RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2017.43.319

ScienceAsia 43 (2017): **319–325**

Commutator subgroups of Vershik-Kerov groups for infinite symplectic groups

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Received 9 Jun 2017
Accepted 24 Aug 2017

ABSTRACT: Let R be a commutative ring with identity 1. We describe two kinds of Vershik-Kerov groups for the symplectic case: $\mathrm{Sp}_{\mathrm{VK}}(2,\infty,R)$ and $\mathrm{GSp}_{\mathrm{VK}}(2,\infty,R)$. We also determine the commutator subgroups of these groups over a wide class of commutative rings. For an arbitrary infinite field, we find the bounds for the commutator width of the groups $\mathrm{Sp}_{\mathrm{VK}}(2,\infty,K)$ and $\mathrm{GSp}_{\mathrm{VK}}(2,\infty,K)$.

KEYWORDS: infinite triangular matrices, infinite unitriangular matrices, commutator width, lower central series

MSC2010: 20F12 20F14

INTRODUCTION

Let R be an associative ring with identity 1. By $\mathrm{GL}_c(\infty,R)$, $\mathrm{GL}_{\mathrm{r}}(\infty,R)$, $\mathrm{GL}_{\mathrm{rc}}(\infty,R)$ we denote the groups of all infinite dimensional (indexed by $\mathbb{N}$) column-finite, row-finite, row-column-finite invertible matrices over R, respectively. By $\mathrm{GL}_{\mathrm{VK}}(\infty,R)$ we denote the Vershik-Kerov group which is the subgroup of $\mathrm{GL}_c(\infty,R)$ consisting of matrices having only a finite number of non-zero entries below the main diagonal. The group $\mathrm{GL}_{\mathrm{VK}}(\infty,R)$ stems from asymptotic representation theory which connects functional analysis, algebra, and combinatorial probability theory, and is related to classical groups of infinite dimensions [1–3]. In recent years, some important subgroups of Vershik-Kerov group have been studied. Gupta and Hołubowski determined the commutator subgroup of Vershik-Kerov group over an infinite field [4] and a wide class of associative rings [5]. Parabolic subgroups of Vershik-Kerov group are described in Refs. 6, 7. Słowik studied the lower central series of subgroups of the Vershik-Kerov group in Ref. 8.

Let $\mathrm{Mat}_n(R)$ be the set of all $n\times n$ matrices over R. $\mathrm{Mat}_\infty(R)$ stands for the set of all infinite dimensional matrices (indexed by $\mathbb{N}$). Denote by $\mathrm{Mat}_{2,\infty}(R)$ the set $\mathrm{Mat}_2(\mathrm{Mat}_\infty(R))$ of 2×2 matrices with coefficients in $\mathrm{Mat}_\infty(R)$. Denote by $\mathrm{Mat}^{\mathrm{fin}}_{2,\infty}(R)$ the set of all the matrices below

$$M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in \mathrm{Mat}_{2,\infty}(R)$$

where A is column-finite, D is row-finite and B, C are row-column-finite matrices. When R is a commutative ring with identity 1, we define

$$\mathrm{Sp}^{\mathrm{fin}}_{2,\infty}(R)=\left\{M\in \mathrm{Mat}^{\mathrm{fin}}_{2,\infty}(R)\;\middle|\; MHM'=H\right\},$$
$$\mathrm{GSp}^{\mathrm{fin}}_{2,\infty}(R)=\left\{M\in \mathrm{Mat}^{\mathrm{fin}}_{2,\infty}(R)\;\middle|\; MHM'=\lambda H\right\}$$

where

$$H=\begin{pmatrix} O & I\\ -I & O\end{pmatrix},$$

$\lambda\in R^*$. M' is the transpose of M, I represents the identity matrices, and O the zero matrices. In this paper, we are concerned about the group $\mathrm{Sp}_{\mathrm{VK}}(2,\infty,R)$, which can be viewed as the symplectic case of the Vershik-Kerov group.

Let R be a commutative ring with identity 1 and $\{v_1,\dots,v_n,v_{n+1},\dots\}$ a basis of an infinite dimensional (indexed by $\mathbb{N}$) linear space over R. By $\mathrm{T}(\infty,R)$ we denote the group of all infinite dimensional (indexed by $\mathbb{N}$) upper triangular matrices whose entries on the main diagonal are invertible in R. We can find that the elements of $\mathrm{T}(\infty,R)$ preserve the complete flag

$$v_1\subset\cdots\subset\langle v_1,\,\dots v_n\rangle\subset\langle v_1,\,\dots v_{n+1}\rangle\subset\cdots.$$

For the case of a $2n$-dimensional symplectic space V with a basis $\{u_1,v_1,u_2,v_2\cdots,u_n,v_n\}$, where u_k, $v_k (1\leqslant k\leqslant n)$ is a hyperbolic pair, there is an orthogonal direct sum decomposition $V=\langle u_1,v_1\rangle\perp\langle u_2,v_2\rangle\perp\cdots\perp\langle u_n,v_n\rangle$. Let $W_k=\langle u_1,\dots u_k\rangle$ be a k-dimensional totally isotropic subspace. Then $W_k^{\perp} = \langle u_1, \dots u_k, u_{k+1}, \dots, u_n, v_{k+1}, \dots, v_n \rangle$ is a $(2n-k)$-dimensional subspace. Thus we can obtain a complete flag of V

$$0 \subset W_1 \subset \cdots \subset W_n = W_n^{\perp} \subset \cdots \subset W_1^{\perp} \subset V.$$

The group preserving the above complete flag should be

$$\left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \mathrm{Sp}(2n, R) \,\middle|\, A \in \mathrm{T}(n, R) \right\},$$

which is a subgroup of $\mathrm{Sp}(2n,R)$. Here we denote it by $\mathrm{TSp}(2n,R)$. If we sequentially select $\{u_1, \dots, u_n, v_n, \dots, v_1\}$ as the basis of V, we can show that all the elements of $\mathrm{TSp}(2n,R)$ are upper triangular invertible matrices.

When we consider the infinite case, the complete flag of V should be

$$0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n \subset \cdots,$$
$$\cdots \subset W_n^{\perp} \subset W_{n-1}^{\perp} \subset \cdots \subset W_1^{\perp} \subset V.$$

And the group preserving this complete flag should be

$$\left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \mathrm{Sp}^{\mathrm{fin}}_{2,\infty}(R) \,\middle|\, A \in \mathrm{T}(\infty, R) \right\}.$$

Denote it by $\mathrm{TSp}(2,\infty,R)$. Let $\mathrm{UT}(\infty,R)$ be the group of all infinite dimensional (indexed by $\mathbb{N}$) upper triangular matrices whose entries on the main diagonal are identities. We can define a subgroup of $\mathrm{TSp}(2,\infty,R)$

$$\left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \mathrm{Sp}^{\mathrm{fin}}_{2,\infty}(R) \,\middle|\, A \in \mathrm{UT}(\infty, R) \right\} = \mathrm{USp}(2,\infty,R).$$

We can also define an overgroup of $\mathrm{TSp}(2,\infty,R)$

$$\left\{ \begin{pmatrix} A & B \\ O & \lambda(A')^{-1} \end{pmatrix} \in \mathrm{GSp}^{\mathrm{fin}}_{2,\infty}(R) \,\middle|\, A \in \mathrm{T}(\infty, R) \right\} = \mathrm{TGSp}(2,\infty,R).$$

For an associative ring R with identity 1, we denote by $\mathrm{GL}(n,R)$ the general linear group of $n \times n$ invertible matrices over R. $\mathrm{E}(n,R)$ stands for the subgroup of $\mathrm{GL}(n,R)$ generated by all elementary transvections $t_{ij}(\alpha) = I + \alpha E_{ij}$, with $1 \leqslant i \neq j \leqslant n$, $\alpha \in R$, where E_{ij} denotes the matrix with 1 at the position (i,j) and zeros elsewhere. When R is a field, we know that the elementary subgroup $\mathrm{E}(n,R)$ coincides with the special linear group $\mathrm{SL}(n,R)$ over R. By $\mathrm{GL}(\infty,n,R)$ we denote the subgroup of $\mathrm{GL}_{\mathrm{VK}}(\infty,R)$ consisting of all matrices of the form

$$\begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} \end{pmatrix}$$

where $M_{11} \in \mathrm{GL}(n,R)$, $M_{22} \in \mathrm{T}(\infty,R)$. And by $\mathrm{E}(\infty,n,R)$ we denote the subgroup of $\mathrm{GL}(\infty,n,R)$ consisting of all matrices of the same form satisfying $M_{11} \in \mathrm{E}(n,R)$ and $M_{22} \in \mathrm{UT}(\infty,R)$. It is clear that

$$\mathrm{GL}(\infty,n,R) \subseteq \mathrm{GL}(\infty,n+1,R),$$
$$\mathrm{GL}_{\mathrm{VK}}(\infty,R) = \bigcup_{n>1} \mathrm{GL}(\infty,n,R),$$

and

$$\mathrm{E}(\infty,n,R) \subseteq \mathrm{E}(\infty,n+1,R),$$
$$\mathrm{E}_{\mathrm{VK}}(\infty,R) = \bigcup_{n>1} \mathrm{E}(\infty,n,R).$$

For infinite dimensional symplectic groups, we can define the Vershik-Kerov groups as follows:

$$\left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \mathrm{Sp}^{\mathrm{fin}}_{2,\infty}(R) \,\middle|\, A \in \mathrm{GL}_{\mathrm{VK}}(\infty, R) \right\} = \mathrm{Sp}_{\mathrm{VK}}(2,\infty,R),$$

$$\left\{ \begin{pmatrix} A & B \\ O & \lambda(A')^{-1} \end{pmatrix} \in \mathrm{GSp}^{\mathrm{fin}}_{2,\infty}(R) \,\middle|\, A \in \mathrm{GL}_{\mathrm{VK}}(\infty, R) \right\} = \mathrm{GSp}_{\mathrm{VK}}(2,\infty,R),$$

$$\left\{ \begin{pmatrix} A & B \\ O & (A')^{-1} \end{pmatrix} \in \mathrm{Sp}^{\mathrm{fin}}_{2,\infty}(R) \,\middle|\, A \in \mathrm{E}_{\mathrm{VK}}(\infty, R) \right\} = \mathrm{USp}_{\mathrm{VK}}(2,\infty,R).$$

Now we give some notation which will be used in this paper. For a group G and elements a and b of G, we write $[a,b] = a^{-1}b^{-1}ab$ as the commutator of a and b. $[G,G]$ stands for the commutator subgroup of G generated by all the commutators of the elements in G. Suppose H is a subgroup of G, by $[H,G]$ we denote the subgroup of G generated by all commutators $[h,g]$, where $h \in H$, $g \in G$. The lower central series of G is defined inductively as

$$\gamma_0(G) = G,\ \gamma_{n+1}(G) = [\gamma_n(G), G] \quad \text{for } n \geqslant 0.$$

Denote by $c(G)$ the commutator width of G, which is the least integer s such that every element of the commutator subgroup of G is the product of at most s commutators. If such an s does not exist, we set $c(G) = \infty$.

The following problem has been discussed in Refs. 4, 5, 7, 8.

Problem 1 *Does* $\mathrm{E_{VK}}(\infty,R)$ *coincide with the commutator subgroup of* $\mathrm{GL_{VK}}(\infty,R)$*?*

The above problem was posed by Sushchanskii at the conference, Groups and Their Actions, Bedlewo 2010. Gupta and Hołubowski gave a positive answer for fields and a wide class of associative rings[4,5]. For the symplectic case, we can investigate the following two problems.

Problem 2 *Does* $\mathrm{USp_{VK}}(2,\infty,R)$ *coincide with the commutator subgroup of* $\mathrm{GSp_{VK}}(2,\infty,R)$*?*

Problem 3 *Does* $\mathrm{USp_{VK}}(2,\infty,R)$ *coincide with the commutator subgroup of* $\mathrm{Sp_{VK}}(2,\infty,R)$*?*

Our main results are the following.

Theorem 1 *Assume that R is a commutative ring such that* 1 *is a sum of two invertible elements. Then the commutator subgroup of* $\mathrm{Sp_{VK}}(2,\infty,R)$ *coincides with the group* $\mathrm{USp_{VK}}(2,\infty,R)$.

Theorem 2 *Assume that R is a commutative ring such that* 1 *is a sum of two invertible elements. Then the commutator subgroup of* $\mathrm{GSp_{VK}}(2,\infty,R)$ *coincides with the group* $\mathrm{USp_{VK}}(2,\infty,R)$.

When we consider these kinds of symplectic groups over an infinite field, we have the following two theorems.

Theorem 3 *Assume that K is an infinite field. Then the commutator subgroup of* $\mathrm{Sp_{VK}}(2,\infty,K)$ *coincides with the group* $\mathrm{USp_{VK}}(2,\infty,K)$ *and* $c(\mathrm{Sp_{VK}}(2,\infty,K))\leqslant 3$.

Theorem 4 *Assume that K is an infinite field. Then the commutator subgroup of* $\mathrm{GSp_{VK}}(2,\infty,K)$ *coincides with the group* $\mathrm{USp_{VK}}(2,\infty,K)$ *and* $c(\mathrm{GSp_{VK}}(2,\infty,K))\leqslant 3$.

To prove the results above, we will use the following important theorems.

Theorem 5 (Ref. 5) *Assume that R is an associative ring with a commutative group of invertible elements such that* 1 *is a sum of two invertible elements. Then the commutator subgroup of the group* $\mathrm{GL_{VK}}(\infty,R)$ *coincides with the group* $\mathrm{E_{VK}}(\infty,R)$.

Theorem 6 (Ref. 5) *Assume that R is an associative ring with commutative group of invertible elements such that* 1 *is a sum of two invertible elements. Then the commutator subgroup of the group* $\mathrm{T}(\infty,R)$ *coincides with the group* $\mathrm{UT}(\infty,R)$ *and* $c(\mathrm{T}(\infty,R))\leqslant 2$. *Furthermore the lower central series of the group* $\mathrm{T}(\infty,R)$ *is*

$$\gamma_0(\mathrm{T}(\infty,R))=\mathrm{T}(\infty,R),$$
$$\gamma_k(\mathrm{T}(\infty,R))=\mathrm{UT}(\infty,R),\quad \text{for all}\quad k\geqslant 1,$$

i.e., it stabilizes on the group $\mathrm{UT}(\infty,R)$.

PROOFS OF THE MAIN RESULTS

We first define the following subgroups. Let

$$\left\{\begin{pmatrix} I & B\\ O & I\end{pmatrix}\in \mathrm{Sp}_{2,\infty}^{\mathrm{fin}}(R)\;\middle|\; B=B'\right\}=\mathbb{U},$$
$$\left\{\begin{pmatrix} A & O\\ O & (A')^{-1}\end{pmatrix}\in \mathrm{Sp}_{2,\infty}^{\mathrm{fin}}(R)\;\middle|\; A\in\mathrm{UT}(\infty,R)\right\}=\mathbb{UT},$$
$$\left\{\begin{pmatrix} A & O\\ O & (A')^{-1}\end{pmatrix}\in \mathrm{Sp}_{2,\infty}^{\mathrm{fin}}(R)\;\middle|\; A\in\mathrm{T}(\infty,R)\right\}=\mathbb{T},$$
$$\left\{\begin{pmatrix} A & O\\ O & (A')^{-1}\end{pmatrix}\in \mathrm{Sp}_{2,\infty}^{\mathrm{fin}}(R)\;\middle|\; A\in\mathrm{E_{VK}}(\infty,R)\right\}=\mathbb{E}_{\mathrm{VK}},$$
$$\left\{\begin{pmatrix} A & O\\ O & (A')^{-1}\end{pmatrix}\in \mathrm{Sp}_{2,\infty}^{\mathrm{fin}}(R)\;\middle|\; A\in\mathrm{GL_{VK}}(\infty,R)\right\}=\mathbb{GL}_{\mathrm{VK}}.$$

It is clear that $\mathbb{U}$ and $\mathbb{UT}$ are two subgroups of $\mathrm{USp}(2,\infty,R)$ and $\mathbb{U}$ is a normal subgroup of $\mathrm{USp}(2,\infty,R)$. So it is easy to verify the following lemma.

Lemma 1 $\mathrm{USp}(2,\infty,R)=\mathbb{U}\rtimes\mathbb{UT}$.

In the same way, we can obtain the following conclusion.

Lemma 2 $\mathrm{TSp}(2,\infty,R)=\mathbb{U}\rtimes\mathbb{T}$. *Furthermore,* $\mathrm{USp_{VK}}(2,\infty,R)=\mathbb{U}\rtimes\mathbb{E}_{\mathrm{VK}}$ *and* $\mathrm{Sp_{VK}}(2,\infty,R)=\mathbb{U}\rtimes\mathbb{GL}_{\mathrm{VK}}$.

To prove Theorem 1 and Theorem 2, we need to use Lemma 3, Corollary 1 and Theorem 7 which will be proved below.

Lemma 3 *For any commutative ring R with* 1*, every element of the group $\mathbb{U}$ can be written as a commutator of* $\mathrm{USp}(2,\infty,R)$.

Proof: For any element H of $\mathbb{U}$ we can write

$$H=\begin{pmatrix} I & X\\ O & I\end{pmatrix},$$

where $X = (x_{ij})$ is a row-column-finite matrix in $\mathrm{Mat}_\infty(R)$ with $X' = X$. Let J be an infinite Jordan matrix

$$J = \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \end{pmatrix}.$$

All blank entries are equal to 0. For each $H \in \mathbb{U}$, we will find $X = (x_{ij}) \in \mathrm{Mat}_\infty(R)$ such that

$$\begin{pmatrix} I & X \\ O & I \end{pmatrix}^{-1} \begin{pmatrix} J^{-1} & O \\ O & J' \end{pmatrix}^{-1} \begin{pmatrix} I & X \\ O & I \end{pmatrix} \begin{pmatrix} J^{-1} & O \\ O & J' \end{pmatrix} = \begin{pmatrix} I & JXJ' - X \\ O & I \end{pmatrix} = \begin{pmatrix} I & B \\ O & I \end{pmatrix}.$$

Note that $B' = B$ and $X' = X$. We only need to find x_{ij} for all $i \leqslant j \in \mathbb{N}$. Comparing entries of two sides of $B = JXJ' - X$ we obtain for all $i \in \mathbb{N}$

$$\begin{aligned} b_{ii} &= x_{i+1,i} + x_{i,i+1} + x_{i+1,i+1} \\ &= 2x_{i,i+1} + x_{i+1,i+1}, \end{aligned}$$

and for all $k \in \mathbb{N}$

$$b_{i,i+k} = x_{i+1,i+k} + x_{i,i+1+k} + x_{i+1,i+1+k},$$

which is equivalent to

$$\begin{aligned} x_{i+1,i+1} &= b_{ii} - 2x_{i,i+1}, \\ x_{i+1,i+1+k} &= b_{i,i+k} - x_{i+1,i+k} - x_{i,i+1+k}. \end{aligned}$$

We can choose the elements in the first row of X to be arbitrary. Then all the elements in first column of X are obtained from $X' = X$. Next, from the equations above, we can find x_{22}, $x_{23} = x_{32}$, $x_{24} = x_{42}$, $x_{25} = x_{52}$, and so on. In this way, row by row and column by column, we can find any element x_{ij} of X in finite number of steps. □

Lemma 4 *Assume that R is a commutative ring such that* 1 *is a sum of two invertible elements. Then the commutator subgroup of the group* $\mathbb{T}$ *coincides with the group* $\mathbb{UT}$ *and* $c(\mathbb{T}) \leqslant 2$. *Furthermore, the lower central series of the group* $\mathbb{T}$ *is*

$$\gamma_0(\mathbb{T}) = \mathbb{T}, \quad \gamma_k(\mathbb{T}) = \mathbb{UT}, \quad \textit{for all} \quad k \geqslant 1,$$

i.e., it stabilizes on the group $\mathbb{UT}$.

Proof: Note that there exists a group isomorphism from $\mathrm{UT}(\infty, R)$ to $\mathbb{UT}$:

$$A \mapsto \begin{pmatrix} A & O \\ O & (A')^{-1} \end{pmatrix}.$$

From Theorem 6 we can easily obtain the conclusion. □

In the same way, from Theorem 5 we have the following result.

Corollary 1 *Assume that R is a commutative ring such that* 1 *is a sum of two invertible elements. Then the commutator subgroup of the group* $\mathbb{GL}_{\mathrm{VK}}$ *coincides with the group* $\mathbb{E}_{\mathrm{VK}}$.

Theorem 7 *Assume that R is a commutative ring such that* 1 *is a sum of two invertible elements. Then the commutator subgroup of* $\mathrm{TGSp}(2,\infty,R)$ *coincides with* $\mathrm{USp}(2,\infty,R)$ *and* $c(\mathrm{TGSp}(2,\infty,R)) \leqslant 3$. *Furthermore, the lower central series of the group* $\mathrm{TGSp}(2,\infty,R)$ *is*

$$\begin{aligned} &\gamma_0(\mathrm{TGSp}(2,\infty,R)) = \mathrm{TGSp}(2,\infty,R), \\ &\gamma_k(\mathrm{TGSp}(2,\infty,R)) = \mathrm{USp}(2,\infty,R), \\ &\qquad \textit{for all} \quad k \geqslant 1, \end{aligned}$$

i.e., it stabilizes on the group $\mathrm{USp}(2,\infty,R)$.

Proof: For any two elements in $\mathrm{TGSp}(2,\infty,R)$ having the form

$$\begin{pmatrix} A_1 & B_1 \\ O & \lambda_1 (A_1')^{-1} \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ O & \lambda_2 (A_2')^{-1} \end{pmatrix},$$

the commutator is

$$\left[\begin{pmatrix} A_1 & B_1 \\ O & \lambda_1 (A_1')^{-1} \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ O & \lambda_2 (A_2')^{-1} \end{pmatrix} \right] = \begin{pmatrix} [A_1, A_2] & B_3 \\ O & ([A_1, A_2]')^{-1} \end{pmatrix},$$

where λ_1, $\lambda_2 \in R^*$, $A_1, A_2 \in \mathrm{T}(\infty, R)$ and

$$\begin{aligned} B_3 = {} & A_1^{-1} A_2^{-1} (A_1 B_2 + \lambda_2 B_1 (A_2')^{-1}) \\ & - (\lambda_1 A_1^{-1} A_2^{-1} B_2 A_2' + A_1^{-1} B_1 A_1' A_2') \\ & \cdot (A_1')^{-1} (A_2')^{-1}. \end{aligned}$$

From

$$[\mathrm{T}(\infty, R), \mathrm{T}(\infty, R)] \subseteq \mathrm{UT}(\infty, R),$$

we can easily obtain

$$\begin{aligned} &[\mathrm{TGSp}(2,\infty,R), \mathrm{TGSp}(2,\infty,R)] \\ &\qquad \subseteq \mathrm{USp}(2,\infty,R). \end{aligned} \tag{1}$$

From Lemma 1, $\mathrm{USp}(2,\infty,R) = \mathbb{U} \rtimes \mathbb{UT}$. Thus we know that for all elements G in $\mathrm{USp}(2,\infty,R)$, there exists a unique H in $\mathbb{U}$ and K in $\mathbb{UT}$ such that $G = HK$. For each $G \in \mathrm{USp}(2,\infty,R)$, we can write

$$H = \begin{pmatrix} I & B \\ O & I \end{pmatrix}, \; K = \begin{pmatrix} A & O \\ O & (A')^{-1} \end{pmatrix},$$

$$G=\begin{pmatrix}A & B_0\\ O & (A')^{-1}\end{pmatrix}=\begin{pmatrix}I & B\\ O & I\end{pmatrix}\begin{pmatrix}A & O\\ O & (A')^{-1}\end{pmatrix}=HK,$$

where $A\in \mathrm{UT}(\infty,R)$ and $B=B'=B_0A'=A'B_0$, $H\in \mathbb{U}$, and $K\in \mathbb{UT}$. From Lemma 3 we know H is a commutator of $\mathrm{USp}(2,\infty,R)$. And from Lemma 4 we obtain that K can be written as a product of two commutators of $\mathbb{T}$. So each element in $\mathrm{USp}(2,\infty,R)$ can be written as a product of three commutators of $\mathrm{USp}(2,\infty,R)$.

$$\begin{aligned}&\mathrm{USp}(2,\infty,R)\\ &\quad\subseteq[\mathrm{USp}(2,\infty,R),\mathrm{USp}(2,\infty,R)]\\ &\quad\subseteq[\mathrm{USp}(2,\infty,R),\mathrm{TGSp}(2,\infty,R)]\\ &\quad\subseteq[\mathrm{TGSp}(2,\infty,R),\mathrm{TGSp}(2,\infty,R)].\end{aligned}\tag{2}$$

Thus from (1) we obtain

$$\begin{aligned}&\mathrm{USp}(2,\infty,R)\\ &\quad=[\mathrm{USp}(2,\infty,R),\mathrm{USp}(2,\infty,R)]\\ &\quad=[\mathrm{USp}(2,\infty,R),\mathrm{TGSp}(2,\infty,R)]\\ &\quad=[\mathrm{TGSp}(2,\infty,R),\mathrm{TGSp}(2,\infty,R)].\end{aligned}$$

Then the lower central series of $\mathrm{TGSp}(2,\infty,R)$ is

$$\begin{aligned}&\gamma_0(\mathrm{TGSp}(2,\infty,R))=\mathrm{TGSp}(2,\infty,R),\\ &\gamma_1(\mathrm{TGSp}(2,\infty,R))=\mathrm{USp}(2,\infty,R),\\ &\qquad\gamma_2(\mathrm{TGSp}(2,\infty,R))\\ &\quad=[\gamma_1(\mathrm{TGSp}(2,\infty,R)),\mathrm{TGSp}(2,\infty,R)]\\ &\quad=\mathrm{USp}(2,\infty,R)\end{aligned}$$

and so on. □

When we chose $\lambda_1=\lambda_2=1$, the two elements

$$\begin{pmatrix}A_1 & B_1\\ O & \lambda_1(A_1')^{-1}\end{pmatrix},\begin{pmatrix}A_2 & B_2\\ O & \lambda_2(A_2')^{-1}\end{pmatrix}$$

in $\mathrm{TGSp}(2,\infty,R)$ are also two elements in the group $\mathrm{TSp}(2,\infty,R)$. Then (1) and (2) in the proof of Theorem 7 are changed to

$$[\mathrm{TSp}(2,\infty,R),\mathrm{TSp}(2,\infty,R)]\subseteq\mathrm{USp}(2,\infty,R)$$

and

$$\begin{aligned}\mathrm{USp}(2,\infty,R)&\subseteq[\mathrm{USp}(2,\infty,R),\mathrm{USp}(2,\infty,R)]\\ &\subseteq[\mathrm{USp}(2,\infty,R),\mathrm{TSp}(2,\infty,R)]\\ &\subseteq[\mathrm{TSp}(2,\infty,R),\mathrm{TSp}(2,\infty,R)],\end{aligned}$$

respectively. So

$$\begin{aligned}\mathrm{USp}(2,\infty,R)&=[\mathrm{USp}(2,\infty,R),\mathrm{USp}(2,\infty,R)]\\ &=[\mathrm{USp}(2,\infty,R),\mathrm{TSp}(2,\infty,R)]\\ &=[\mathrm{TSp}(2,\infty,R),\mathrm{TSp}(2,\infty,R)].\end{aligned}$$

And the lower central series of $\mathrm{TSp}(2,\infty,R)$ is

$$\begin{aligned}&\gamma_0(\mathrm{TSp}(2,\infty,R))=\mathrm{TSp}(2,\infty,R),\\ &\gamma_1(\mathrm{TSp}(2,\infty,R))=\mathrm{USp}(2,\infty,R),\\ &\qquad\gamma_2(\mathrm{TSp}(2,\infty,R))\\ &\quad=[\gamma_1(\mathrm{TSp}(2,\infty,R)),\mathrm{TSp}(2,\infty,R)]\\ &\quad=\mathrm{USp}(2,\infty,R).\end{aligned}$$

Thus we can obtain the following corollary.

Corollary 2 *Assume that R is a commutative ring such that 1 is a sum of two invertible elements. Then the commutator subgroup of $\mathrm{TSp}(2,\infty,R)$ coincides with the group $\mathrm{USp}(2,\infty,R)$ and $c(\mathrm{TSp}(2,\infty,R))\leqslant 3$. Furthermore, the lower central series of the group $\mathrm{TSp}(2,\infty,R)$ is*

$$\begin{aligned}&\gamma_0(\mathrm{TSp}(2,\infty,R))=\mathrm{TSp}(2,\infty,R),\\ &\gamma_k(\mathrm{TSp}(2,\infty,R))=\mathrm{USp}(2,\infty,R),\quad\forall k\geqslant 1,\end{aligned}$$

i.e., it stabilizes on the group $\mathrm{USp}(2,\infty,R)$.

Now we finish the proof of Theorem 1 and Theorem 2. *Proof*: Using the method in Theorem 7, we can easily obtain

$$[\mathrm{GSp}_{\mathrm{VK}}(2,\infty,R),\mathrm{GSp}_{\mathrm{VK}}(2,\infty,R)]\subseteq\mathrm{USp}_{\mathrm{VK}}(2,\infty,R)$$

and

$$[\mathrm{Sp}_{\mathrm{VK}}(2,\infty,R),\mathrm{Sp}_{\mathrm{VK}}(2,\infty,R)]\subseteq\mathrm{USp}_{\mathrm{VK}}(2,\infty,R),$$

which are similar to (1). From Lemma 2 we know that $\mathrm{USp}_{VK}(2,\infty,R)=\mathbb{U}\rtimes\mathbb{E}_{\mathrm{VK}}$. So for each $G\in\mathrm{USp}_{\mathrm{VK}}(2,\infty,R)$, there exists a decomposition

$$G=\begin{pmatrix}I & B\\ O & I\end{pmatrix}\begin{pmatrix}A & O\\ O & (A')^{-1}\end{pmatrix},$$

where

$$\begin{pmatrix}I & B\\ O & I\end{pmatrix}\in\mathbb{U},\quad\begin{pmatrix}A & O\\ O & (A')^{-1}\end{pmatrix}\in\mathbb{E}_{\mathrm{VK}}.$$

Then from Lemma 3 and Corollary 1 we obtain

$$[\mathrm{GSp}_{\mathrm{VK}}(2,\infty,R),\mathrm{GSp}_{\mathrm{VK}}(2,\infty,R)]\supseteq\mathrm{USp}_{\mathrm{VK}}(2,\infty,R)$$

and

$$[\mathrm{Sp}_{\mathrm{VK}}(2,\infty,R),\mathrm{Sp}_{\mathrm{VK}}(2,\infty,R)]\supseteq\mathrm{USp}_{\mathrm{VK}}(2,\infty,R).$$

Thus we obtain the conclusions of Theorem 1 and Theorem 2. □

To prove Theorem 3 and Theorem 4, Corollary 3 of Lemma 5 will be used. Next we show Lemma 5 (which is also proved in Ref. 9 in a different way) and two corollaries.

Lemma 5 *Assume that R is an associative ring with an infinite field K in the centre $Z(R)$ of R. Every element $C \in \mathrm{UT}(\infty,R)$ is a commutator of $\mathrm{T}(\infty,R)$.*

Proof: Let $A = \mathrm{diag}(a_1, a_2, \ldots, a_n, \ldots)$ be a diagonal matrix with pairwise distinct non-zero elements $a_1, a_2, \ldots, a_n, \ldots$ of K in its diagonal. We will find $X = (x_{ij}) \in \mathrm{UT}(\infty,R)$ such that $C = X^{-1}A^{-1}XA$. Since every unitriangular matrix is invertible, this equation is equivalent to

$$AXC = XA.$$

We use induction on $n = j - i$ (i.e., n is a number of the superdiagonal of X above the main diagonal). When $n = 1$, comparing the $(i, i+1)$ entries of both sides of the matrix equation we can obtain

$$a_i(c_{i,i+1} + x_{i,i+1}) = a_{i+1}x_{i,i+1},$$

which implies

$$(a_{i+1} - a_i)x_{i,i+1} = a_i c_{i,i+1}.$$

Now we suppose that x_{ij} for all $j - i < n$ has been found. Comparing the $(i, i+n)$ entries of both sides of the matrix equation we obtain

$$a_i(c_{i,i+n} + x_{i,i+1}c_{i+1,i+n} + x_{i,i+2}c_{i+2,i+n} + \cdots + x_{i,i+n-1}c_{i+n-1,i+n} + x_{i,i+n}) = a_{i+n}x_{i,i+n}$$

which is equivalent to

$$(a_{i+n} - a_i)x_{i,i+n} = a_i(c_{i,i+n} + x_{i,i+1}c_{i+1,i+n} + x_{i,i+2}c_{i+2,i+n} + \cdots + x_{i,i+n-1}c_{i+n-1,i+n}).$$

Thus we can find $x_{i,i+n}$ for all $i \in \mathbb{N}$. □

Corollary 3 *Assume that K is an infinite field. Then every element $C \in \mathrm{UT}(\infty,K)$ is a commutator of $\mathrm{T}(\infty,K)$.*

Corollary 4 *Assume that K is an infinite field. Then the commutator subgroup of $\mathrm{TSp}(2,\infty,R)$ coincides with the group $\mathrm{USp}(2,\infty,R)$ and $c(\mathrm{TSp}(2,\infty,R)) \leqslant 2$. Furthermore the lower central series of the group $\mathrm{TSp}(2,\infty,R)$ is*

$$\gamma_0(\mathrm{TSp}(2,\infty,R)) = \mathrm{TSp}(2,\infty,R),$$
$$\gamma_k(\mathrm{TSp}(2,\infty,R)) = \mathrm{USp}(2,\infty,R), \quad \forall k \geqslant 1,$$

i.e., it stabilizes on the group $\mathrm{USp}(2,\infty,R)$.

Now we finish the proof of Theorem 3 and Theorem 4. *Proof*: From Theorem 1 and Theorem 2, we know that the commutator subgroup of $\mathrm{Sp}_{\mathrm{VK}}(2,\infty,K)$ coincides with $\mathrm{USp}_{\mathrm{VK}}(2,\infty,K)$. So does the commutator subgroup of $\mathrm{GSp}_{\mathrm{VK}}(2,\infty,K)$. Next we will determine the commutator width of $\mathrm{Sp}_{\mathrm{VK}}(2,\infty,K)$ and $\mathrm{GSp}_{\mathrm{VK}}(2,\infty,K)$.

From Lemma 2 we know that every element of $\mathrm{USp}_{VK}(2,\infty,K)$ can be written as a product of an element of $\mathbb{U}$ and an element of $\mathbb{E}_{VK}$. Each element of $\mathrm{E}_{\mathrm{VK}}(\infty,K)$ has the following decomposition:

$$\begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} \end{pmatrix} = \begin{pmatrix} I_n & M_{12} \\ O & M_{22} \end{pmatrix} \begin{pmatrix} M_{11} & O \\ O & I \end{pmatrix},$$

where $M_{11} \in \mathrm{E}(n,K)$ and $M_{11} \in \mathrm{UT}(\infty,K)$. From Theorems 1 and 2 of Ref. 10, we know that for any field K except $\mathbb{F}_2$ and $\mathbb{F}_3$, every element of $\mathrm{SL}(n,K)$ (coinciding with $\mathrm{E}(n,K)$) is a commutator of $\mathrm{GL}(n,K)$. Note that

$$\left[\begin{pmatrix} A_1 & O \\ O & I \end{pmatrix}, \begin{pmatrix} A_2 & O \\ O & I \end{pmatrix}\right] = \begin{pmatrix} [A_1, A_2] & O \\ O & I \end{pmatrix},$$

and the matrix

$$\begin{pmatrix} M_{11} & O \\ O & I \end{pmatrix}$$

is a commutator. From Corollary 3 it follows that

$$\begin{pmatrix} I_n & M_{12} \\ O & M_{22} \end{pmatrix}$$

is a commutator of $\mathrm{GL}(n,K)$. So

$$\begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} \end{pmatrix}$$

is a product of at most 2 commutators. Note that there is a group isomorphism from $\mathrm{E}_{\mathrm{VK}}(\infty,R)$ to $\mathbb{E}_{\mathrm{VK}}$:

$$A \mapsto \begin{pmatrix} A & O \\ O & (A')^{-1} \end{pmatrix},$$

and any element of $\mathbb{E}_{\mathrm{VK}}$ is a product of at most 2 commutators. Finally, from Lemma 3, every element of $\mathrm{USp}_{\mathrm{VK}}(2,\infty,K)$ can be written as a product of at most 3 commutators. □

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