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On new inequalities of Fejér-Hermite-Hadamard type for differentiable (α, m)-preinvex mappings

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ABSTRACT: The authors establish several new inequalities of the Fejér-Hermite-Hadamard type for mappings which have absolute values of the first derivatives which are (α, m)-preinvex. The results presented provide extensions of some known results. These new established inequalities are also applied to construct inequalities for special means.

KEYWORDS: integral inequalities, convex mappings

MSC2010: 26A51 26D07 26D15 26D20

INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2} \quad (1)$$

is referred to as Hermite-Hadamard's inequality and is one of the most famous results for convex mappings.

Fejér provided a weighted generalization of (1) [1]:

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,\mathrm{d}x \leqslant \int_a^b f(x)w(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b w(x)\,\mathrm{d}x \quad (2)$$

where $f : [a, b] \to \mathbb{R}$ is a convex function and $f : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetrical about $\frac{1}{2}(a+b)$.

Definition 1 A set $S \subseteq \mathbb{R}^n$ is said to be an invex set with respect to the mapping $\eta : S \times S \to \mathbb{R}^n$ if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$ [2].

Definition 2 The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η if for every $x, y \in K$ and $t \in [0, 1]$ [2]

$$f(x + t\eta(y, x)) \leqslant (1-t)f(x) + tf(y). \quad (3)$$

Definition 3 The function f on the invex set $K \subseteq [0, b^*]$, $b^* > 0$, is said to be (α, m)-preinvex with respect to η if

$$f(x + t\eta(y, x)) \leqslant (1-t^\alpha)f(x) + mt^\alpha f\left(\frac{y}{m}\right) \quad (4)$$

holds for all $x, y \in K$, $t \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$ [3].

Theorem 1 (Ref. 4) *Let $f : [a, a+\eta(b, a)] \to (0, \infty)$ be an open preinvex function on the interval of real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a \leqslant a + \eta(b, a)$. Then*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leqslant \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}. \quad (5)$$

In Ref. 5, they found the right-sided integral inequalities of Fejér type concerning the product of an s-convex mapping and a symmetric function. In Ref. 6 some left-sided Fejér-Hermite-Hadamard type inequalities for preinvex mappings were also established.

In recent years, many researchers have studied bounds for both Hermite-Hadamard and Fejér type inequalities via different classes of convex mappings; for generalizations, refinements, variations and new inequalities for them, see Refs. 7–19. Based on this literature and especially the idea in Refs. 5, 6, by discovering a weighted identity involving a symmetric mapping and a differentiable preinvex mapping defined on open invex subset, we derive the right-sided new Fejér-Hermite-Hadamard type inequalities for mappings which have absolute values of the first derivatives which are (α,m)-preinvex. Our results give extensions of some known results. The new integral inequalities are also applied to special means.

MAIN RESULTS

Lemma 1 *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}$ and $a,b \in K$, $a<b$ with $\eta(b,a)>0$. Assume that $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L_1([a,a+\eta(b,a)])$. If $g:[a,a+\eta(b,a)]\to[0,\infty)$ is an integrable mapping and symmetrical about $a+\frac{1}{2}\eta(b,a)$, then*

$$\left[\frac{f(a)+f(a+\eta(b,a))}{2}\right]\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x = \frac{\eta(b,a)}{4}\left\{\int_0^1\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,\mathrm{d}x\right]\times\left[f'(\varphi(t))-f'(\psi(t))\right]\mathrm{d}t\right\} \quad (6)$$

where $\varphi(t)=a+\frac{1}{2}t\eta(b,a)$ and $\psi(t)=a+(1-\frac{1}{2}t)\eta(b,a)$. In particular,

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta(b,a)}{4}\left\{\int_0^1\left|\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,\mathrm{d}x\right]\right|\times\left[\left|f'(\varphi(t))\right|+\left|f'(\psi(t))\right|\right]\mathrm{d}t\right\} \quad (7)$$

and

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{4}\|g\|_\infty \times\int_0^1 (1-t)\left[\left|f'(\varphi(t))\right|+\left|f'(\psi(t))\right|\right]\mathrm{d}t \quad (8)$$

where $\|g\|_\infty = \sup_{t\in[a,a+\eta(b,a)]} g(t)$.

Proof: Since $g(x)$ is symmetrical about $a+\frac{1}{2}\eta(b,a)$, $g(\psi(t))=g(\varphi(t))$ for all $t\in[0,1]$. Hence

$$\begin{aligned} I^* &= \frac{\eta(b,a)}{4}\int_0^1\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,\mathrm{d}x\right]\times\left[f'(\varphi(t))-f'(\psi(t))\right]\mathrm{d}t \\ &= \frac{\eta(b,a)}{4}\int_0^1\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,\mathrm{d}x\right]f'(\varphi(t))\,\mathrm{d}t - \frac{\eta(b,a)}{4}\int_0^1\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,\mathrm{d}x\right]f'(\psi(t))\,\mathrm{d}t \\ &:= I_1 - I_2. \end{aligned}$$

Via integration by parts we obtain

$$\begin{aligned} I_1 &= \frac{\eta(b,a)}{4}\int_0^1\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,\mathrm{d}x\right]f'(\varphi(t))\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^1\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,\mathrm{d}x\right]\mathrm{d}[f(\varphi(t))] \\ &= \frac{1}{2}\left\{\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,\mathrm{d}x\right]f(\varphi(t))\Big|_0^1 - \frac{\eta(b,a)}{2}\int_0^1\left[g(\varphi(t))+g(\psi(t))\right]f(\varphi(t))\,\mathrm{d}t\right\} \\ &= \frac{1}{2}\left\{f(a)\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \eta(b,a)\int_0^1 g(\varphi(t))f(\varphi(t))\,\mathrm{d}t\right\} \\ &= \frac{1}{2}\left\{f(a)\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - 2\int_a^{a+(1/2)\eta(b,a)} g(x)f(x)\,\mathrm{d}x\right\} \end{aligned}$$

and similarly,

$$I_2 = -\frac{1}{2}\left\{f(a+\eta(b,a))\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - 2\int_{a+(1/2)\eta(b,a)}^{a+\eta(b,a)} g(x)f(x)\,\mathrm{d}x\right\}.$$

From I_1 and I_2, it follows that

$$I^* = I_1 - I_2$$

$$= \left[\frac{f(a)+f(a+\eta(b,a))}{2}\right]\int_a^{a+\eta(b,a)} g(x)\,dx - \int_a^{a+\eta(b,a)} f(x)g(x)\,dx$$

which is the required result (6). Using Minkowski's inequality, it is straightforward to obtain (7) and (8). □

Remark 1 If $g(x)=1$, $x\in[a,a+\eta(b,a)]$ in Lemma 1, then (6) reduces to

$$\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx = \frac{\eta(b,a)}{4}\int_0^1 (t-1)\left[f'\left(a+\frac{t}{2}\eta(b,a)\right) - f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right]dt. \quad (9)$$

Remark 2 If $\eta(b,a)=b-a$ in Lemma 1, then (6) becomes

$$\left[\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx - \int_a^b f(x)g(x)\,dx = \frac{b-a}{4}\left\{\int_0^1\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,dx\right]\left[f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right) - f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt\right\}$$
$$= \frac{b-a}{4}\left\{\int_0^1\left[\int_{\psi(t)}^{\varphi(t)} g(x)\,dx\right] \times\left[f'\left((1-t)a+t\frac{a+b}{2}\right) - f'\left(t\frac{a+b}{2}+(1-t)b\right)\right]dt\right\} \quad (10)$$

where $\varphi(t)=(1-t)a+\frac{1}{2}t(a+b)$ and $\psi(t)=\frac{1}{2}t(a+b)+(1-t)b$.

The second integral identity in (10) is proved in Ref. 5 [page 754, Lemma 2.1].

With the help of Lemma 1, a new upper bound for the right-hand side of (2) via (α,m)-preinvex mappings is shown with the following inequality.

Theorem 2 *Let $K\subseteq\mathbb{R}_0$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$, $a,b\in K$ and $0\leqslant a<b$ with $\eta(b,a)>0$. Suppose $f:K\to\mathbb{R}_0$ is a differentiable mapping on K such that $f'\in L_1([a,a+\eta(b,a)])$ and $g:[a,a+\eta(b,a)]\to[0,\infty)$ is an integrable mapping and symmetrical about $a+\frac{1}{2}\eta(b,a)$. If $|f'|^q$ for $q\geqslant 1$ is (α,m)-preinvex on $[a,b/m]$, $\alpha\in(0,1]$ and $m\in(0,1]$ with $(b/m)\in K$, then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,dx - \int_a^{a+\eta(b,a)} f(x)g(x)\,dx\right| \leqslant \frac{\eta^2(b,a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\|g\|_\infty \times\left\{\left[\left(\tfrac{1}{2}-\frac{1}{2^\alpha(\alpha+1)(\alpha+2)}\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q} + \left[\left(\tfrac{1}{2}-\frac{1+\alpha 2^{\alpha+1}}{2^\alpha(\alpha+1)(\alpha+2)}\right)\left|f'(a)\right|^q + \frac{m(1+\alpha 2^{\alpha+1})}{2^\alpha(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}. \quad (11)$$

Proof: By (8) in Lemma 1 and the power mean inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,dx - \int_a^{a+\eta(b,a)} f(x)g(x)\,dx\right| \leqslant \frac{\eta^2(b,a)}{4}\|g\|_\infty\left(\int_0^1(1-t)\,dt\right)^{1-1/q} \times\left\{\left[\int_0^1(1-t)\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q dt\right]^{1/q} + \left[\int_0^1(1-t)\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{1/q}\right\}$$
$$= \frac{\eta^2(b,a)}{4}\|g\|_\infty\left(\frac{1}{2}\right)^{1-1/q} \times\left\{\left[\int_0^1(1-t)\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q dt\right]^{1/q} + \left[\int_0^1(1-t)\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{1/q}\right\}. \quad (12)$$

Since $|f'|^q$ is (α,m)-preinvex in the second sense on $[a,b/m]$, for any $t\in[0,1]$, we have

$$\int_0^1 (1-t)\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q dt$$

$$\leqslant \left|f'(a)\right|^q \int_0^1 (1-t)\left(1-\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t + \left|f'\left(\frac{b}{m}\right)\right|^q \int_0^1 m(1-t)\frac{t^\alpha}{2^\alpha}\mathrm{d}t$$

$$= \left(\tfrac{1}{2} - \frac{1}{2^\alpha(\alpha+1)(\alpha+2)}\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q \quad (13)$$

and

$$\int_0^1 (1-t)\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t$$

$$\leqslant \left|f'(a)\right|^q \int_0^1 (1-t)\left[1-\left(1-\frac{t}{2}\right)^\alpha\right]\mathrm{d}t + \left|f'\left(\frac{b}{m}\right)\right|^q \int_0^1 m(1-t)\left(1-\frac{t}{2}\right)^\alpha \mathrm{d}t$$

$$= \left(\tfrac{1}{2} - \frac{1+\alpha 2^{\alpha+1}}{2^\alpha(\alpha+1)(\alpha+2)}\right)\left|f'(a)\right|^q + \frac{m(1+\alpha 2^{\alpha+1})}{2^\alpha(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q. \quad (14)$$

Using (13) and (14) in (12), we deduce the required inequality (11). □

Corollary 1 *If $q=1$ in Theorem 2, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{4}\|g\|_\infty\left[\left(1-\frac{1+\alpha 2^\alpha}{2^{\alpha-1}(\alpha+1)(\alpha+2)}\right)\left|f'(a)\right| + \frac{m(1+\alpha 2^\alpha)}{2^{\alpha-1}(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|\right]. \quad (15)$$

Corollary 2 *If $g(x)=1$ in Theorem 2, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \leqslant \frac{\eta(b,a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\left(\tfrac{1}{2}-\frac{1}{2^\alpha(\alpha+1)(\alpha+2)}\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q} + \left[\left(\tfrac{1}{2}-\frac{1+\alpha 2^{\alpha+1}}{2^\alpha(\alpha+1)(\alpha+2)}\right)\left|f'(a)\right|^q + \frac{m(1+\alpha 2^{\alpha+1})}{2^\alpha(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}. \quad (16)$$

Corollary 3 *With the same assumptions given in Theorem 2, if $|f'(x)| \leqslant \Upsilon$ on $[a, a+\eta(b,a)]$ with $m=1$, we deduce*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{4}\Upsilon\|g\|_\infty. \quad (17)$$

Corollary 4 *If $q=1$, $\alpha=1$, $m=1$, and $g(x)=1$ in Theorem 2, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \leqslant \tfrac{1}{8}\eta(b,a)(|f'(a)|+|f'(b)|) \quad (18)$$

which is Theorem 2.1 from [Ref. 20 page 3].

On the basis of Lemma 1 and by using Hölder's inequality, we obtain the result below.

Theorem 3 *Under the conditions of Theorem 2, if $|f'|^q$ for $q>1$ is (α,m)-preinvex on $[a, b/m]$, $\alpha \in (0,1]$ and $m \in (0,1]$ with $(b/m) \in K$, then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{4}\left(\frac{q-1}{2q-1}\right)^{(q-1)/q}\|g\|_\infty \times \left\{\left[\left(1-\frac{1}{2^\alpha(\alpha+1)}\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q} + \left[\left(1+\frac{1-2^{\alpha+1}}{2^\alpha(\alpha+1)}\right)\left|f'(a)\right|^q + \frac{m(2^{\alpha+1}-1)}{2^\alpha(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}. \quad (19)$$

Proof: By (8) in Lemma 1 and Hölder's inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right|$$

$$\leqslant \frac{\eta^2(b,a)}{4}\left(\int_0^1 (1-t)^{q/(q-1)}\mathrm{d}t\right)^{1-1/q}\|g\|_\infty \times\left\{\left[\int_0^1 \left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q} + \left[\int_0^1 \left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}$$

$$= \frac{\eta^2(b,a)}{4}\left(\frac{q-1}{2q-1}\right)^{(q-1)/q}\|g\|_\infty \times\left\{\left[\int_0^1 \left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q} + \left[\int_0^1 \left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}. \quad (20)$$

Since $|f'|^q$ is (α,m)-preinvex in the second sense on $[a,b/m]$, for any $t\in[0,1]$, we have

$$\int_0^1 \left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t \leqslant \left|f'(a)\right|^q \int_0^1\left(1-\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t + \left|f'\left(\frac{b}{m}\right)\right|^q \int_0^1 m\frac{t^\alpha}{2^\alpha}\mathrm{d}t = \left(1-\frac{1}{2^\alpha(\alpha+1)}\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^q \quad (21)$$

and

$$\int_0^1 \left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t \leqslant \left|f'(a)\right|^q\int_0^1\left[1-\left(1-\frac{t}{2}\right)^\alpha\right]\mathrm{d}t + \left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1 m\left(1-\frac{t}{2}\right)^\alpha \mathrm{d}t = \left(1+\frac{1-2^{\alpha+1}}{2^\alpha(\alpha+1)}\right)\left|f'(a)\right|^q + \frac{m(2^{\alpha+1}-1)}{2^\alpha(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^q. \quad (22)$$

Using (21), (22) in (20), we deduce the result (19). □

Corollary 5 *If $g(x)=1$ with $\alpha=m=1$ in Theorem 3, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \leqslant \frac{\eta(b,a)}{4}\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \times\left\{\left[\frac{3}{4}\left|f'(a)\right|^q+\frac{1}{4}\left|f'(b)\right|^q\right]^{1/q} + \left[\frac{1}{4}\left|f'(a)\right|^q+\frac{3}{4}\left|f'(b)\right|^q\right]^{1/q}\right\} \leqslant \frac{\eta(b,a)}{4}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\frac{3^{1/q}+1}{4^{1/q}}(|f'(a)|+|f'(b)|). \quad (23)$$

Here, $0<1/q<1$ for $q>1$. To prove the second inequality above, we use the fact that $\sum_{i=1}^{n}(a_i+b_i)^r \leqslant \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r$, for $0\leqslant r<1, a_1,\dots,a_n\geqslant 0$ and $b_1,\dots,b_n\geqslant 0$.

Corollary 6 *With the same assumptions given in Theorem 3, if $|f'(x)|\leqslant \Upsilon$ on $[a,a+\eta(b,a)]$ with $m=1$, we deduce*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{2}\left(\frac{q-1}{2q-1}\right)^{(q-1)/q}\Upsilon\|g\|_\infty. \quad (24)$$

Theorem 4 *Let $K\subseteq\mathbb{R}_0$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$, $a,b\in K$ and $0\leqslant a<b$ with $\eta(b,a)>0$. Suppose $f:K\to\mathbb{R}_0$ is a differentiable mapping on K such that $f'\in L_1([a,a+\eta(b,a)])$ and $g:[a,a+\eta(b,a)]\to[0,\infty)$ is an integrable mapping and symmetrical about $a+\frac{1}{2}\eta(b,a)$. If $|f'|^q$ is (α,m)-preinvex on $[a,b/m]$ with $q=p/(p-1)$, $p>1$, $\alpha\in(0,1]$, $m\in(0,1]$ and $(b/m)\in K$, then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{4}\left(\frac{1}{p+1}\right)^{1/p}\|g\|_\infty \times\left\{\left[\left(1-\frac{1}{2^\alpha(\alpha+1)}\right)|f'(a)|^q + \frac{m}{2^\alpha(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q} + \left[\left(1+\frac{1-2^{\alpha+1}}{2^\alpha(\alpha+1)}\right)|f'(a)|^q\right.\right.$$

$$+\frac{m(2^{\alpha+1}-1)}{2^{\alpha}(\alpha+1)}\left|f'\left(\frac{b}{m}\right)\right|^{q}\Bigg]^{1/q}\Bigg\}. \quad (25)$$

Proof: By (8) in Lemma 1 and Hölder's inequality for $p>1$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \\
&\leqslant \frac{\eta^2(b,a)}{4}\|g\|_\infty\left(\int_0^1 (1-t)^p\,\mathrm{d}t\right)^{1/p} \\
&\quad\times\left\{\left[\int_0^1 \left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q} + \left[\int_0^1 \left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q}\right\} \\
&= \frac{\eta^2(b,a)}{4}\|g\|_\infty\left(\frac{1}{p+1}\right)^{1/p} \\
&\quad\times\left\{\left[\int_0^1 \left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q} + \left[\int_0^1 \left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}.
\end{aligned} \quad (26)$$

Using (21) and (22) in (26), we obtain the result (25). □

Corollary 7 *If taking $g(x)=1$ with $\alpha=m=1$ in Theorem 4, we obtain*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \\
&\leqslant \frac{\eta(b,a)}{4}\left(\frac{1}{p+1}\right)^{1/p}\left\{\left[\frac{3}{4}\left|f'(a)\right|^q + \frac{1}{4}\left|f'(b)\right|^q\right]^{1/q} + \left[\frac{1}{4}\left|f'(a)\right|^q + \frac{3}{4}\left|f'(b)\right|^q\right]^{1/q}\right\} \\
&\leqslant \frac{\eta(b,a)}{4}\left(\frac{1}{p+1}\right)^{1/p}\frac{3^{1/q}+1}{4^{1/q}}(|f'(a)|+|f'(b)|)
\end{aligned} \quad (27)$$

where $p^{-1}+q^{-1}=1$. To prove the second inequality above, we use the same method as in Corollary 5.

Corollary 8 *With the same assumptions given in Theorem 4, if $|f'(x)|\leqslant\Upsilon$ on $[a,a+\eta(b,a)]$ with $m=1$, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{2}\left(\frac{1}{p+1}\right)^{1/p}\Upsilon\|g\|_\infty, \quad (28)$$

where $p^{-1}+q^{-1}=1$.

Theorem 5 *Suppose that all the assumptions of Theorem 4 are satisfied. Then*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \\
&\leqslant \frac{\eta^2(b,a)}{4}\left(\frac{q-1}{2q-p-1}\right)^{(q-1)/q}\|g\|_\infty \\
&\quad\times\left\{\left[\left(\frac{1}{p+1}-\frac{1}{2^\alpha}\beta(\alpha+1,p+1)\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha}\beta(\alpha+1,p+1)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right. \\
&\quad+\left[\left(\frac{1}{p+1}+\frac{(-1)^q}{2^\alpha}\beta(\alpha+1,p+1)\right)\left|f'(a)\right|^q \left.- \frac{m(-1)^q}{2^\alpha}\beta(\alpha+1,p+1)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}
\end{aligned} \quad (29)$$

where

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad \forall x,y>0.$$

Proof: By (8) in Lemma 1 and Hölder's inequality for $p>1$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \\
&\leqslant \frac{\eta^2(b,a)}{4}\|g\|_\infty\left(\int_0^1 (1-t)^{(q-p)/(q-1)}\mathrm{d}t\right)^{(q-1)/q} \\
&\quad\times\left\{\left[\int_0^1 (1-t)^p\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q} + \left[\int_0^1 (1-t)^p\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}
\end{aligned}$$

$$= \frac{\eta^2(b,a)}{4}\|g\|_\infty\left(\frac{q-1}{2q-p-1}\right)^{(q-1)/q} \times\left\{\left[\int_0^1 (1-t)^p\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q} + \left[\int_0^1 (1-t)^p\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}. \tag{30}$$

Since $|f'|^q$ is (α,m)-preinvex in the second sense on $[a,b/m]$, for any $t\in[0,1]$, we have

$$\int_0^1 (1-t)^p\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t \leqslant \left|f'(a)\right|^q\int_0^1 (1-t)^p\left(1-\frac{t^\alpha}{2^\alpha}\right)\mathrm{d}t + \left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1 m(1-t)^p\frac{t^\alpha}{2^\alpha}\mathrm{d}t = \left(\frac{1}{p+1}-\frac{1}{2^\alpha}\beta(\alpha+1,p+1)\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha}\beta(\alpha+1,p+1)\left|f'\left(\frac{b}{m}\right)\right|^q \tag{31}$$

and

$$\int_0^1 (1-t)^p\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t \leqslant \left|f'(a)\right|^q\int_0^1 (1-t)^p\left[1-\left(1-\frac{t}{2}\right)^\alpha\right]\mathrm{d}t + \left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1 m(1-t)^p\left(1-\frac{t}{2}\right)^\alpha \mathrm{d}t = \left(\frac{1}{p+1}+\frac{(-1)^q}{2^\alpha}\beta(\alpha+1,p+1)\right)\left|f'(a)\right|^q - \frac{m(-1)^q}{2^\alpha}\beta(\alpha+1,p+1)\left|f'\left(\frac{b}{m}\right)\right|^q. \tag{32}$$

Using (31) and (32) in (30), we obtain (29). □

Corollary 9 *If $g(x)=1$ in Theorem 5, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \leqslant \frac{\eta(b,a)}{4}\left(\frac{q-1}{2q-p-1}\right)^{(q-1)/q} \times\left\{\left[\left(\frac{1}{p+1}-\frac{1}{2^\alpha}\beta(\alpha+1,p+1)\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha}\beta(\alpha+1,p+1)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q} + \left[\left(\frac{1}{p+1}+\frac{(-1)^p}{2^\alpha}\beta(\alpha+1,p+1)\right)\left|f'(a)\right|^q - \frac{m(-1)^p}{2^\alpha}\beta(\alpha+1,p+1)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\} \tag{33}$$

where $p^{-1}+q^{-1}=1$.

Corollary 10 *With the same assumptions given in Theorem 5, if $|f'(x)|\leqslant\Upsilon$ on $[a,a+\eta(b,a)]$ with $m=1$, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{2}\left(\frac{q-1}{2q-p-1}\right)^{(q-1)/q}\left(\frac{1}{p+1}\right)^{1/q}\Upsilon\|g\|_\infty \tag{34}$$

where $p^{-1}+q^{-1}=1$.

Theorem 6 *Suppose that all the assumptions of Theorem 4 are satisfied. Then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{4}\|g\|_\infty \times\left\{\left[\left(\frac{1}{q+1}-\frac{1}{2^\alpha}\beta(\alpha+1,q+1)\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha}\beta(\alpha+1,q+1)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q} + \left[\left(\frac{1}{q+1}+\frac{(-1)^q}{2^\alpha}\beta(\alpha+1,q+1)\right)\left|f'(a)\right|^q - \frac{m(-1)^q}{2^\alpha}\beta(\alpha+1,q+1)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\} \tag{35}$$

where

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad \forall x,y>0.$$

Proof: Using Lemma 1 and Hölder's integral inequality for $p>1$, we obtain

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right|$$

$$\leqslant \frac{\eta^2(b,a)}{4}\|g\|_\infty\left(\int_0^1 1\,\mathrm{d}t\right)^{1/p} \times\left\{\left[\int_0^1 (1-t)^q\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q} +\left[\int_0^1 (1-t)^q\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}. \quad (36)$$

Replacing p in (31) and(32) by q and substituting them into (36), we deduce (35). □

Corollary 11 *If $g(x)=1$ in Theorem 6, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \leqslant \frac{\eta(b,a)}{4}\left\{\left[\left(\frac{1}{q+1}-\frac{1}{2^\alpha}\beta(\alpha+1,q+1)\right)\left|f'(a)\right|^q + \frac{m}{2^\alpha}\beta(\alpha+1,q+1)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q} +\left[\left(\frac{1}{q+1}+\frac{(-1)^q}{2^\alpha}\beta(\alpha+1,q+1)\right)\left|f'(a)\right|^q - \frac{m(-1)^q}{2^\alpha}\beta(\alpha+1,q+1)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}. \quad (37)$$

Corollary 12 *With the same assumptions given in Theorem 6, if $|f'(x)|\leqslant\Upsilon$ on $[a,a+\eta(b,a)]$ with $m=1$, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \frac{\eta^2(b,a)}{2}\left(\frac{1}{q+1}\right)^{1/q}\Upsilon\|g\|_\infty. \quad (38)$$

Corollary 13 *From Corollaries 8, 10 and 12, we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} g(x)\,\mathrm{d}x - \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}x\right| \leqslant \min\{K_1,K_2,K_3\}$$

where

$$K_1=\frac{\eta^2(b,a)}{2}\left(\frac{1}{p+1}\right)^{1/p}\Upsilon\|g\|_\infty,$$

$$K_2=\frac{\eta^2(b,a)}{2}\left(\frac{q-1}{2q-p-1}\right)^{(q-1)/q}\left(\frac{1}{p+1}\right)^{1/q}\Upsilon\|g\|_\infty,$$

and

$$K_3=\frac{\eta^2(b,a)}{2}\left(\frac{1}{q+1}\right)^{1/q}\Upsilon\|g\|_\infty.$$

APPLICATION TO SPECIAL MEANS

Definition 4 [Ref. 21] A function $M:\mathbb{R}_+^2\to\mathbb{R}_+$ is called a mean function if it has the following properties.

(i) Homogeneity: $M(ax,ay)=aM(x,y)$ for all $a>0$.
(ii) Symmetry: $M(x,y)=M(y,x)$.
(iii) Reflexivity: $M(x,x)=x$.
(iv) Monotonicity: if $x\leqslant x'$ and $y\leqslant y'$, then $M(x,y)\leqslant M(x',y')$.
(v) Internality: $\min\{x,y\}\leqslant M(x,y)\leqslant\max\{x,y\}$.

For arbitrary positive real numbers $a>0$ and $b>0$, we define $A:=A(a,b)=\frac{1}{2}(a+b)$, $G:=G(a,b)=\sqrt{ab}$, $H:=H(a,b)=2ab/(a+b)$,

$$P_r:=P_r(a,b)=\left(\frac{a^r+b^r}{2}\right)^{1/r},\ r\geqslant 1$$

$$I(a,b)=\begin{cases}\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a\neq b,\\ a, & a=b,\end{cases}$$

$$L(a,b)=\begin{cases}\dfrac{b-a}{\ln b-\ln a}, & a\neq b,\\ a, & a=b,\end{cases}$$

and

$$L_s(a,b)=\begin{cases}\left[\dfrac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}\right]^{\frac{1}{s}}, & s\neq 0,-1,\ a\neq b,\\ L(a,b), & s=-1,\quad a\neq b,\\ I(a,b), & s=0\quad a\neq b,\\ a, & a=b.\end{cases}$$

Clearly, L_p is monotonic nondecreasing over $p\in\mathbb{R}$, with $L_{-1}:=L$ and $L_0:=I$. In particular, we have $H\leqslant G\leqslant L\leqslant I\leqslant A$.

Now, let $0<a<b$. Suppose that the function $M:=M(a,b):[a+\eta(b,a)]\times[a,a+\eta(b,a)]\to\mathbb{R}^+$, which is one of the abovementioned means. Then one can obtain different inequalities below.

Letting $\eta(b,a)=M(b,a)$ in (18), (23) and (27), one can derive the following significant inequalities.

$$\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,\mathrm{d}x\right| \leqslant \frac{M(b,a)}{8}(|f'(a)|+|f'(b)|) \quad (39)$$

$$\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,\mathrm{d}x\right| \leqslant \frac{M(b,a)}{4}\left(\frac{q-1}{2q-1}\right)^{(q-1)/q} \times \frac{3^{1/q}+1}{4^{1/q}}(|f'(a)|+|f'(b)|) \quad (40)$$

and

$$\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,\mathrm{d}x\right| \leqslant \frac{M(b,a)}{4}\left(\frac{1}{p+1}\right)^{1/p} \frac{3^{1/q}+1}{4^{1/q}}(|f'(a)|+|f'(b)|). \quad (41)$$

Letting $M = A$, G, H, P_r, I, L, L_s in (39), (40), and (41), one can obtain the required inequalities. The details are left for the reader to explore.

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