \notin D'$. No matter whether v_{4k} is in D' or not, v_{4k-1} must be in D'. Then $D' \backslash \{v_{4k+3}, v_{4k+4}, v_{4k+5}\} \cup \{v_{4k+1}\}$ is a γ-total dominating set of P_{4k+2} containing v_{4k-1}. Since v_{4k-1} is only in O_{k+1}, $D' \backslash \{v_{4k+3}, v_{4k+4}, v_{4k+5}\} \cup \{v_{4k+1}\} = D_{k+1,j}$ for some $j \in \{1, 2, \dots, k+1\}$. Thus $D' = D_{k+1,j} \backslash \{v_{4k+1}\} \cup \{v_{4k+3}, v_{4k+4}, v_{4k+5}\} = D'_{k+2,j}$, a contradiction.

Subcase 2.1.2: $v_{4k+2} \notin D'$. If $v_{4k+1} \in D'$, D' is not minimal. Thus $v_{4k+1} \notin D'$, so $v_{4k-1}, v_{4k} \in D'$. Hence $D' \backslash \{v_{4k+3}, v_{4k+4}, v_{4k+5}\} \cup \{v_{4k+1}\}$ is a γ-total dominating set of P_{4k+2} containing v_{4k-1}. Thus $D' \backslash \{v_{4k+3}, v_{4k+4}, v_{4k+5}\} \cup \{v_{4k+1}\} = D_{k+1,j}$ for some $j \in \{1, 2, \dots, k+1\}$. Hence $D' = D_{k+1,j} \backslash \{v_{4k+1}\} \cup \{v_{4k+3}, v_{4k+4}, v_{4k+5}\} = D'_{k+2,j}$, a contradiction.

Subcase 2.2: $v_{4k+3}, v_{4k+5}, v_{4k+6} \in D'$, but $v_{4k+4} \notin D'$. Then $v_{4k+2} \in D'$. If $v_{4k+1} \in D'$, D' is not minimal. Thus $v_{4k+1} \notin D'$. No matter whether v_{4k} is in D' or not, v_{4k-1} must be in D'. Thus $D' \backslash \{v_{4k+3}, v_{4k+5}, v_{4k+6}\} \cup \{v_{4k+1}\}$ is a γ-total dominating set of P_{4k+2}, containing v_{4k-1} and v_{4k+2}. Thus $D' \backslash \{v_{4k+3}, v_{4k+5}, v_{4k+6}\} \cup \{v_{4k+1}\} = D_{k+1,k+1}$. Thus $D' = D_{k+1,k+1} \backslash \{v_{4k+1}\} \cup \{v_{4k+3}, v_{4k+5}, v_{4k+6}\} = D'_{k+2,k+2}$, a contradiction.

Subcase 2.3: $v_{4k+4}, v_{4k+5}, v_{4k+6} \in D'$, but $v_{4k+3} \notin D'$. This case is impossible since D' is not minimal.

Case 3: $|D' \cap \{v_{4k+3}, v_{4k+4}, v_{4k+5}, v_{4k+6}\}| = 4$. This case is impossible since D' is not minimal. □

Theorem 5 *Let* $k \geqslant 0$ *be an integer. Then* $TD_\gamma(P_{4k+3}) \cong P_{k+2}$.

Proof: We prove by induction on k. It is easy to obtain $TD_\gamma(P_3) \cong P_2$ and $TD_\gamma(P_7) \cong P_3$.

Let $k \geqslant 1$. Suppose that $TD_\gamma(P_{4k+3}) \cong P_{k+2}$. Without loss of generality, we may assume that $TD_\gamma(P_{4k+3}) = D_1 D_2 \dots D_{k+2}$, where $D_1 = \{v_1, v_2, v_5, v_6, \dots, v_{4k-3}, v_{4k-2}, v_{4k+1}, v_{4k+2}\}$ and $D_l = D_1 \backslash \{v_{4i+1} \mid i = 0, 1, \dots, l-2\} \cup \{v_{4i+3} \mid i = 0, 1, \dots, l-2\}$ for each $l = 2, 3, \dots, k+2$. It is easy to check that $v_{4k+3} \in D_l$ if and only if $l = k+2$.

We show that $TD_\gamma(P_{4k+7}) \cong P_{k+3}$. For each $l = 1, 2, \dots, k+2$, let $D'_l = D_l \cup \{v_{4k+5}, v_{4k+6}\}$ and $D'_{k+3} = D_{k+2} \cup \{v_{4k+6}, v_{4k+7}\}$. Hence D'_l is a γ-total dominating set of P_{4k+7} for all $l = 1, 2, \dots, k+3$. Clearly, $D'_1 D'_2 \dots D'_{k+3}$ forms a path with $k+3$ vertices in $TD_\gamma(P_{4k+7})$. Suppose for a contradiction that there is another γ-total dominating set D' of P_{4k+7}, which is different from these γ-total dominating sets. Note that $\gamma_t(P_{4k+3}) = 2k+2$ and $\gamma_t(P_{4k+7}) = 2k+4$, so $|D'| = 2k+4$. Furthermore, $|D' \cap \{v_{4k+4}, v_{4k+5}, v_{4k+6}, v_{4k+7}\}| \geqslant 2$ and $v_{4k+6} \in D'$. We consider the following 3 cases.

Case 1: $|D' \cap \{v_{4k+4}, v_{4k+5}, v_{4k+6}, v_{4k+7}\}| = 2$.

Subcase 1.1: $v_{4k+5}, v_{4k+6} \in D'$, but $v_{4k+4}, v_{4k+7} \notin D'$. Hence $D' \backslash \{v_{4k+5}, v_{4k+6}\}$ is a γ-total dominating set of P_{4k+3}. Thus $D' \backslash \{v_{4k+5}, v_{4k+6}\} = D_l$ for some $l \in \{1, 2, \dots, k+2\}$. Hence $D' = D_l \cup \{v_{4k+5}, v_{4k+6}\} = D'_l$, a contradiction.

Subcase 1.2: $v_{4k+6}, v_{4k+7} \in D'$, but $v_{4k+4}, v_{4k+5} \notin D'$. Then $v_{4k+3} \in D'$. Thus $D' \backslash \{v_{4k+6}, v_{4k+7}\}$ is a γ-total dominating set of P_{4k+3}, which contains v_{4k+3}. Hence $D' \backslash \{v_{4k+6}, v_{4k+7}\} = D_{k+2}$ since v_{4k+3} is only in D_{k+2}. Hence $D' = D_{k+2} \cup \{v_{4k+6}, v_{4k+7}\} = D'_{k+3}$, a contradiction.

Case 2: $|D' \cap \{v_{4k+4}, v_{4k+5}, v_{4k+6}, v_{4k+7}\}| = 3$.

Subcase 2.1: $v_{4k+4}, v_{4k+5}, v_{4k+6} \in D'$, but $v_{4k+7} \notin D'$.

Subcase 2.1.1: $v_{4k+3} \in D'$. Thus $v_{4k+2} \notin D'$. Hence $D' \backslash \{v_{4k+3}, v_{4k+4}, v_{4k+5}, v_{4k+6}\}$ is a total dominating subset of D'. Since v_{4k+3}, v_{4k+4}, v_{4k+5}, and v_{4k+6} dominate 6 vertices, the other $2k$ vertices in D' must dominate at least $4k+1$ vertices. This contradicts Lemma 1.

Subcase 2.1.2: $v_{4k+3} \notin D'$. Hence $D' \backslash \{v_{4k+4}, v_{4k+5}, v_{4k+6}\}$ is a total dominating subset of D'. As with Subcase 2.1.1, there is a contradiction.

Subcase 2.2: $v_{4k+4}, v_{4k+6}, v_{4k+7} \in D'$, but $v_{4k+5} \notin D'$. Then $v_{4k+3} \in D'$. If $v_{4k+2} \in D'$, then

D' is not minimal. Hence $v_{4k+2} \notin D'$. Hence $D' \backslash \{v_{4k+3}, v_{4k+4}, v_{4k+6}, v_{4k+7}\}$ is a total dominating subset of D'. Similarly, we then obtain a contradiction to Lemma 1.

Subcase 2.3: $v_{4k+5}, v_{4k+6}, v_{4k+7} \in D'$, but $v_{4k+4} \notin D'$. This case is impossible since D' is not minimal.

Case 3: $|D' \cap \{v_{4k+4}, v_{4k+5}, v_{4k+6}, v_{4k+7}\}| = 4$. This case is impossible since D' is not minimal. □

TOTAL DOMINATING GRAPH OF CYCLES

In this section, we always let $C_n = v_0 v_1 \dots v_{n-1} v_0$ be a cycle with $n \geqslant 3$ vertices. It easy to see that $TD_\gamma(C_3) \cong C_3$ and $TD_\gamma(C_4) \cong C_4$. For $n \geqslant 5$, we obtain the following theorems.

Theorem 6 *Let $k \geqslant 2$ be an integer. Then $TD_\gamma(C_{4k}) \cong 4K_1$.*

Proof: We claim that each γ-total dominating set of C_{4k} cannot contain three or more consecutive vertices of C_{4k}. Suppose for a contradiction that there is a γ-total dominating set D of C_{4k}, which contains three or more consecutive vertices of C_{4k}. Let l be the largest number of these consecutive vertices, so $l \geqslant 3$. Let S be the set obtained from D by removing these l vertices. Then S is a total dominating subset of D. By Theorem 1, $|D| = 2k$. Since these l vertices dominate $l+2$ vertices of C_{4k}, the other $2k - l$ vertices in D must dominate at least $4k - (l+2) = 4k - l - 2$ vertices of C_{4k}. By Lemma 1, the $2k - l$ vertices in S can dominate at most $4k - 2l$ vertices of C_{4k}, which is less than $4k - l - 2$ since $l \geqslant 3$. This is a contradiction. Thus every γ-total dominating set must contain k groups of two consecutive vertices of C_{4k}. It is easy to see that there are only four γ-total dominating sets, which are $\{v_0, v_1, v_4, v_5, \dots, v_{4k-4}, v_{4k-3}\}$, $\{v_1, v_2, v_5, v_6, \dots, v_{4k-3}, v_{4k-2}\}$, $\{v_2, v_3, v_6, v_7, \dots, v_{4k-2}, v_{4k-1}\}$, and $\{v_0, v_3, v_4, \dots, v_{4k-5}, v_{4k-4}, v_{4k-1}\}$. □

Theorem 7 *Let $k \geqslant 1$ be an integer. Then $TD_\gamma(C_{4k+1}) \cong C_{4k+1}$.*

Proof: For $k = 1$, it is easy to obtain $TD_\gamma(C_5) \cong C_5$. Let $k \geqslant 2$.

Claim 1: each γ-total dominating set of C_{4k+1} cannot contain four or more consecutive vertices of C_{4k+1}. Suppose for a contradiction that there is a γ-total dominating set D of C_{4k+1}, which contains four or more consecutive vertices of C_{4k+1}. Let l be the largest number of these consecutive vertices, so $l \geqslant 4$. The set obtained from D by removing these l vertices forms a total dominating subset of D. We then obtain a contradiction to Lemma 1.

Claim 2: Each γ-total dominating set of C_{4k+1} contains only one group of three consecutive vertices of C_{4k+1}. Since $\gamma_t(C_{4k+1}) = 2k+1$ is an odd integer, each γ-total dominating set of C_{4k+1} contains at least one group of three consecutive vertices. Suppose for a contradiction that there is a γ-total dominating set D of C_{4k+1}, which contains l groups of three consecutive vertices of C_{4k+1}, where $l \geqslant 2$. These $3l$ vertices dominate at most $5l$ vertices. Thus the other $2k+1-3l$ vertices in D must dominate at least $4k+1-5l$ vertices of C_{4k+1}. By Lemma 1, these $2k+1-3l$ vertices in D can dominate at most $4k+2-6l$ vertices of C_{4k+1}, which is less than $4k+1-5l$ since $l \geqslant 3$. This is a contradiction.

Let D be any γ-total dominating set, so D contains one group of 3 consecutive vertices, which dominates 5 vertices of C_{4k+1}. We may consider the other $4k-4$ vertices in C_{4k+1} which are not dominated as a path. Apart from the 3 consecutive vertices in D, the other $2k$—2 vertices must dominate all $4k-4$ vertices on this path. By Theorem 2, there is only one γ-total dominating set of this path. Hence there is only one γ-total dominating set of C_{4k-4} containing these 3 consecutive vertices. To find all γ-total dominating sets of C_{4k+1}, it suffices to find 3 consecutive vertices on the cycle. Clearly, there are $4k+1$ γ-total dominating sets. Recall that $C_{4k+1} = v_0 v_1 \dots v_{4k} v_0$. Let $D_0 = \{v_0, v_1, v_2, v_5, v_6, v_9, v_{10}, \dots, v_{4k-3}, v_{4k-2}\}$ and $D_l = D_{l-1} \backslash \{v_{(4l-2) \pmod{4k+1}}\} \cup \{v_{(4l) \pmod{4k+1}}\}$ for each $l = 1, 2, \dots, 4k$. Then $D_0 D_1 \dots D_{4k} D_0$ forms a cycle with $4k+1$ vertices. □

Theorem 8 *Let $k \geqslant 1$ be an integer. Then $TD_\gamma(C_{4k+2}) \cong C_{2k+1} \square C_{2k+1}$.*

Proof:

We prove by induction on k. For $k = 1$ and $k = 2$, the graph $TD_\gamma(C_6)$ and $TD_\gamma(C_{10})$ are shown in Fig. 4 and Fig. 5, respectively.

Let $k \geqslant 2$. Suppose that $TD_\gamma(C_{4k+2}) \cong C_{2k+1} \square C_{2k+1}$. Without loss of generality, we may assume that $TD_\gamma(C_{4k+2})$ is the graph shown in Fig. 6, whose vertices are $D_{i,j} = O_i \cup E_j$ for all integers $1 \leqslant i, j \leqslant 2k+1$, where $O_1 = \{v_1\} \cup \{v_{4i-1} \mid i = 1, 2, \dots, k\}$, $E_1 = \{v_0\} \cup \{v_{4i+2} \mid i = 0, 1, \dots, k-1\}$, and for each $l = 2, 3, \dots, 2k+1$, $O_l = O_{l-1} \backslash \{v_{(4l-5) \pmod{4k+2}}\} \cup \{v_{(4l-3) \pmod{4k+2}}\}$ and $E_l = E_{l-1} \backslash \{v_{(4l-6) \pmod{4k+2}}\} \cup \{v_{(4l-4) \pmod{4k+2}}\}$.

Recall that $C_{4k+6} = v_0 v_1 \dots v_{4k+5} v_0$. We prove that $TD_\gamma(C_{4k+6}) \cong C_{2k+3} \square C_{2k+3}$. For each $i, j =$

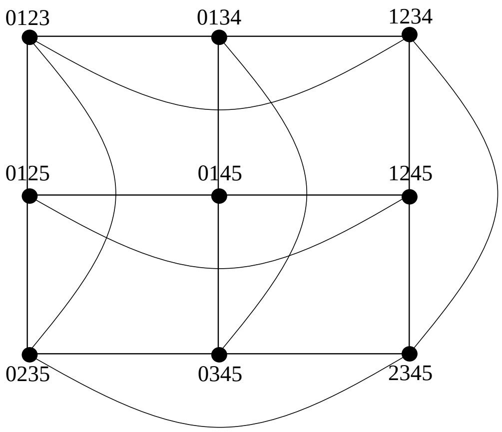

Fig. 4 The γ-total dominating graph of a cycle with 6 vertices.

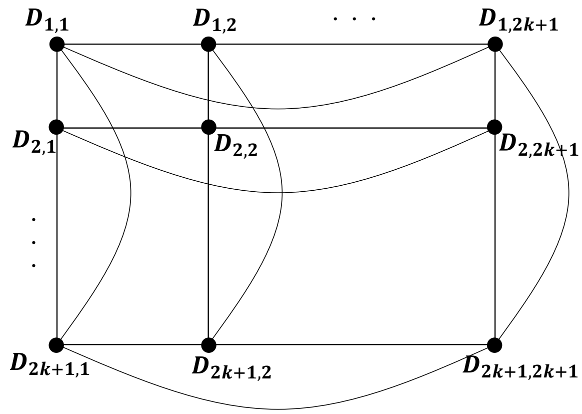

Fig. 6 The γ-total dominating graph of a cycle with $4k+2$ vertices.

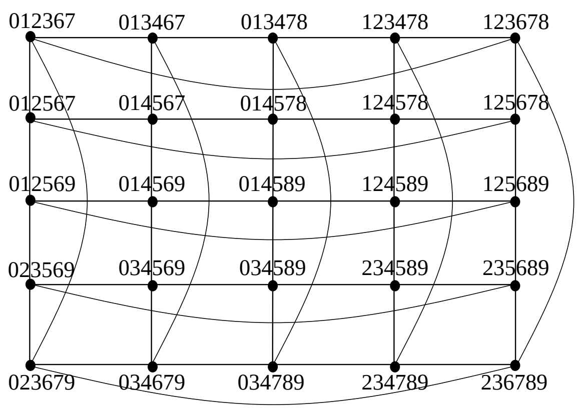

Fig. 5 The γ-total dominating graph of a cycle with 10 vertices.

$1,2,\dots,k+1$, let $D'_{i,j} = D_{i,j} \cup \{v_{4k+2}, v_{4k+3}\}$. For each $i = 1,2,\dots,k+1$, let $D'_{i,k+2} = D'_{i,k+1} \setminus \{v_{4k+2}\} \cup \{v_{4k+4}\}$. For each $i = 1,2,\dots,k+1$ and $j = k+3, k+4,\dots,2k+2$, let $D'_{i,j} = D_{i,j-1} \cup \{v_{4k+3}, v_{4k+4}\}$. For each $i = 1,2,\dots,k+1$, let $D'_{i,2k+3} = D'_{i,2k+2} \setminus \{v_{4k}\} \cup \{v_{4k+2}\}$. For each $j = 1,2,\dots,2k+3$, let $D'_{k+2,j} = D'_{k+1,j} \setminus \{v_{4k+3}\} \cup \{v_{4k+5}\}$. For each $i = k+3, k+4,\dots,2k+2$ and $j = 1,2,\dots,k+1$, let $D'_{i,j} = D_{i-1,j} \cup \{v_{4k+2}, v_{4k+5}\}$. For each $i = k+3, k+4,\dots,2k+2$, let $D'_{i,k+2} = D'_{i,k+1} \setminus \{v_{4k+2}\} \cup \{v_{4k+4}\}$. For each $i,j = k+3, k+4,\dots,2k+2$, let $D'_{i,j} = D_{i-1,j-1} \cup \{v_{4k+4}, v_{4k+5}\}$. For each $i = k+3, k+4,\dots,2k+2$, let $D'_{i,2k+3} = D'_{i,2k+2} \setminus \{v_{4k}\} \cup \{v_{4k+2}\}$. For each $j = 1,2,\dots,2k+3$, let $D'_{2k+3,j} = D'_{2k+2,j} \setminus \{v_{4k+1}\} \cup \{v_{4k+3}\}$. Note that $\gamma_t(C_{4k+6}) = 2k+4 = (2k+2)+2 = \gamma_t(C_{4k+2})+2$. It is easy to check that $D'_{i,j}$ is a γ-total dominating set of C_{4k+6} for all $i,j = 1,2,\dots,2k+3$. Furthermore, these $D'_{i,j}$'s form the graph $C_{2k+3} \square C_{2k+3}$ in $TD_\gamma(C_{4k+6})$.

Claim 1. For each $i = 1,2,\dots,2k+3$, $D'_{i,1} \setminus \{v_0\} \cup \{v_{4k+4}\} = D'_{i,2k+3}$. Let $i \in \{1,2,\dots,2k+3\}$. Then we have $D'_{i,1}$ and $D'_{i,2k+3}$ are adjacent in $TD_\gamma(C_{4k+6})$. Furthermore, $v_0 \in D'_{i,1}$, $v_0 \notin D'_{i,2k+3}$, $v_{4k+4} \in D'_{i,2k+3}$, and $v_{4k+4} \notin D'_{i,1}$. Thus $D'_{i,1} \setminus \{v_0\} \cup \{v_{4k+4}\} = D'_{i,2k+3}$.

Claim 2. For each $j = 1,2,\dots,2k+3$, $D'_{1,j} \setminus \{v_1\} \cup \{v_{4k+5}\} = D'_{2k+3,j}$. Let $j \in \{1,2,\dots,2k+3\}$. Then we have $D'_{1,j}$ and $D'_{2k+3,j}$ are adjacent in $TD_\gamma(C_{4k+6})$. Furthermore, $v_1 \in D'_{1,j}$, $v_1 \notin D'_{2k+3,j}$, $v_{4k+5} \in D'_{2k+3,j}$, and $v_{4k+5} \notin D'_{1,j}$. Thus $D'_{1,j} \setminus \{v_1\} \cup \{v_{4k+5}\} = D'_{2k+3,j}$.

Next, we prove that there are no other vertices in $TD_\gamma(C_{4k+6})$. Suppose for a contradiction that there is another γ-total dominating set D' of C_{4k+6}, which is different from these γ-total dominating sets. Note that $|D'| = 2k+4$ and $|D' \cap \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}| \geqslant 2$. We consider the following 3 cases.

Case 1: $|D' \cap \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}| = 4$. If $v_0 \in D'$, D' is not minimal. Hence $v_0 \notin D'$. Similarly, $v_{4k+1} \notin D'$. Since v_{4k+2}, v_{4k+3}, v_{4k+4} and v_{4k+5} dominate 6 vertices, the other $2k$ vertices in D' must dominate the vertices $v_1, v_2, v_3, \dots, v_{4k}$. We now consider these $4k$ vertices as a path. By Theorem 2, $D' \setminus \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}$ is the only γ-total dominating set of this path which is $\{v_2, v_3, v_6, v_7, \dots, v_{4k-2}, v_{4k-1}\}$. Thus $D' \setminus \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\} \cup \{v_{4k}, v_{4k+1}\} = \{v_2, v_3, v_6, v_7, \dots, v_{4k-2}, v_{4k-1}\} \cup \{v_{4k}, v_{4k+1}\} = D_{2k+1,2k+1}$ (Fig. 6). Hence $D' = D_{2k+1,2k+1} \setminus \{v_{4k}, v_{4k+1}\} \cup \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\} = [(D_{2k+1,2k+1} \cup \{v_{4k+4}, v_{4k+5}\}) \setminus \{v_{4k}\} \cup$

$\{v_{4k+2}\}]\backslash\{v_{4k+1}\} \cup \{v_{4k+3}\} = [D'_{2k+2,2k+2}\backslash\{v_{4k}\} \cup \{v_{4k+2}\}]\backslash\{v_{4k+1}\} \cup \{v_{4k+3}\} = D'_{2k+2,2k+3}\backslash\{v_{4k+1}\} \cup \{v_{4k+3}\} = D'_{2k+3,2k+3}$, a contradiction.

Case 2: $|D' \cap \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}| = 3$.

Subcase 2.1: $v_{4k+2}, v_{4k+3}, v_{4k+4} \in D'$, but $v_{4k+5} \notin D'$.

Subcase 2.1.1: $v_{4k+1} \in D'$. Since D' contains four consecutive vertices of the cycle, by repeating the process in Case 1, D' must be equal to $D'_{i,j}$ for some i, j, a contradiction.

Subcase 2.1.2: $v_{4k+1} \notin D'$. Thus $v_0 \notin D'$ (otherwise, D' is not minimal). Then $v_1, v_2 \in D'$. Similarly, $v_{4k} \notin D'$, so $v_{4k-1}, v_{4k-2} \in D'$. Hence $D'\backslash\{v_{4k+2}, v_{4k+3}, v_{4k+4}\} \cup \{v_{4k}\}$ is a γ-total dominating set of C_{4k+2}. Thus $D'\backslash\{v_{4k+2}, v_{4k+3}, v_{4k+4}\} \cup \{v_{4k}\} = D_{i,j} = O_i \cup E_j$ for some $1 \leqslant i, j \leqslant 2k+1$. Since $v_1 \in O_i$, $1 \leqslant i \leqslant k+1$. Since $v_{4k-2}, v_{4k} \in E_j$, $j = 2k+1$. Hence $D'\backslash\{v_{4k+2}, v_{4k+3}, v_{4k+4}\} \cup \{v_{4k}\} = D_{i,2k+1}$ for some $i \in \{1, 2, \ldots, k+1\}$. Hence $D' = D_{i,2k+1}\backslash\{v_{4k}\} \cup \{v_{4k+2}, v_{4k+3}, v_{4k+4}\} = (D_{i,2k+1} \cup \{v_{4k+3}, v_{4k+4}\})\backslash\{v_{4k}\} \cup \{v_{4k+2}\} = D'_{i,2k+2}\backslash\{v_{4k}\} \cup \{v_{4k+2}\} = D'_{i,2k+3}$, a contradiction.

Subcase 2.2: $v_{4k+2}, v_{4k+3}, v_{4k+5} \in D'$, but $v_{4k+4} \notin D'$. Hence $v_0 \in D'$. If $v_{4k+1} \in D'$, D' is not minimal. Hence $v_{4k+1} \notin D'$. No matter whether v_{4k} is in D' or not, v_{4k-1} must be in D'. Thus $D'\backslash\{v_{4k+2}, v_{4k+3}, v_{4k+5}\} \cup \{v_{4k+1}\}$ is a γ-total dominating set of C_{4k+2}. Thus $D'\backslash\{v_{4k+2}, v_{4k+3}, v_{4k+5}\} \cup \{v_{4k+1}\} = D_{i,j} = O_i \cup E_j$ for some $1 \leqslant i, j \leqslant 2k+1$. Since $v_{4k-1}, v_{4k+1} \in O_i$, $i = 2k+1$. Since $v_0 \in E_j$, $1 \leqslant j \leqslant k+1$. Thus $D'\backslash\{v_{4k+2}, v_{4k+3}, v_{4k+5}\} \cup \{v_{4k+1}\} = D_{2k+1,j}$ for some $j \in \{1, 2, \ldots, k+1\}$. Hence $D' = D_{2k+1,j}\backslash\{v_{4k+1}\} \cup \{v_{4k+2}, v_{4k+3}, v_{4k+5}\} = (D_{2k+1,j} \cup \{v_{4k+2}, v_{4k+5}\})\backslash\{v_{4k+1}\} \cup \{v_{4k+3}\} = D'_{2k+2,j}\backslash\{v_{4k+1}\} \cup \{v_{4k+3}\} = D'_{2k+3,j}$, a contradiction.

Subcase 2.3: $v_{4k+2}, v_{4k+4}, v_{4k+5} \in D'$, but $v_{4k+3} \notin D'$. It is easy to obtain $v_0, v_{4k} \notin D'$ but $v_2, v_{4k-2}, v_{4k+1} \in D'$. Thus $D'\backslash\{v_{4k+2}, v_{4k+4}, v_{4k+5}\} \cup \{v_0\}$ is a γ-total dominating set of C_{4k+2}. Thus $D'\backslash\{v_{4k+2}, v_{4k+4}, v_{4k+5}\} \cup \{v_0\}\} = D_{i,j} = O_i \cup E_j$ for some $1 \leqslant i, j \leqslant 2k+1$. Since $v_{4k+1} \in O_i$, $k+1 \leqslant i \leqslant 2k+1$. Since $v_0, v_2 \in E_j$, $j = 1$. Thus $D'\backslash\{v_{4k+2}, v_{4k+4}, v_{4k+5}\} \cup \{v_0\} = D_{i,1}$ for some $i \in \{k+1, k+2, \ldots, 2k+1\}$.

Subcase 2.3.1: $i = k+1$. Then $D' = D_{k+1,1} \cup \{v_{4k+2}, v_{4k+5}\}\backslash\{v_0\} \cup \{v_{4k+4}\} = [(D_{k+1,1} \cup \{v_{4k+2}, v_{4k+3}\})\backslash\{v_{4k+3}\} \cup \{v_{4k+5}\}]\backslash\{v_0\} \cup \{v_{4k+4}\} = [D'_{k+1,1}\backslash\{v_{4k+3}\} \cup \{v_{4k+5}\}]\backslash\{v_0\} \cup \{v_{4k+4}\} = D'_{k+2,1}\backslash\{v_0\} \cup \{v_{4k+4}\} = D'_{k+2,2k+3}$ by Claim 1.

Subcase 2.3.2: $i \in \{k+2, k+3, \ldots, 2k+1\}$. Then $D' = D_{i,1} \cup \{v_{4k+2}, v_{4k+5}\}\backslash\{v_0\} \cup \{v_{4k+4}\} = D'_{i+1,1}\backslash\{v_0\} \cup \{v_{4k+4}\} = D'_{i+1,2k+3}$.

Subcase 2.4: $v_{4k+3}, v_{4k+4}, v_{4k+5} \in D'$, but $v_{4k+2} \notin D'$.

Subcase 2.4.1: $v_0 \in D'$. Then D' contains four consecutive vertices of the cycle. Again, we repeat the process in case 1, so we are done.

Subcase 2.4.2: $v_0 \notin D'$. It is easy to obtain $v_2, v_3, v_{4k} \in D'$, but $v_1, v_{4k+1}, v_{4k+2} \notin D'$. Hence $D'\backslash\{v_{4k+3}, v_{4k+4}, v_{4k+5}\} \cup \{v_1\}$ is a γ-total dominating set of C_{4k+2}. Thus $D'\backslash\{v_{4k+3}, v_{4k+4}, v_{4k+5}\} \cup \{v_1\} = D_{i,j} = O_i \cup E_j$ for some $1 \leqslant i, j \leqslant 2k+1$. Since $v_1, v_3 \in O_i$ and $v_2, v_{4k} \in E_j$, $i = 1$ and $j \in \{k+2, k+3, \ldots, 2k+1\}$. Hence $D'\backslash\{v_{4k+3}, v_{4k+4}, v_{4k+5}\} \cup \{v_1\} = D_{1,j}$ for some $j \in \{k+2, k+3, \ldots, 2k+1\}$. Hence $D' = D_{1,j}\backslash\{v_1\} \cup \{v_{4k+3}, v_{4k+4}, v_{4k+5}\} = (D_{1,j} \cup \{v_{4k+3}, v_{4k+4}\})\backslash\{v_1\} \cup \{v_{4k+5}\} = D'_{1,j+1}\backslash\{v_1\} \cup \{v_{4k+5}\} = D'_{2k+3,j+1}$ by Claim 2.

Case 3: $|D' \cap \{v_{4k+2}, v_{4k+3}, v_{4k+4}, v_{4k+5}\}| = 2$.

Subcase 3.1: $v_{4k+3}, v_{4k+4} \in D'$, but $v_{4k+2}, v_{4k+5} \notin D'$. It is easy to obtain $v_1, v_{4k} \in D'$. Since $D'\backslash\{v_{4k+3}, v_{4k+4}\}$ is a γ-total dominating set of C_{4k+2}, $D'\backslash\{v_{4k+3}, v_{4k+4}\} = D_{i,j} = O_i \cup E_j$ for some integers $1 \leqslant i, j \leqslant 2k+1$. Since $v_1 \in D'\backslash\{v_{4k+3}, v_{4k+4}\}$, $v_1 \in O_i$. Thus $i \in \{1, 2, \ldots, k+1\}$.

Subcase 3.1.1: $v_0 \in D'$. Since $v_0, v_{4k} \in E_j$, $j = k+1$. Hence $D' = D_{i,k+1} \cup \{v_{4k+3}, v_{4k+4}\} = (D_{i,k+1} \cup \{v_{4k+2}, v_{4k+3}\})\backslash\{v_{4k+2}\} \cup \{v_{4k+4}\} = D'_{i,k+1}\backslash\{v_{4k+2}\} \cup \{v_{4k+4}\} = D'_{i,k+2}$.

Subcase 3.1.2: $v_0 \notin D'$. Hence $v_2 \in D'$. Since $v_2, v_{4k} \in E_j$, $j \in \{k+2, k+3, \ldots, 2k+1\}$. Hence $D' = D_{i,j} \cup \{v_{4k+3}, v_{4k+4}\} = D'_{i,j+1}$.

Subcase 3.2: $v_{4k+2}, v_{4k+3} \in D'$, but $v_{4k+4}, v_{4k+5} \notin D'$. If $v_{4k+1} \in D'$, we repeat the process in Subcase 2.1; otherwise, we repeat the process in Subcase 3.1.

Subcase 3.3: $v_{4k+4}, v_{4k+5} \in D'$, but $v_{4k+2}, v_{4k+3} \notin D'$. If $v_0 \in D'$, we repeat the process in Subcase 2.4; otherwise, we repeat the process in Subcase 3.1.

Subcase 3.4: $v_{4k+2}, v_{4k+5} \in D'$, but $v_{4k+3}, v_{4k+4} \notin D'$. Then $v_0 \in D'$. If $v_1 \in D'$, we repeat the process in Subcase 2.4; otherwise, we repeat the process in Subcase 3.1. □

Theorem 9 *Let $k \geqslant 1$ be an integer. Then $TD_\gamma(C_{4k+3}) \cong C_{4k+3}$.*

Proof: First, we show that each γ-total dominating set of C_{4k+3} cannot contain three or more consecutive vertices of C_{4k+3}. Suppose for a contradiction that there is a γ-total dominating set D of C_{4k+3}, which contains three or more consecutive vertices of C_{4k+3}. Let l be the largest number of these consecutive vertices, so $l \geqslant 3$. By Theorem 1, $|D| =$

$2k+2$. Since these l vertices dominate $l+2$ vertices of C_{4k+3}, the other $2k+2-l$ vertices in D must dominate at least $4k+3-(l+2)=4k-l+1$ vertices of C_{4k+3}. By Lemma 1, if $l=3$, these $2k+2-l$ vertices can dominate at most $2(2k+2-l)-1=4k+3-2l$ vertices of C_{4k+3}, which is less than $4k-l+1$. Suppose $l \geqslant 4$. Then these $2k+2-l$ vertices can dominate at most $4k+4-2l$ vertices of C_{4k+3}, which is less than $4k-l+1$. This is a contradiction. Hence every γ-total dominating set must contain $k+1$ groups of two consecutive vertices of C_{4k+3}. This means there is only one vertex in C_{4k+3}, which is dominated by 2 vertices in such a γ-total dominating set. To find all γ-total dominating sets, it suffices to find such a vertex on the cycle dominated by two vertices. Clearly, there are exactly $4k+3$ γ-total dominating sets. Recall that $C_{4k+3}=v_0v_1\dots v_{4k+2}v_0$. Let $D_0=\{v_0,v_1,v_3,v_4,v_7,v_8,v_{11},v_{12},\dots,v_{4k-1},v_{4k}\}$ and $D_l=D_{l-1}\setminus\{v_{(4l-1) \pmod{4k+3}}\}\cup\{v_{(4l+1) \pmod{4k+3}}\}$ for each $l=1,2,\dots,4k+2$. Then $D_0D_1\dots D_{4k+2}D_0$ forms a cycle with $4k+3$ vertices. □

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