RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2017.43.047

ScienceAsia 43 (2017): **47–55**

Nonlocal boundary value problems for fractional differential inclusions with Erdélyi-Kober fractional integral boundary conditions

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Received 2 Mar 2016
Accepted 16 Mar 2017

ABSTRACT: We study a new class of boundary value problems consisting of a fractional differential inclusion of Riemann-Liouville type and Erdélyi-Kober fractional integral conditions. Some new existence results for convex as well as nonconvex multivalued maps are obtained by using standard fixed-point theorems. Some illustrative examples are also presented.

KEYWORDS: Riemann-Liouville fractional derivative, fixed point theorem

MSC2010: 34A60 34A08 34A12 34B15

INTRODUCTION

There is considerable interest in fractional differential equations due to their numerous applications, e.g., modelling of viscoelastic behaviour[1], anomalous diffusion[2], cellular neural networks[3], and long-time memory in financial time series via fractional Langevin equations[4]. For further details and examples see Refs. 5–10. For some recent development of the topic see Refs. 11–14 and the references cited therein. Most of the work on the topic involves the Riemann-Liouville or Caputo type fractional derivative. The so-called Erdélyi-Kober fractional derivative, which is a generalization of the Riemann-Liouville fractional derivative, is often used too. An Erdélyi-Kober operator is a fractional integration operation[15]. These operators have been used by many authors, in particular, to obtain solutions of the single, dual, and triple integral equations possessing special functions as their kernels[6,15–18].

We consider the boundary value problem

$$\left.\begin{aligned} &D^q x(t) \in F(t, x(t)),\ 0<t<T,\ 1<q<2,\\ &x(0)=0, \quad \alpha x(T) = \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} x(\xi_i), \end{aligned}\right\} \quad (1)$$

where D^q is the standard Riemann-Liouville fractional derivative of order q, $I_{\eta_i}^{\gamma_i,\delta_i}$ is the Erdélyi-Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $F : [0, T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\alpha, \beta_i \in \mathbb{R}$, $\xi_i \in (0, T)$, $i = 1, 2, \ldots, m$ are given constants.

The present paper is motivated by Ref. 19 where the problem (1) was studied for single-valued maps, i.e., $F = \{f\}$. Here we cover the multivalued case. We establish some existence results for the problem (1) when the right-hand side has convex as well as nonconvex values. In the case of convex values (upper semicontinuous case) we use the nonlinear alternative of Leray-Schauder type. When the right-hand side is not necessarily convex valued (lower semicontinuous case) we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem for lower semicontinuous multivalued maps with nonempty closed and decomposable values. Finally, we prove the existence of solutions for the problem (1) with a not necessarily nonconvex-valued right-hand side by applying a fixed-point theorem for contractive multivalued maps Although the methods used are well known, their exposition in the framework of problem (1) is new.

PRELIMINARIES

Basic material for fractional calculus

In this section, we introduce some notation and definitions of fractional calculus and present results needed later.

The *Riemann-Liouville fractional derivative* of order $q>0$ of a continuous function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$D^q f(t)=\frac{1}{\Gamma(n-q)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-s)^{n-q-1}f(s)\,\mathrm{d}s,$$

where $\Gamma(x)$ is the Gamma function of x, n is a positive integer, and $n-1<q<n$.

The *Riemann-Liouville fractional integral* of order $q>0$ of a continuous function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$J^q f(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)\,\mathrm{d}s,$$

provided the integral exists.

The *Erdélyi-Kober fractional integral* of order $\delta>0$ with $\eta>0$ and $\gamma\in\mathbb{R}$ of a continuous function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$I_\eta^{\gamma,\delta}f(t)=\frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int_0^t\frac{s^{\eta\gamma+\eta-1}f(s)}{(t^\eta-s^\eta)^{1-\delta}}\,\mathrm{d}s$$

provided the right-hand side is pointwise defined on $\mathbb{R}_+$. For $\eta=1$ this operator reduces to the Kober operator[15]

$$I_\gamma^{\delta}f(t)=\frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)}\int_0^t\frac{s^{\gamma}f(s)}{(t-s)^{1-\delta}}\,\mathrm{d}s,\quad \gamma,\delta>0.$$

For $\gamma=0$, the Kober operator reduces to the Riemann-Liouville fractional integral with a power weight:

$$I_0^{\delta}f(t)=\frac{t^{-\delta}}{\Gamma(\delta)}\int_0^t\frac{f(s)}{(t-s)^{1-\delta}}\,\mathrm{d}s,\quad \delta>0.$$

Lemma 1 (Ref. 6) *Let $q>0$ and $y\in C(0,T)\cap L(0,T)$. Then the fractional differential equation $D^q y(t)=0$ has a unique solution*

$$y(t)=c_1t^{q-1}+c_2t^{q-2}+\cdots+c_nt^{q-n},$$

where $c_i\in\mathbb{R}$, $i=1,2,\dots,n$ and $n-1<q<n$.

Lemma 2 (Ref. 6) *Let $q>0$. Then for $y\in C(0,T)\cap L(0,T)$,*

$$J^qD^qy(t)=y(t)+c_1t^{q-1}+c_2t^{q-2}+\cdots+c_nt^{q-n},$$

where $c_i\in\mathbb{R}$, $i=1,2,\dots,n$ and $n-1<q<n$.

Some auxiliary lemmas

Lemma 3 (Ref. 19) *Let δ, $\eta>0$ and $\gamma,q\in\mathbb{R}$. Then*

$$I_\eta^{\gamma,\delta}t^q=\frac{t^q\Gamma(\gamma+q/\eta+1)}{\Gamma(\gamma+q/\eta+\delta+1)}. \tag{2}$$

Lemma 4 (Ref. 19) *Let $1<q<2$, $\delta_i,\eta_i>0$, $\alpha,\gamma_i,\beta_i\in\mathbb{R}$, $\xi_i\in(0,T)$, $i=1,2,\dots,m$ and $h\in C([0,T],\mathbb{R})$. Then the linear Riemann-Liouville fractional differential equation subject to the Erdélyi-Kober fractional integral boundary conditions*

$$\left.\begin{aligned}&D^qx(t)=h(t),\quad t\in(0,T),\\ &x(0)=0,\quad \alpha x(T)=\sum_{i=1}^m\beta_iI_{\eta_i}^{\gamma_i,\delta_i}x(\xi_i),\end{aligned}\right\} \tag{3}$$

is equivalent to the integral equation

$$x(t)=J^qh(t)-\frac{t^{q-1}}{\Lambda}\Big(\alpha J^qh(T)-\sum_{i=1}^m\beta_iI_{\eta_i}^{\gamma_i,\delta_i}J^qh(\xi_i)\Big), \tag{4}$$

where

$$\Lambda:=\alpha T^{q-1}-\sum_{i=1}^m\frac{\beta_i\xi_i^{q-1}\Gamma(\gamma_i+(q-1)/\eta_i+1)}{\Gamma(\gamma_i+(q-1)/\eta_i+\delta_i+1)}\neq0. \tag{5}$$

Basic material for multivalued maps

Here we outline some basic concepts of multivalued analysis[20,21]. Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\|=\sup\{|x(t)|,t\in[0,T]\}$. Also by $L^1([0,T],\mathbb{R})$ we denote the space of functions $x:[0,T]\to\mathbb{R}$ such that $\|x\|_{L^1}=\int_0^T|x(t)|\,\mathrm{d}t$.

For a normed space $(X,\|\cdot\|)$, let

$$\begin{aligned}\mathscr{P}_{\mathrm{cl}}(X)&=\{Y\in\mathscr{P}(X):Y\text{ is closed}\},\\ \mathscr{P}_{\mathrm{b}}(X)&=\{Y\in\mathscr{P}(X):Y\text{ is bounded}\},\\ \mathscr{P}_{\mathrm{cl,b}}(X)&=\{Y\in\mathscr{P}(X):Y\text{ is closed and bounded}\},\\ \mathscr{P}_{\mathrm{cp}}(X)&=\{Y\in\mathscr{P}(X):Y\text{ is compact}\},\\ \mathscr{P}_{\mathrm{cp,c}}(X)&=\{Y\in\mathscr{P}(X):Y\text{ is compact and convex}\}.\end{aligned}$$

A multivalued map $G:X\to\mathscr{P}(X)$ is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x\in X$. G is *bounded* on bounded sets if $G(Y)=\cup_{x\in Y}G(x)$ is bounded in X for all $Y\in\mathscr{P}_{\mathrm{b}}(X)$ (i.e., $\sup_{x\in Y}\{\sup\{|y|:y\in G(x)\}\}<\infty$). G is called *upper semicontinuous* (u.s.c.) on X if for each $x_0\in X$

the set $G(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(x_0)$ there exists an open neighbourhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is *lower semicontinuous* if the set $\{y \in X : G(y) \cap Y \neq \varnothing\}$ is open for any open set Y in X. G is said to be *completely continuous* if $G(B)$ is relatively compact for every $B \in \mathscr{P}_b(X)$; if the multivalued map G is completely continuous with nonempty compact values then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G is said to be *measurable* if for every $y \in X$, the function

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. *G has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$.

EXISTENCE RESULTS

The Carathéodory case

In this subsection we consider the case when F has convex values and prove an existence result based on a nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

A multivalued map $F : [0,T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be *Carathéodory* if (i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$ and (ii) $x \mapsto F(t,x)$ is u.s.c. for almost all $t \in [0,T]$. A Carathéodory function F is called L^1-Carathéodory if for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leqslant \varphi_\rho(t)$$

for all $\|x\| \leqslant \rho$ and for almost all $t \in [0,T]$.

For each $y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ on } [0,T]\}.$$

We define the graph of G to be the set $\operatorname{Gr}(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper semicontinuity.

Lemma 5 (Ref. 20) *If $G : X \to \mathscr{P}_{cl}(Y)$ is u.s.c. then $\operatorname{Gr}(G)$ is a closed subset of $X \times Y$, i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is u.s.c.*

Lemma 6 (Ref. 22) *Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathscr{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathscr{P}_{cp,c}(C(J,X)),$$
$$x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

We recall the well-known nonlinear alternative of Leray-Schauder type for multivalued (Kakutani) maps.

Lemma 7 (Ref. 23) *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \bar{U} \to \mathscr{P}_{cp,c}(C)$ is a u.s.c. compact map. Then either (i) F has a fixed point in $\bar{U}$, or (ii) there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Throughout this paper, for convenience, we use the following expressions

$$J^q f(s)(z) = \frac{1}{\Gamma(q)} \int_0^z (z-s)^{q-1} f(s)\,\mathrm{d}s, \quad z \in \{t, T\},$$

for $t \in [0,T]$ and

$$I_{\eta_i}^{\gamma_i,\delta_i} J^q f(s)(\xi_i) = \frac{\eta_i \xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(q)\Gamma(\delta_i)} \times \int_0^{\xi_i} \int_0^r \frac{r^{\eta_i\gamma_i+\eta_i-1}(r-s)^{q-1}}{(\xi_i^{\eta_i} - r^{\eta_i})^{1-\delta_i}} f(s)\,\mathrm{d}s\,\mathrm{d}r,$$

where $\xi_i \in (0,T)$ for $i = 1,2,\ldots,m$.

Theorem 1 *Assume that*

(H_1) *$F : [0,T] \times \mathbb{R} \to \mathscr{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leqslant p(t)\psi(\|x\|)$$

for each $(t,x) \in [0,T] \times \mathbb{R}$;

(H_3) *there exists a constant $M > 0$ such that $M/\psi(M)\|p\|\Psi > 1$, where*

$$\Psi = \frac{T^q}{\Gamma(q+1)} + \frac{|\alpha| T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)} \times \sum_{i=1}^m \frac{|\beta_i| \xi_i^q \Gamma(\gamma_i + q/\eta_i + 1)}{\Gamma(\gamma_i + q/\eta_i + \delta_i + 1)}. \quad (6)$$

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof: Define the operator $\mathscr{F} : C([0,T],\mathbb{R}) \to \mathscr{P}(C([0,T],\mathbb{R}))$ by

$$\mathscr{F}(x) = \{h \in C([0,T],\mathbb{R}) : h(t) = v(t)\} \tag{7}$$

where

$$v(t) = J^q v(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v(s)(\xi_i)\Big)$$

for $v \in S_{F,x}$. It is obvious that the fixed points of $\mathscr{F}$ are solutions of the boundary value problem (1).

We now show that $\mathscr{F}$ satisfies the assumptions of Leray-Schauder Nonlinear alternative (Lemma 7). The proof consists of several steps.

Step 1: $\mathscr{F}(x)$ is convex for each $x \in C([0,T],\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values).

Step 2: $\mathscr{F}$ maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([0,T],\mathbb{R}) : \|x\| \leqslant \rho\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathscr{F}(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$h(t) = J^q v(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v(s)(\xi_i)\Big).$$

Then we have

$$\begin{aligned}
|h(x)| &\leqslant J^q|v(s)|(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|} J^q|v(s)|(T) \\
&\quad + \frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q |v(s)|(\xi_i) \\
&\leqslant \psi(\|x\|)\Big(J^q p(s)(T) + \frac{|\alpha|T^{q-1}}{|\Lambda|} J^q p(s)(T) \\
&\quad + \frac{T^{q-1}}{|\Lambda|}\sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q p(s)(\xi_i)\Big) \\
&\leqslant \psi(\|x\|)\|p\|\Big(\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} \\
&\quad + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m} \frac{|\beta_i|\xi_i^q \Gamma(\gamma_i + q/\eta_i + 1)}{\Gamma(\gamma_i + q/\eta_i + \delta_i + 1)}\Big)
\end{aligned}$$

and consequently, $\|h\| \leqslant \psi(r)\|p\|\Psi$.

Step 3: $\mathscr{F}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_\rho$. For each $h \in \mathscr{F}(x)$ we obtain

$$\begin{aligned}
&|h(\tau_2) - h(\tau_1)| \\
&\leqslant |J^q v(s)(\tau_2) - J^q v(s)(\tau_1)| \\
&\quad + \frac{|\alpha||\tau_2^{q-1} - \tau_1^{q-1}|}{|\Lambda|} J^q |v(s)|(T) \\
&\quad + \frac{|\tau_2^{q-1} - \tau_1^{q-1}|}{|\Lambda|}\sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q |v(s)|(\xi_i) \\
&\leqslant \frac{\psi(r)}{\Gamma(q)}\Big|\int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}] p(s)\,\mathrm{d}s \\
&\quad + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1} p(s)\,\mathrm{d}s\Big| \\
&\quad + \frac{|\tau_2^{q-1} - \tau_1^{q-1}|\psi(r)}{|\Lambda|}\Big(|\alpha| J^q p(s)(T) \\
&\quad + \sum_{i=1}^{m} |\beta_i| I_{\eta_i}^{\gamma_i,\delta_i} J^q p(s)(\xi_i)\Big).
\end{aligned}$$

Clearly, the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $\tau_2 - \tau_1 \to 0$. As $\mathscr{F}$ satisfies the above three assumptions, it follows by the Ascoli-Arzelá theorem that $\mathscr{F} : C([0,T],\mathbb{R}) \to \mathscr{P}(C([0,T],\mathbb{R}))$ is completely continuous. Since $\mathscr{F}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph.

Step 4: $\mathscr{F}$ has a closed graph. Let $x_n \to x_*, h_n \in \mathscr{F}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathscr{F}(x_*)$. Associated with $h_n \in \mathscr{F}(x_n)$ there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$h_n(t) = J^q v_n(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v_n(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v_n(s)(\xi_i)\Big).$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$h_*(t) = J^q v_*(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v_*(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v_*(s)(\xi_i)\Big).$$

Consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to$

$C([0,T],\mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = J^q v(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v(s)(\xi_i)\Big).$$

Observe that

$$\|h_n(t) - h_*(t)\| \to 0, \quad \text{as } n \to \infty.$$

Thus it follows by Lemma 6 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have

$$h_*(t) = J^q v_*(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v_*(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v_*(s)(\xi_i)\Big),$$

for some $v_* \in S_{F,x_*}$.

Step 5: there exists an open set $U \subseteq C([0,T],\mathbb{R})$ with $x \notin \lambda\mathscr{F}(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\mathscr{F}(x)$. Then there exists $v \in L^1([0,T],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,T]$,

$$x(t) = \lambda J^q v(s)(t) - \lambda\frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v(s)(\xi_i)\Big).$$

Using the computations of the second step above,

$$\begin{aligned}\|x\| &\leqslant \psi(\|x\|)\|p\|\Big\{\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} \\ &\quad + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i+q/\eta_i+1)}{\Gamma(\gamma_i+q/\eta_i+\delta_i+1)}\Big\} \\ &= \psi(\|x\|)\|p\|\Psi,\end{aligned}$$

which implies that

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Psi} \leqslant 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(I,\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\mathscr{F} : \bar{U} \to \mathscr{P}(C(I,\mathbb{R}))$ is a compact multivalued map and u.s.c. with convex closed values. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\mathscr{F}(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 7), we deduce that $\mathscr{F}$ has a fixed point $x \in \bar{U}$ which is a solution of the problem (1). □

The lower semicontinuous case

In the next result, F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo[24] for lower semicontinuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathscr{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semicontinuous if the set $\{y \in X : G(y) \cap B \neq \varnothing\}$ is open for any open set B in E. Let A be a subset of $[0,T]\times\mathbb{R}$. A is $\mathscr{L}\otimes\mathscr{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathscr{J}\times\mathscr{D}$, where $\mathscr{J}$ is Lebesgue measurable in $[0,T]$ and $\mathscr{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathscr{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v \in \mathscr{A}$ and measurable $\mathscr{J} \subset [0,T] = J$, the function $u\chi_{\mathscr{J}} + v\chi_{J-\mathscr{J}} \in \mathscr{A}$, where $\chi_{\mathscr{J}}$ stands for the characteristic function of $\mathscr{J}$.

Let $F : [0,T]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathscr{F} : C([0,T]\times\mathbb{R}) \to \mathscr{P}(L^1([0,T],\mathbb{R}))$ associated with F by

$$\mathscr{F}(x) = \{w \in L^1([0,T],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for almost all } t \in [0,T]\},$$

which is called the Nemytskii operator associated with F.

Let $F : [0,T]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semicontinuous type if its associated Nemytskii operator $\mathscr{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

Lemma 8 (Ref. 25) *Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator and is lower semicontinuous and has nonempty closed and decomposable values. Then N has a continuous selection, i.e., there exists a continuous (single-valued) function $g : Y \to L^1([0,T],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 2 *Assume that* (H_2), (H_3), *and the following condition holds.*

(H_4) $F : [0,T]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
   (i) $(t,x) \mapsto F(t,x)$ *is $\mathscr{L}\otimes\mathscr{B}$ measurable;*
   (ii) $x \mapsto F(t,x)$ *is lower semicontinuous for each $t \in [0,T]$.*

Then the boundary value problem (1) *has at least one solution on* $[0,T]$.

Proof: It follows from (H_2) and (H_4) that F is of lower semicontinuous type. Then from Lemma 8 there exists a continuous function $f : C^2([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(x) \in \mathscr{F}(x)$ for all $x \in C([0,T],\mathbb{R})$.

Consider the problem

$$\left.\begin{aligned} &D^q x(t) = f(x(t)),\ 0 < t < T,\\ &x(0) = 0, \quad \alpha x(T) = \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} x(\xi_i). \end{aligned}\right\} \tag{8}$$

Observe that if $x \in C^2([0,T],\mathbb{R})$ is a solution of (8), then x is a solution to (1). To transform (8) into a fixed point problem, we define the operator $\bar{\mathscr{F}}$ by

$$\bar{\mathscr{F}} x(t) = J^q f(x(t)) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q f(x(T)) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q f((\xi_i))\Big).$$

It can easily be shown that $\bar{\mathscr{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 1. □

The Lipschitz case

In this subsection we prove the existence of solutions to (1) with a not necessarily nonconvex-valued right-hand side by applying a fixed-point theorem for multivalued maps due to Covitz and Nadler[26].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathscr{P}_{\mathrm{cl,b}}(X), H_d)$ is a metric space[27].

A multivalued operator $N : X \to \mathscr{P}_{\mathrm{cl}}(X)$ is called

(i) γ-Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x),N(y)) \leqslant \gamma d(x,y)$ for each $x,y \in X$;

(ii) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 9 (Ref. 26) *Let (X,d) be a complete metric space. If $N : X \to \mathscr{P}_{\mathrm{cl}}(X)$ is a contraction then* Fix$N \neq$ *∅.*

Theorem 3 *Assume that*

(A_1) *$F : [0,T] \times \mathbb{R} \to \mathscr{P}_{\mathrm{cp}}(\mathbb{R})$ is such that $F(\cdot,x) : [0,T] \to \mathscr{P}_{\mathrm{cp}}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(A_2) *$H_d(F(t,x),F(t,\bar{x})) \leqslant m(t)|x-\bar{x}|$ for almost all $t \in [0,T]$ and $x,\bar{x} \in \mathbb{R}$ with $m \in C([0,T],\mathbb{R}^+)$ and $d(0,F(t,0)) \leqslant m(t)$ for almost all $t \in [0,T]$.*

Then the boundary value problem (1) has at least one solution on $[0,T]$ if $\|m\|\Psi < 1$, i.e.,

$$\|m\|\left\{\frac{T^q}{\Gamma(q+1)} + \frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)} \sum_{i=1}^{m} \frac{|\beta_i|\xi_i^q \Gamma(\gamma_i + q/\eta_i + 1)}{\Gamma(\gamma_i + q/\eta_i + \delta_i + 1)}\right\} < 1.$$

Proof: Consider the operator $\mathscr{F}$ defined by (7). Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,T],\mathbb{R})$ by the assumption (A_1). Thus F has a measurable selection (see Theorem III.6 of Ref. 28). Now we show that the operator $\mathscr{F}$ is closed and a contraction. We show that $\mathscr{F}(x) \in \mathscr{P}_{\mathrm{cl}}((C[0,T],\mathbb{R}))$ for each $x \in C([0,T],\mathbb{R})$. Let $\{u_n\}_{n\geqslant 0} \in \mathscr{F}(x)$ be such that $u_n \to u (n \to \infty)$ in $C([0,T],\mathbb{R})$. Then $u \in C([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,T]$,

$$u_n(t) = J^q v_n(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v_n(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v_n(s)(\xi_i)\Big).$$

As F has compact values, it follows that the sequence v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in [0,T]$ we have

$$u_n(t) \to v(t) = J^q v(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v(s)(\xi_i)\Big).$$

Hence $u \in \mathscr{F}(x)$.

Next we show that there exists $\delta < 1$ ($\delta = \|m\|\Psi$) such that

$$H_d(\mathscr{F}(x), \mathscr{F}(\bar{x})) \leqslant \delta\|x - \bar{x}\|$$

for each $x,\bar{x} \in C^2([0,T],\mathbb{R})$. Let $x,\bar{x} \in C^2([0,T],\mathbb{R})$ and $h_1 \in \mathscr{F}(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,T]$,

$$h_1(t) = J^q v_1(s)(t) - \frac{t^{q-1}}{\Lambda}\Big(\alpha J^q v_1(s)(T) - \sum_{i=1}^{m} \beta_i I_{\eta_i}^{\gamma_i,\delta_i} J^q v_1(s)(\xi_i)\Big).$$

By (A_2) we have

$$H_d(F(t,x),F(t,\bar{x})) \leqslant m(t)|x(t)-\bar{x}(t)|.$$

Hence there exists $w \in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w(t)| \leqslant m(t)|x(t)-\bar{x}(t)|, \quad t\in[0,T].$$

Define $U:[0,T]\to\mathscr{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\leqslant m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t)\cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 of Ref. 28), there exists a function $v_2(t)$ which is a measurable selection for U. Thus $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in[0,T]$ we have $|v_1(t)-v_2(t)|\leqslant m(t)|x(t)-\bar{x}(t)|$. For each $t\in[0,T]$, let us define

$$h_2(t)=J^q v_2(s)(t)-\frac{t^{q-1}}{\Lambda}\Bigl(\alpha J^q v_2(s)(T) - \sum_{i=1}^{m}\beta_i I_{\eta_i}^{\gamma_i,\delta_i}J^q v_2(s)(\xi_i)\Bigr).$$

Thus

$$\begin{aligned}
&|h_1(t)-h_2(t)|\\
&\leqslant J^q|v_1(s)-v_2(s)|(t)+\frac{t^{q-1}}{|\Lambda|}\Bigl(\alpha J^q|v_1(s) - v_2(s)|(T)+\sum_{i=1}^{m}\beta_i I_{\eta_i}^{\gamma_i,\delta_i}J^q|v_1(s)-v_2(s)|(\xi_i)\Bigr)\\
&\leqslant \|m\|\,\|x-\bar{x}\|\Bigl(\frac{T^q}{\Gamma(q+1)}+\frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i+q/\eta_i+1)}{\Gamma(\gamma_i+q/\eta_i+\delta_i+1)}\Bigr).
\end{aligned}$$

Hence

$$\begin{aligned}
&\|h_1-h_2\|\\
&\leqslant \|m\|\,\|x-\bar{x}\|\Bigl(\frac{T^q}{\Gamma(q+1)}+\frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i+q/\eta_i+1)}{\Gamma(\gamma_i+q/\eta_i+\delta_i+1)}\Bigr).
\end{aligned}$$

Analogously, interchanging the roles of x and $\bar{x}$,

$$H_d(\mathscr{F}(x),\mathscr{F}(\bar{x}))\leqslant\delta\|x-\bar{x}\|$$

with $\delta<1$ where

$$\delta=\|m\|\Bigl(\frac{T^q}{\Gamma(q+1)}+\frac{|\alpha|T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)}\sum_{i=1}^{m}\frac{|\beta_i|\xi_i^q\Gamma(\gamma_i+q/\eta_i+1)}{\Gamma(\gamma_i+q/\eta_i+\delta_i+1)}\Bigr).$$

Thus $\mathscr{F}$ is a contraction. Hence it follows by Lemma 9 that $\mathscr{F}$ has a fixed point x which is a solution of (1). □

Examples

We now illustrate our main theorems with the help of three examples. Consider the following nonlocal boundary value problems for fractional differential inclusions with Erdélyi-Kober fractional integral boundary conditions:

$$\left.\begin{aligned}
&D^{3/2}x(t)\in F(t,x(t)),\ t\in(0,2),\ x(0)=0,\\
&\tfrac{2}{3}x(2)=\tfrac{3}{2}I_{\sqrt{2}/3}^{5/4,1/2}x(\tfrac14)\\
&\quad+\frac{\ln 2}{\sqrt{3}}I_{\sqrt{3}/4}^{4/9,3/2}x(\tfrac12)+\tfrac{5}{3}I_{\sqrt{5}/6}^{7/2,5/2}x(\tfrac34)\\
&\quad+\frac{3}{\sqrt{5}}I_{3/8}^{8/3,2/9}x(\tfrac54)+\frac{e}{\pi}I_{4/7}^{3/8,6/\sqrt{7}}x(\tfrac32)\\
&\quad+\frac{\pi}{6}I_{7/9}^{13/11,\sqrt{2}/\sqrt{3}}x(\tfrac74).
\end{aligned}\right\}\quad(9)$$

Here we have $q=\frac32$, $m=6$, $T=2$, $\alpha=\frac23$, $\beta_1=\frac32$, $\beta_2=\ln 2/\sqrt{3}$, $\beta_3=\frac53$, $\beta_4=3/\sqrt{5}$, $\beta_5=e/\pi$, $\beta_6=\pi/6$, $\eta_1=\sqrt{2}/3$, $\eta_2=\sqrt{3}/4$, $\eta_3=\sqrt{5}/6$, $\eta_4=\frac38$, $\eta_5=\frac47$, $\eta_6=\frac79$, $\gamma_1=\frac54$, $\gamma_2=\frac49$, $\gamma_3=\frac72$, $\gamma_4=\frac83$, $\gamma_5=\frac38$, $\gamma_6=\frac{13}{11}$, $\delta_1=\frac12$, $\delta_2=\frac32$, $\delta_3=\frac52$, $\delta_4=\frac29$, $\delta_5=6/\sqrt{7}$, $\delta_6=\sqrt{2}/\sqrt{3}$, $\xi_1=\frac14$, $\xi_2=\frac12$, $\xi_3=\frac34$, $\xi_4=\frac54$, $\xi_5=\frac32$, and $\xi_6=\frac74$. This gives $|\Lambda|=1.030325363$ and $\Psi=6.111985724$.

(a) Let $F:[0,2]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x\to F(t,x) = \left[\frac{\sin^2x+\mathrm{e}^{-|x|}}{10(2+\cos t)},\ \frac{1}{t+3}\Bigl(\frac{x^2}{5(1+|x|)}+3\Bigr)\right].\quad(10)$$

For $f\in F$ we have

$$|f|\leqslant\max\Bigl(\frac{\sin^2x+\mathrm{e}^{-|x|}}{10(2+\cos t)},\ \frac{1}{t+3}\Bigl(\frac{x^2}{5(1+|x|)}+3\Bigr)\Bigr) \leqslant\frac{1}{t+3}\Bigl(\frac{|x|}{5}+3\Bigr),\quad x\in\mathbb{R}.$$

Thus for $x\in\mathbb{R}$,

$$\|F(t,x)\|_{\mathscr{P}}:=\sup\{|y|:y\in F(t,x)\} \leqslant\frac{1}{t+3}\Bigl(\frac{|x|}{5}+3\Bigr)=p(t)\psi(|x|),$$

with $p(t)=1/(t+3)$, $\psi(|x|)=(|x|/5)+3$. Hence there exists a constant $M>10.31499084$ satisfying (H_3). Thus all the conditions of Theorem 1 are satisfied. Hence the problem (9) with $F(t,x)$ given by (10) has at least one solution on $[0,2]$.

(b) Let $F:[0,2]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t,x)=\left[\mathrm{e}^{-x^4}+\frac{t}{3},\frac{|x|}{1+2|x|}+t+\tfrac{3}{2}\right]. \quad (11)$$

For $f\in F$ we have

$$|f|\leqslant \max\left(\mathrm{e}^{-x^4}+\frac{t}{3},\frac{|x|}{1+2|x|}+t+\tfrac{3}{2}\right) \leqslant 2+t, \quad x\in\mathbb{R}.$$

Here,

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y\in F(t,x)\} \leqslant (2+t)=p(t)\psi(\|x\|), \quad x\in\mathbb{R},$$

with $p(t)=2+t$, $\psi(\|x\|)=1$. By computing directly, there exists a constant $M>24.44794290$ satisfying (H_3). Then, by Theorem 2, the problem (9) with $F(t,x)$ given by (11) has at least one solution on $[0,2]$.

(c) Consider the multivalued map $F:[0,2]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ given by

$$x\to F(t,x)=\left[0,\frac{1}{8+t^2}\left(\frac{x^2+2|x|}{2(1+|x|)}+1\right)\right]. \quad (12)$$

Then we have

$$H_d(F(t,x),F(t,\bar{x}))\leqslant \frac{1}{8+t^2}|x-\bar{x}|.$$

Let $m(t)=1/(8+t^2)$. Then $H_d(F(t,x),F(t,\bar{x}))\leqslant m(t)|x-\bar{x}|$ with $d(0,F(t,0)\leqslant m(t)$ and $\|m\|=\frac{1}{8}$. We can show that

$$\|m\|\Psi = 0.7639982155 < 1.$$

Thus all the conditions of Theorem 3 are fulfilled. Hence by the conclusion of Theorem 3, the problem (9) with $F(t,x)$ given by (12) has at least one solution on $[0,2]$.

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