RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2017.43.201

ScienceAsia 43 (2017): **201–206**

Positive solutions for nonlinear Hadamard fractional differential equations with integral boundary conditions

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Received 18 Sep 2016
Accepted 19 Jun 2017

ABSTRACT: We investigate the existence of positive solutions for a class of nonlinear Hadamard fractional differential equations with integral boundary conditions. By using the properties of Green's functions and the Krasnoselskii-Zabreiko fixed point theorem, two new existence results for at least one positive solution are obtained. Two examples are given to illustrate the main results.

KEYWORDS: fixed point theorem, Green's function

MSC2010: 26A33 34A08 34B18

INTRODUCTION

Fractional differential equations can describe phenomena in fields such as control, porous media, electrochemistry, viscoelasticity, and electromagnetism[1–4]. Zhou and Peng[5,6] obtained the existence and uniqueness of local and global mild solutions for the time-fractional Navier-Stokes equations by using fixed point theory. Some authors studied the existence and multiplicity of solutions or positive solutions for nonlinear boundary value problems involving fractional differential equations with various kinds of boundary value conditions[7–10]. and quoted the references therein. For example, the solutions of fractional integrodifferential equations with boundary value conditions have been investigated[11,12]. Yang[13] obtained the existence and multiplicity of positive solutions for nonlinear Caputo fractional differential equations with integral boundary conditions. Henderson and Luca investigated the positive solutions of nonlinear boundary value problems for systems of fractional differential equations[14].

In the past ten years, most of the work on the topic has been based on Riemann-Liouville and Caputo type fractional differential equations. Recently, more studies have looked at the boundary value problems of nonlinear Hadamard fractional differential equations[15–18]. Ahmad and Ntouyas[19,20] studied the existence and uniqueness of solutions for fractional integral boundary value problem involving Hadamard-type fractional differential equations/systems with integral boundary conditions by applying some standard fixed point theorems. Ahmad et al[21] investigated the existence of solutions for fractional boundary value problem involving Hadamard-type fractional differential inclusions and integral boundary conditions based on standard fixed point theorems for multivalued maps. By applying some inequalties with Green's functions and Guo-Krasnoselskii fixed point theorems, Yang[22,23] considered the existence of positive solution for nonlinear Hadamard fractional differential equations with four-point coupled and coupled integral boundary conditions, respectively. Aljoudi et al studied a nonlocal boundary value problem of Hadamard type coupled sequential fractional differential equations supplemented with coupled strip conditions[24].

Motivated by above results, the main aim of this paper is to investigate the following nonlinear Hadamard fractional differential equation with integral boundary conditions:

$$\begin{gathered} D^q u(t) + f(t,u(t)) = 0, \quad t \in [1,\mathrm{e}], \\ u^{(m)}(1) = 0, \quad u(\mathrm{e}) = \int_1^{\mathrm{e}} g(t)u(t)\frac{\mathrm{d}t}{t}, \end{gathered} \tag{1}$$

where $0 \leqslant m \leqslant n-2$, $n \in \mathbb{N}$, $n \geqslant 3$, $q \in (n-1,n]$ is a real number, D^q is the Hadamard fractional derivative of fractional order q, $f \in C([1,\mathrm{e}]\times\mathbb{R}^+,\mathbb{R})$, and $g \in C([1,\mathrm{e}],\mathbb{R}^+)$. The nonlinear term f may grow both superlinearly and sublinearly at ∞. In this paper, by using the properties of Green's functions and the Krasnoselskii-Zabreiko fixed point theorem, two new existence results for at least one positive solution for (1) are obtained.

PRELIMINARIES

Definition 1 The Hadamard derivative of fractional order q for a function $g:[1,\infty)\to\mathbb{R}$ is defined[3] as

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}\,\mathrm{d}s,$$

where $n-1<q<n$, $n=[q]+1$, and $[q]$ denotes the integer part of the real number q.

Definition 2 The Hadamard fractional integral of order q for a function $g:[1,\infty)\to\mathbb{R}$ is defined[3] as

$$I^q g(t) = \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}\,\mathrm{d}s,\quad q>0,$$

provided the integral exists.

Set $\rho(t)=(\log t)^{q-1}(1-\log t)$ and $\hat{\rho}(t)=(1-\log t)^{q-1}\log t$, for $q>2$, $t\in[1,\mathrm{e}]$, and

$$G(t,s)=\frac{1}{\Gamma(q)}\begin{cases}\Lambda-(\log(t/s))^{q-1}, & 1\leqslant s\leqslant t\leqslant \mathrm{e},\\ \Lambda, & 1\leqslant t\leqslant s\leqslant \mathrm{e},\end{cases}\tag{2}$$

where $\Lambda=(\log t)^{q-1}(1-\log s)^{q-1}$.

Lemma 1 (Ref. 22) *The function $G(t,s)$ defined by (2) has the following properties.*

(P_1) *$G(t,s)$ is a continuous function on $(t,s)\in[1,\mathrm{e}]^2$ and $G(t,s)>0$, for $t,s\in(1,\mathrm{e})$.*

(P_2) *$\rho(t)\hat{\rho}(s)\leqslant\Gamma(q)G(t,s)\leqslant(q-1)\hat{\rho}(s)$, for $t,s\in[1,\mathrm{e}]$.*

(P_3) *$\rho(t)\hat{\rho}(s)\leqslant\Gamma(q)G(t,s)\leqslant(q-1)\rho(t)$, for $t,s\in[1,\mathrm{e}]$.*

For the sake of simplicity, we always assume that the following conditions hold.

(H_1) $\kappa=1-\int_1^{\mathrm{e}}(\log t)^{q-1}g(t)\,\mathrm{d}t/t>0$;

(H_2) there exists a positive constant M such that $f(t,u(t))\geqslant -M$, for any $(t,u)\in[1,\mathrm{e}]\times\mathbb{R}^+$.

Lemma 2 *Let $x\in C[1,\mathrm{e}]$. Then the Hadamard fractional boundary value problem*

$$D^q u(t)+x(t)=0,\quad t\in[1,\mathrm{e}],$$
$$u^{(m)}(1)=0,\quad u(\mathrm{e})=\int_1^{\mathrm{e}} g(t)u(t)\frac{\mathrm{d}t}{t},\tag{3}$$

where $0\leqslant m\leqslant n-2$, $n\in\mathbb{N}$, $n\geqslant 3$, $q\in(n-1,n]$ is a real number, has a unique solution in the form

$$u(t)=\int_1^{\mathrm{e}} H(t,s)x(s)\frac{\mathrm{d}s}{s},$$

where

$$H(t,s)=G(t,s)+\frac{(\log t)^{q-1}}{\kappa}\int_1^{\mathrm{e}} G(t,s)g(t)\frac{\mathrm{d}t}{t}.$$

Proof: As argued in Ref. 3, the solution of the Hadamard differential equation in (3) can be written as the equivalent integral equation

$$u(t)=c_1(\log t)^{q-1}+c_2(\log t)^{q-2}+\cdots+c_n(\log t)^{q-n} - \frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}x(s)\frac{\mathrm{d}s}{s}.\tag{4}$$

From $u^{(m)}(1)=0$, $0\leqslant m\leqslant n-2$, we have $c_n=c_{n-1}=\cdots=c_2=0$. Thus (4) reduces to

$$u(t)=c_1(\log t)^{q-1}-\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}x(s)\frac{\mathrm{d}s}{s}.\tag{5}$$

Using the integral boundary condition $u(\mathrm{e})=\int_1^{\mathrm{e}} g(t)u(t)\,\mathrm{d}t/t$ in (5), we have

$$c_1=\int_1^{\mathrm{e}} g(t)u(t)\frac{\mathrm{d}t}{t}+\frac{1}{\Gamma(q)}\int_1^{\mathrm{e}}\left(\log\frac{\mathrm{e}}{s}\right)^{q-1}x(s)\frac{\mathrm{d}s}{s}.\tag{6}$$

Substituting (6) into (5), we obtain

$$\begin{aligned}u(t)&=(\log t)^{q-1}\int_1^{\mathrm{e}} g(t)u(t)\frac{\mathrm{d}t}{t}\\&\quad+(\log t)^{q-1}\cdot\frac{1}{\Gamma(q)}\int_1^{\mathrm{e}}\left(\log\frac{\mathrm{e}}{s}\right)^{q-1}x(s)\frac{\mathrm{d}s}{s}\\&\quad-\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}x(s)\frac{\mathrm{d}s}{s}\\&=(\log t)^{q-1}\int_1^{\mathrm{e}} g(t)u(t)\frac{\mathrm{d}t}{t}+\int_1^{\mathrm{e}} G(t,s)x(s)\frac{\mathrm{d}s}{s}.\end{aligned}\tag{7}$$

Multiplying (7) by $g(t)/t$ and integrating the resulting identity with respect to t from 1 to e, we obtain

$$\int_1^{\mathrm{e}} g(t)u(t)\frac{\mathrm{d}t}{t}=\int_1^{\mathrm{e}}(\log t)^{q-1}g(t)\frac{\mathrm{d}t}{t}\int_1^{\mathrm{e}} g(t)u(t)\frac{\mathrm{d}t}{t} + \int_1^{\mathrm{e}} g(t)\int_1^{\mathrm{e}} G(t,s)x(s)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}.$$

Solving for $\int_1^{\mathrm{e}} g(t)u(t)\,\mathrm{d}t/t$, we obtain

$$\int_1^{\mathrm{e}} g(t)u(t)\frac{\mathrm{d}t}{t} = \frac{1}{\kappa}\int_1^{\mathrm{e}} g(t)\int_1^{\mathrm{e}} G(t,s)x(s)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}. \tag{8}$$

Combining (7) and (8) gives

$$u(t) = \frac{(\log t)^{q-1}}{\kappa}\int_1^{\mathrm{e}} g(t)\int_1^{\mathrm{e}} G(t,s)x(s)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} + \int_1^{\mathrm{e}} G(t,s)x(s)\frac{\mathrm{d}s}{s} = \int_1^{\mathrm{e}} H(t,s)x(s)\frac{\mathrm{d}s}{s}.$$

□

Lemma 3 *The following inequalities hold:*

$$\mathscr{K}_1\hat{\rho}(s) \leqslant \int_1^{\mathrm{e}} H(t,s)\hat{\rho}(t)\frac{\mathrm{d}t}{t} \leqslant \mathscr{K}_2\hat{\rho}(s), \quad s\in[1,\mathrm{e}],$$

where

$$\mathscr{K}_1 = \frac{q\Gamma(q)}{\Gamma(2q+1)}\left(\frac{q}{2q+1}+\frac{1}{\kappa}\int_1^{\mathrm{e}}\rho(t)g(t)\frac{\mathrm{d}t}{t}\right),$$
$$\mathscr{K}_2 = \frac{q-1}{\Gamma(q+2)}\left(1+\frac{1}{\kappa}\int_1^{\mathrm{e}} g(t)\frac{\mathrm{d}t}{t}\right).$$

Proof: Combining Lemmas 1 and 2, we obtain

$$\frac{1}{\Gamma(q)}\left(\rho(t)+\frac{(\log t)^{q-1}}{\kappa}\int_1^{\mathrm{e}}\rho(t)g(t)\frac{\mathrm{d}t}{t}\right)\hat{\rho}(s) \leqslant H(t,s) \leqslant \frac{q-1}{\Gamma(q)}\left(1+\frac{1}{\kappa}\int_1^{\mathrm{e}} g(t)\frac{\mathrm{d}t}{t}\right)\hat{\rho}(s). \tag{9}$$

Multiplying the above equation by $\hat{\rho}(t)/t$ and integrating the resulting identity with respect to t from 1 to e, we obtain the desired results. □

Let $\mathscr{E}=C([1,\mathrm{e}],\mathbb{R})$, $\|u\|=\max_{t\in[1,\mathrm{e}]}|u(t)|$, $\mathscr{P}=\{u\in\mathscr{E}: u(t)\geqslant \mathfrak{L}^{-1}\omega(t)\|u\|, \forall t\in[1,\mathrm{e}]\}$, where $\mathfrak{L}=(q-1)(1+\int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)$, $\omega(t)=(\rho(t)+(\log t)^{q-1}\int_1^{\mathrm{e}}(\rho(t)g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)$. Then $(\mathscr{E},\|\cdot\|)$ becomes a real Banach space with the norm $\|u\|=\max_{t\in[1,\mathrm{e}]}|u(t)|$ and $\mathscr{P}$ is a cone on $\mathscr{E}$. We denote $\mathscr{B}_\tau=\{u\in\mathscr{E}:\|u\|<\tau\}$ for $\tau>0$ in the following.

We now note that u is the solution of (1) if and only if u is a fixed point of the operator

$$(\mathbb{A}u)(t) = \int_1^{\mathrm{e}} H(t,s)f(s,u(s))\frac{\mathrm{d}s}{s}.$$

Clearly, from the Arzelà-Ascoli Theorem, $\mathbb{A}:\mathscr{E}\to\mathscr{E}$ is a completely continuous operator. We now show the relation between the fixed point of $\mathbb{A}$ and the fixed point of the operator $\mathbb{B}$ defined by

$$(\mathbb{B}u)(t) = \int_1^{\mathrm{e}} H(t,s)F(s,u(s)-w(t))\frac{\mathrm{d}s}{s},$$

where

$$F(t,x) = \begin{cases}\bar{f}(t,x), & t\in[1,\mathrm{e}],\ x\geqslant 0,\\ \bar{f}(t,0), & t\in[1,\mathrm{e}],\ x<0,\end{cases}$$

the function $\bar{f}(t,x)+M$, $\bar{f}:[1,\mathrm{e}]\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous and $w(t)=M\int_1^{\mathrm{e}}\{H(t,s)/s\}\,\mathrm{d}s$, $t\in[1,\mathrm{e}]$. Clearly, $\mathbb{B}:\mathscr{E}\to\mathscr{E}$ is also a completely continuous operator. From Lemma 3, we can easily obtain $\mathbb{B}(\mathscr{P})\subset\mathscr{P}$. By Lemma 2 in Ref. 25, we easily have the following lemma.

Lemma 4 *If u^* is a positive fixed point of $\mathbb{A}$, then u^*+w is a positive fixed point of $\mathbb{B}$. Conversely, if u is a positive fixed point of $\mathbb{B}$ and $u(t)\geqslant w(t)$, $t\in[1,\mathrm{e}]$, then $u^*=u-w$ is a positive fixed point of $\mathbb{A}$.*

Lemma 5 (Ref. 26) *Let $\mathscr{E}$ be a real Banach space and $\mathscr{P}$ a cone of $\mathscr{E}$. Suppose that $\mathbb{A}:(\bar{\mathscr{B}}_R\backslash\mathscr{B}_r)\cap\mathscr{P}\to\mathscr{P}$ is a completely continuous operator with $0<r<R$. If either*

(i) *$\mathbb{A}u\nleq u$ for each $u\in\partial\mathscr{B}_r\cap\mathscr{P}$ and $\mathbb{A}u\ngeq u$ for each $u\in\partial\mathscr{B}_R\cap\mathscr{P}$ or*
(ii) *$\mathbb{A}u\ngeq u$ for each $u\in\partial\mathscr{B}_r\cap\mathscr{P}$ and $\mathbb{A}u\nleq u$ for each $u\in\partial\mathscr{B}_R\cap\mathscr{P}$,*

then $\mathbb{A}$ has at least one fixed point on $(\mathscr{B}_R\backslash\bar{\mathscr{B}}_r)\cap\mathscr{P}$.

MAIN RESULTS

In this section, by the Krasnoselskii-Zabreiko fixed point theorem in Lemma 5, we obtain two new existence results for at least one positive solution for the boundary value problem (1). To obtain a positive fixed point of the operator $\mathbb{A}$, by Lemma 4, we need only seek the positive fixed point u of $\mathbb{B}$ and $u\geqslant w$. It follows from (P_3) in Lemma 1 that

$$\begin{aligned}\int_1^{\mathrm{e}} H(t,s)\frac{\mathrm{d}s}{s} &= \int_1^{\mathrm{e}}\left(G(t,s)+\frac{(\log t)^{q-1}}{\kappa}\int_1^{\mathrm{e}} G(t,s)g(t)\frac{\mathrm{d}t}{t}\right)\frac{\mathrm{d}s}{s}\\ &\leqslant \frac{q-1}{\Gamma(q)}\int_1^{\mathrm{e}}\left(\rho(t)+\frac{(\log t)^{q-1}}{\kappa}\int_1^{\mathrm{e}}\rho(t)g(t)\frac{\mathrm{d}t}{t}\right)\frac{\mathrm{d}s}{s}\\ &= (q-1)\omega(t).\end{aligned}$$

For any $u \in \mathscr{P}$ and $t \in [1, \mathrm{e}]$, we have

$$\begin{aligned} u(t) - w(t) &= u(t) - M\int_1^{\mathrm{e}} H(t,s)\frac{\mathrm{d}s}{s} \\ &= u(t) - M(q-1)\omega(t) \\ &\geqslant u(t) - M\mathfrak{L}(q-1)u(t)\|u\|^{-1}. \end{aligned}$$

Hence $\|u\| \geqslant M\mathfrak{L}(q-1) = M(q-1)^2(1 + \int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)$ leads to $u(t) \geqslant w(t)$ for $t \in [1, \mathrm{e}]$.

Next, let $\mathfrak{M}_1^{-1} = \mathscr{K}_1$ and $\mathfrak{M}_2^{-1} = \mathscr{K}_2$. Meanwhile, we list the following assumptions on f:

(H$_3$) $\liminf_{u\to\infty}(f(t,u)/u) > \mathfrak{M}_1$ uniformly with respect to $t \in [1, \mathrm{e}]$;

(H$_4$) there exists $Q(t) : [1, \mathrm{e}] \to [0, +\infty)$ such that $f(t,u) + M \leqslant Q(t)$, for any $u \in [0, M(q-1)^2(1 + \int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)]$ and $t \in [1, \mathrm{e}]$, $\int_1^{\mathrm{e}}(\hat{\rho}(s)Q(s)/s)\,\mathrm{d}s \leqslant M(q-1)/\Gamma(q)$;

(H$_5$) $\liminf_{u\to\infty}(f(t,u)/u) < \mathfrak{M}_2$ uniformly with respect to $t \in [1, \mathrm{e}]$;

(H$_6$) there exists $Q(t) : [1, \mathrm{e}] \to [0, +\infty)$, $\theta \in (0, (\mathrm{e}-1)/2)$ and $t_0 \in [1+\theta, \mathrm{e}-\theta]$ such that $f(t,u) + M \geqslant Q(t)$, for any $u \in [0, M(q-1)^2(1 + \int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)]$ and $t \in [1+\theta, \mathrm{e}-\theta]$, $\int_{1+\theta}^{\mathrm{e}-\theta}(\omega(t_0)\hat{\rho}(s)Q(s)/s)\,\mathrm{d}s \geqslant M(q-1)^2(1 + \int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)$.

Theorem 1 *Let* (H$_1$)–(H$_4$) *hold. Then the boundary value problem* (1) *has at least one positive solution.*

Proof: (H$_3$) implies that $\liminf_{u\to\infty}(f(t,u)/u) > \mathfrak{M}_1$ uniformly with respect to $t \in [1, \mathrm{e}]$. Consequently, there exist $\varepsilon > 0$ and $b > 0$ such that $f(t,u) + M \geqslant (\mathfrak{M}_1 + \varepsilon)u - b$ for all $u \in \mathbb{R}^+$ and $t \in [1, \mathrm{e}]$. We now show that there exists a large enough positive number $R > M(q-1)^2(1 + \int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)$ such that

$$u \ngeq \mathbb{B}u, \quad \forall u \in \partial\mathscr{B}_R \cap \mathscr{P}. \tag{10}$$

Indeed, if the claim is false, there exists $u \in \partial\mathscr{B}_R \cap \mathscr{P}$ such that $u \geqslant \mathbb{B}u$. This yields, for all $t \in [1, \mathrm{e}]$:

$$\begin{aligned} &u(t) \geqslant (\mathbb{B}u)(t) \\ &\quad\geqslant \int_1^{\mathrm{e}} H(t,s)\big((\mathfrak{M}_1 + \varepsilon)(u(s) - w(s)) - b\big)\frac{\mathrm{d}s}{s}. \end{aligned}$$

Multiplying this by $\hat{\rho}(t)/t$ and integrating over $[1, \mathrm{e}]$, we obtain

$$\begin{aligned} &\int_1^{\mathrm{e}} u(t)\hat{\rho}(t)\frac{\mathrm{d}t}{t} + \int_1^{\mathrm{e}}\hat{\rho}(t)\int_1^{\mathrm{e}} H(t,s)\big((\mathfrak{M}_1 + \varepsilon)w(s) \\ &\quad + b\big)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \geqslant \int_1^{\mathrm{e}}\hat{\rho}(t)\int_1^{\mathrm{e}} H(t,s)(\mathfrak{M}_1 + \varepsilon)u(s)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}. \end{aligned}$$

It follows from the above inequality and Lemma 3 that

$$\begin{aligned} &\int_1^{\mathrm{e}} u(t)\hat{\rho}(t)\frac{\mathrm{d}t}{t} + \frac{\mathscr{K}_{\in}-(\text{Ш})}{\Gamma(q+2)}(b + M(\mathfrak{M}_1 + \varepsilon)\mathscr{K}_2) \\ &\qquad\qquad \geqslant (\mathfrak{M}_1 + \varepsilon)\mathscr{K}_1\int_1^{\mathrm{e}} u(t)\hat{\rho}(t)\frac{\mathrm{d}t}{t}. \end{aligned}$$

The preceding inequality and $u \in \mathscr{P}$ imply that

$$\begin{aligned} \frac{\mathscr{K}_\infty}{\mathfrak{L}}\|u\| &= \int_1^{\mathrm{e}}\mathfrak{L}^{-1}\|u\|\omega(t)\hat{\rho}(t)\frac{\mathrm{d}t}{t} \leqslant \int_1^{\mathrm{e}} u(t)\hat{\rho}(t)\frac{\mathrm{d}t}{t} \\ &\leqslant \frac{\Gamma(q)\mathscr{K}_2(b + M(\mathscr{K}_1^{-1} + \varepsilon)\mathscr{K}_2)}{\varepsilon\mathscr{K}_1\Gamma(q+2)}. \end{aligned} \tag{11}$$

From (11), we immediately have

$$\|u\| \leqslant \frac{\mathfrak{L}\Gamma(q)\mathscr{K}_2(b + M(\mathscr{K}_1^{-1} + \varepsilon)\mathscr{K}_2)}{\varepsilon\mathscr{K}_1^2\Gamma(q+2)} := N_1 > 0.$$

If $R > \max\{N_1, M(q-1)^2(1 + \int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)\}$, this contradicts $u \in \partial\mathscr{B}_R \cap \mathscr{P}$. As a result, (10) is true. On the other hand, by (H$_4$) and (9), we have

$$\begin{aligned} (\mathbb{B}u)(t) &= \int_1^{\mathrm{e}} H(t,s)F(t, u(s) - w(s))\frac{\mathrm{d}s}{s} \\ &\leqslant \int_1^{\mathrm{e}} H(t,s)Q(s)\frac{\mathrm{d}s}{s} \leqslant \int_1^{\mathrm{e}}\mathfrak{L}\hat{\rho}(s)Q(s)\frac{\mathrm{d}s}{s} \\ &\leqslant \frac{M(q-1)^2}{\Gamma(q)}\left(1 + \frac{1}{\kappa}\int_1^{\mathrm{e}} g(t)\frac{\mathrm{d}t}{t}\right) = \|u\|, \end{aligned}$$

for any $(t,u) \in [1, \mathrm{e}] \times \partial\mathscr{B}_r$, where $r = M(q-1)^2(1 + \int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)$. Then we have $\|\mathbb{B}u\| \leqslant \|u\|$, for any $u \in \partial\mathscr{B}_r \cap \mathscr{P}$. This leads to $u \nleq \mathbb{B}u$, for any $u \in \partial\mathscr{B}_r \cap \mathscr{P}$. Now Lemma 5 implies that $\mathbb{B}$ has at least one fixed point on $(\mathscr{B}_R \backslash \bar{\mathscr{B}}_r) \cap \mathscr{P}$. Hence (1) has at least one positive solution. □

Theorem 2 *Let* (H$_1$), (H$_2$), (H$_5$), *and* (H$_6$) *hold. Then the boundary value problem* (1) *has at least one positive solution.*

Proof: (H$_5$) implies that $\liminf_{u\to\infty}(f(t,u)/u) < \mathfrak{M}_2$ uniformly with respect to $t \in [1, \mathrm{e}]$. Hence there exist $\varepsilon \in (0, \mathfrak{M}_2)$ and $b > 0$ such that $f(t,u) + M \leqslant (\mathfrak{M}_2 - \varepsilon)u + b$ for all $u \in \mathbb{R}^+$ and $t \in [1, \mathrm{e}]$. We now show that there exists a large enough positive number $R > M(q-1)^2(1 + \int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)$ such that

$$u \nleq \mathbb{B}u, \quad \forall u \in \partial\mathscr{B}_R \cap \mathscr{P}. \tag{12}$$

Indeed, if the claim (12) is false, there exists $u \in \partial\mathscr{B}_R \cap \mathscr{P}$ such that $u \leqslant \mathbb{B}u$. This yields

$$\begin{aligned} u(t) &\leqslant (\mathbb{B}u)(t) \\ &\leqslant \int_1^{\mathrm{e}} H(t,s)\big((\mathfrak{M}_2-\varepsilon)(u(s)-w(s))+b\big)\frac{\mathrm{d}s}{s} \\ &\leqslant \int_1^{\mathrm{e}} H(t,s)\big((\mathfrak{M}_2-\varepsilon)u(s)+b\big)\frac{\mathrm{d}s}{s}, \end{aligned}$$

for any $t \in [1,\mathrm{e}]$. Multiplying the above inequality by $\hat{\rho}(t)/t$ and integrating over $[1,\mathrm{e}]$, we obtain

$$\begin{aligned} &\int_1^{\mathrm{e}} u(t)\hat{\rho}(t)\frac{\mathrm{d}t}{t} \\ &\leqslant \int_1^{\mathrm{e}} \hat{\rho}(t)\int_1^{\mathrm{e}} H(t,s)\big((\mathfrak{M}_2-\varepsilon)u(s)+b\big)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} \\ &\leqslant \int_1^{\mathrm{e}} \mathscr{K}_2\hat{\rho}(t)\big((\mathfrak{M}_2-\varepsilon)u(t)+b\big)\frac{\mathrm{d}t}{t}. \end{aligned}$$

It follows from the above inequality and Lemma 3 that

$$\int_1^{\mathrm{e}} u(t)\hat{\rho}(t)\frac{\mathrm{d}t}{t} \leqslant \frac{b\Gamma(q)}{\varepsilon\Gamma(q+2)}.$$

As with (11), we immediately have $\|u\| \leqslant b\mathfrak{L}\Gamma(q)/(\varepsilon\mathscr{K}_1\Gamma(q+2)) := N_2 > 0$. If $R > \max\{N_2, M(q-1)^2(1+\int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)\}$, this contradicts $u \in \partial\mathscr{B}_R \cap \mathscr{P}$. As a result, (12) is true. On the other hand, by (H_6) and (9), we have

$$\begin{aligned} (\mathbb{B}u)(t_0) &= \int_1^{\mathrm{e}} H(t,s)F(t_0,u(s)-w(s))\frac{\mathrm{d}s}{s} \\ &\geqslant \int_{1+\theta}^{\mathrm{e}-\theta} H(t_0,s)Q(s)\frac{\mathrm{d}s}{s} \\ &\geqslant \int_{1+\theta}^{\mathrm{e}-\theta} \omega(t_0)\hat{\rho}(s)Q(s)\frac{\mathrm{d}s}{s} \\ &\geqslant \frac{M(q-1)^2}{\Gamma(q)}\left(1+\frac{1}{\kappa}\int_1^{\mathrm{e}} g(t)\frac{\mathrm{d}t}{t}\right) = \|u\|, \end{aligned}$$

for any $(t,u) \in [1,\mathrm{e}]\times\partial\mathscr{B}_r$, where $r = M(q-1)^2(1+\int_1^{\mathrm{e}}(g(t)/t)\,\mathrm{d}t/\kappa)/\Gamma(q)$. Then we have $\|\mathbb{B}u\| \geqslant \|u\|$, for any $u \in \partial\mathscr{B}_r \cap \mathscr{P}$. This leads to $u \ngeq \mathbb{B}u$, for any $u \in \partial\mathscr{B}_r \cap \mathscr{P}$. Now Lemma 5 implies that $\mathbb{B}$ has at least one fixed point on $(\mathscr{B}_R \backslash \bar{\mathscr{B}}_r)\cap\mathscr{P}$. Hence the boundary value problem (1) has at least one positive solution. □

TWO EXAMPLES

Consider the Hadamard fractional boundary value problem

$$\begin{aligned} &D^{2.5}u(t)+f(t,u(t)) = 0, \quad t \in [1,\mathrm{e}], \\ &u(1) = u'(0) = 0, \quad u(\mathrm{e}) = \int_1^{\mathrm{e}} u(t)\frac{\mathrm{d}t}{t}. \end{aligned} \tag{13}$$

(i) Suppose that $f(t,u) = M_1(u/8)^\beta - \sqrt{\pi}t\sin u$, where $M_1 > 0$ and $\beta > 1$. Then for any $M_1 \leqslant 8\sqrt{\pi}$, (13) has at least one positive solution.

(ii) Suppose that $f(t,u) = M_2\exp(16/\sqrt{\pi}-u) - 2t\cos u$, where $M_2 > 0$. Then for any $M_2 \geqslant 757$, (13) has at least one positive solution.

Proof: By direct calculation, we have $\kappa = \frac{3}{5}$, which implies (H_1). Since $q = 2.5$, then $\Gamma(2.5) = 3\sqrt{\pi}/4$, $\Gamma(4.5) = 105\sqrt{\pi}/16$, $\Gamma(6) = 120$, $\mathfrak{L} = 16/(3\sqrt{\pi})$, $\mathscr{K}_1 = 17\sqrt{\pi}/1792$, $\mathscr{K}_2 = 64/105\sqrt{\pi}$.

(i) Fix $M = \sqrt{\pi}$. Then $f(t,u) = M_1(u/8)^\beta - \sqrt{\pi}t\sin u \geqslant -\sqrt{\pi} = -M$ for all $t \in [1,\mathrm{e}]$, $u \in [0,+\infty)$. Then the condition (H_2) holds. By simple computation, we obtain

$$\begin{aligned} \lim_{u\to\infty}\frac{f(t,u)}{u} &= \lim_{u\to\infty}\frac{M_1\left(\frac{u}{8}\right)^\beta - \sqrt{\pi}t\sin u}{u} \\ &= +\infty > \mathfrak{M}_1 \end{aligned}$$

uniformly with respect to $t \in [1,\mathrm{e}]$. Thus (H_3) holds true. Furthermore, $f(t,u)+M = M_1(u/8)^\beta - \sqrt{\pi}t\sin u + \sqrt{\pi} \leqslant M_1(u/8)^\beta + \sqrt{\pi} \leqslant M_1 + \sqrt{\pi} = Q(t)$ for all $u \in [0,8]$, and

$$\begin{aligned} \int_1^{\mathrm{e}} \hat{\rho}(s)Q(s)\frac{\mathrm{d}s}{s} &= \left(M_1+\sqrt{\pi}\right)\int_1^{\mathrm{e}} \hat{\rho}(s)\frac{\mathrm{d}s}{s} \\ &\leqslant \frac{2}{9\sqrt{\pi}}\left(M_1+\sqrt{\pi}\right) \leqslant 2 = \frac{M(q-1)}{\Gamma(q)}. \end{aligned}$$

Hence (H_4) holds. By Theorem 1, the boundary value problem (13) has at least one positive solution.

(ii) Fix $M = 2$. Then we obtain $f(t,u) = M_2\exp(16/\sqrt{\pi}-u) - 2t\cos u \geqslant -2 = -M$ for all $t \in [1,\mathrm{e}]$, $u \in [0,+\infty)$. Then the condition (H2) holds. Next, we shall show that $f(t,u)$ satisfies the conditions (H_5) and (H_6). Since

$$\begin{aligned} \lim_{u\to\infty}\frac{f(t,u)}{u} &= \lim_{u\to\infty}\frac{M_2\exp(16/\sqrt{\pi}-u)-2t\cos u}{u} \\ &= 0 < \mathfrak{M}_2 \end{aligned}$$

uniformly with respect to $t \in [1,\mathrm{e}]$, we have (H_5).

Choosing $\theta = 0.25$ and $t_0 = \sqrt{\mathrm{e}}$, gives $\omega(t_0) = 29/(63\sqrt{2\pi})$, $f(t,u)+M = f(t,u)+2 \geqslant$

$M_2 \exp(16/\sqrt{\pi} - u) \geqslant M_2 = Q(t)$ for all $u \in [0, 16/\sqrt{\pi}]$. Since $\int_{1.25}^{e-0.25} (\hat{\rho}(s)/s)\,ds = \int_{1.25}^{e-0.25} (1 - \log s)^{1.5} \{(\log s)/s\}\,ds \geqslant \int_{1.25}^{e-0.25} (1 - \log s)^2 \log s(\,ds/s) \approx 0.06501$, we have $\int_{1.25}^{e-0.25} \omega(\sqrt{\pi})(\hat{\rho}(s)Q(s)/s)\,ds = M_2\omega(\sqrt{\pi}) \int_{1.25}^{e-0.25} (\hat{\rho}(s)/s)\,ds \geqslant M_2 29/(63\sqrt{2\pi}) \times 0.06501 \geqslant 16/\sqrt{\pi}$. Consequently, ($H_6$) holds. By Theorem 2, the boundary value problem (13) has at least one positive solution. □

Acknowledgements: This work was supported by the Key Scientific Research Programmes of Higher Education of Henan Province-Guidance Plan (No. 16B110011). The authors thank the anonymous referees for their constructive comments and suggestions.

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