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Reverses and variations of the Young inequality

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ABSTRACT: We extend the range of the weighted operator means for $\nu \notin [0,1]$ and obtain some corresponding operator inequalities. We also present several reversed Young-type inequalities.

KEYWORDS: weighted operator, positive operator, binary operation, Hilbert-Schmidt norm, Young-type inequality

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INTRODUCTION

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H equipped with the operator norm, $S(H)$ the set of all bounded self-adjoint operators, and $\mathbb{P} = \mathbb{P}(\mathrm{H})$ the open convex cone of all positive invertible operators. For $X, Y \in S(H)$, we write $X \leqslant Y$ if $Y - X$ is positive, and $X < Y$ if $Y - X$ is positive invertible.

The unitarily invariant norm $\|\cdot\|$ is defined on the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. For $A = (a_{ij}) \in \mathbb{M}_n$, the Hilbert-Schmidt norm of A is defined by $\|A\|_2 = (\Sigma_{j=1}^n s_j^2(A))^{1/2}$, where $s_1(A), s_2(A), \dots, s_n(A)$ are the singular values of A, i.e., the eigenvalues of the positive matrix $|A| = (A^*A)^{1/2}$ where $A^* = (\overline{A})^{\mathrm{T}})$, arranged in decreasing order and repeated according to multiplicity. It is known that the Hilbert-Schmidt norm is unitarily invariant.

Let a, $b > 0$ be two positive real numbers and $\nu \in [0,1]$. The ν-weighted arithmetic and geometric means of a and b, denoted by $A_\nu(a,b)$ and $G_\nu(a,b)$, respectively, are defined as

$$A_\nu(a,b) = (1-\nu)a + \nu b, \quad G_\nu(a,b) = a^{1-\nu} b^\nu.$$

Note that $A_\nu(a,b) \geqslant G_\nu(a,b)$ for all $\nu \in [0,1]$. This is the well-known Young inequality. In particular, if $\nu = \frac{1}{2}$ then $A_{1/2}(a,b) = \frac{1}{2}(a+b)$ and $G_{1/2}(a,b) = \sqrt{ab}$ are the arithmetic and geometric means, respectively. The Heinz mean of a and b is defined as

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}$$

for $\nu \in [0,1]$. For $\nu = 0, 1$, this is equal to arithmetic mean and for $\nu = \frac{1}{2}$ it is the geometric mean.

Let $A, B \in B(H)$ be two positive operators and $\nu \in [0,1]$. The ν-weighted arithmetic mean of A and B, denoted by $A\nabla_\nu B$, is defined as

$$A\nabla_\nu B = (1-\nu)A + \nu B.$$

If A is invertible, the ν-weighted geometric mean of A and B, denoted by $A\sharp_\nu B$, is defined as

$$A\sharp_\nu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2}.$$

For more details, see Ref. 1. When $\nu = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The operator version of the Heinz mean, denoted by $H_\nu(A,B)$, is defined as

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2}, \qquad 0 \leqslant \nu \leqslant 1.$$

It is well known that if A and B are positive invertible operators, then

$$A\nabla_\nu B \geqslant A\sharp_\nu B, \qquad 0 \leqslant \nu \leqslant 1.$$

The Specht ratio[2,3] is defined by

$$S(t) = \frac{t^{1/(t-1)}}{e\log t^{1/(t-1)}} \text{ for } t > 0, t \neq 1,$$

and

$$S(1) = \lim_{t\to 1} S(t) = 1.$$

Furuichi[4] gave the following refined version:

$$A\nabla_\nu B \geqslant S(h^r) A\sharp_\nu B \geqslant A\sharp_\nu B,$$

where $r = \min\{\nu, 1-\nu\}$. Zuo et al[5] gave another one:

$$K(h,2)^r A\sharp_\nu B \leqslant A\nabla_\nu B,$$

where $K(t,2)=(t+1)^2/4t$ for $t>0$ is the Kantorovich constant. In Ref. 6, Furuichi gave another refined version:

$$A\nabla_\nu B \geqslant A\sharp_\nu B + 2r(A\nabla B - A\sharp B) \geqslant A\sharp_\nu B.$$

Recently there have been a number of other studies on similar topics and various improvement versions[7–11].

The Heinz norm inequality, which is one of the essential inequalities in operator theory, states that for any positive operators $A,B\in M_n$, any operator $X\in M_n$ and $\nu\in[0,1]$, the following double inequality holds:

$$2\left\|A^{1/2}XB^{1/2}\right\| \leqslant \left\|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right\| \leqslant \|AX+XB\|. \quad (1)$$

Kittaneh and Manasrah[12] showed a refinement of the right-hand side of inequality (1) for the Hilbert-Schmidt norm as follows:

$$\left\|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right\|_2^2 + 2r_0\|AX-XB\|_2^2 \leqslant \|AX+XB\|_2^2, \quad (2)$$

in which $A,B,X\in M_n$ such that A, B are positive semidefinite, $\nu\in[0,1]$ and $r_0=\min\{\nu,1-\nu\}$. Kaur et al[13], using the convexity of the function $f(\nu)=\left|\left|\left|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right|\right|\right|$ with $\nu\in[0,1]$, presented more refinements of the Heinz inequality.

It was shown in Ref. 14 that a reverse of inequality (2) is

$$\|AX+XB\|_2^2 \leqslant \left\|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right\|_2^2 + 2r_0\|AX-XB\|_2^2, \quad (3)$$

where $A,B,X\in M_n$ such that A, B are positive semidefinite, $\nu\in[0,1]$, and $r_0=\max\{\nu,1-\nu\}$.

In this paper, we extend the range of the weighted operator means for $\nu\notin[0,1]$ and obtain some corresponding operator inequalities. We also present a reverse of (2) and some other operator inequalities.

SOME OPERATOR INEQUALITIES FOR $\nu\notin[0,1]$

For $A,B\in\mathbb{P}$ and $\nu\in[0,1]$, the ν-weighted geometric operator mean is defined as

$$A\sharp_\nu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2}.$$

For convenience, we use the notation $\natural_\nu$ and $H_\nu^\natural$ for the binary operation

$$A\natural_\nu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2},$$
$$H_\nu^\natural(A,B) = \frac{A\natural_\nu B + A\natural_{1-\nu}B}{2},$$

for $\nu\notin[0,1]$. We use the notation $\diamondsuit_\nu$ and $H_\nu^\diamondsuit$ for the binary operation

$$A\diamondsuit_\nu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2},$$
$$H_\nu^\diamondsuit(A,B) = \frac{A\natural_\nu B + A\natural_{1-\nu}B}{2},$$

for $\nu\notin[\frac{1}{2},1]$, whose formulae are the same as $\sharp_\nu$ and $H_\nu(A,B)$. Note that $A\sharp_\nu B$ for $\nu\in[0,1]$ is monotonic, but $A\natural_\nu B$ and $A\diamondsuit_\nu B$ are not.

In this section, we extend the range of the definition of the weighted operator. We also present some operator inequalities for $\nu\notin[0,1]$ and $\nu\notin[\frac{1}{2},1]$. To obtain the results, we need the following lemmas.

Lemma 1 (Ref. 15) *Let $X\in B(H)$ be self-adjoint and let f and g be continuous real functions such that $f(t)\geqslant g(t)$ for all $t\in \mathrm{Sp}(X)$ (the spectrum of X). Then $f(X)\geqslant g(X)$.*

Lemma 2 (Ref. 16) *Let $a,\ b>0$ and $\nu\notin[0,1]$. Then,*

(i)

$$\nu a+(1-\nu)b+(\nu-1)(\sqrt{a}-\sqrt{b})^2 \leqslant a^\nu b^{1-\nu},$$

(ii)

$$(a+b)+2(\nu-1)(\sqrt{a}-\sqrt{b})^2 \leqslant a^\nu b^{1-\nu}+b^\nu a^{1-\nu},$$

(iii)

$$(a+b)^2+2(\nu-1)(a-b)^2 \leqslant (a^\nu b^{1-\nu}+b^\nu a^{1-\nu})^2.$$

Proof: Let $a,b>0$ and $\nu\notin[0,1]$.
(i) Assume that $f(t)=t^{1-\nu}-\nu+(\nu-1)t$ with $t\in(0,\infty)$. It is easy to see that $f(t)$ has a minimum at $t=1$ in the interval $(0,\infty)$. Hence $f(t)\geqslant f(1)=0$ for all $t>0$. Assume that $a,b>0$. Letting $t=b/a$, we get

$$\nu a+(1-\nu)b \leqslant a^\nu b^{1-\nu}.$$

So we have

$$\begin{aligned}&\nu a+(1-\nu)b+(\nu-1)(\sqrt{a}-\sqrt{b})^2\\ &=(2-2\nu)\sqrt{ab}+(2\nu-1)a\\ &\leqslant(\sqrt{ab})^{2-2\nu}a^{2\nu-1}=a^\nu b^{1-\nu}.\end{aligned}$$

(ii) It can be proved in a similar fashion to (i).
(iii) It follows from (ii) by replacing a by a^2 and b by b^2.

□

Theorem 1 *Let $A, B \in \mathbb{P}$ and $\nu \notin [0,1]$. Then:*

$$\nu A + (1-\nu)B + 2(\nu-1)(A\nabla B - A\sharp B) \leqslant A\natural_{1-\nu}B.$$

Proof: By Lemma 2(i), we have

$$\nu + (1-\nu)b + (\nu-1)(1-\sqrt{b})^2 \leqslant b^{1-\nu},$$

for any $b > 0$. If $X = A^{-1/2}BA^{-1/2}$ and thus $\mathrm{Sp}(X) \subseteq (0, +\infty)$, then we have

$$\nu + (1-\nu)t + (\nu-1)(1-\sqrt{t})^2 \leqslant t^{1-\nu},$$

for any $t \in \mathrm{Sp}(X)$. This is the same as

$$\nu I + (1-\nu)X + (\nu-1)(I - X^{1/2})^2 \leqslant X^{1-\nu}. \quad (4)$$

Multiplying both sides of (4) by $A^{1/2}$, we get

$$\nu A + (1-\nu)B + (\nu-1)(A + B - 2A^{1/2}X^{1/2}A^{1/2}) \leqslant A^{1/2}X^{1-\nu}A^{1/2}. \quad (5)$$

If $\nu \notin [0,1]$, then

$$\nu A + (1-\nu)B + 2(\nu-1)(A\nabla B - A\sharp B) \leqslant A\natural_{1-\nu}B.$$

□

Remark 1 In Ref. 12, the authors showed that if $\nu \in (0, \frac{1}{2})$, then

$$\nu A + (1-\nu)B + 2(\nu-1)(A\nabla B - A\sharp B) \leqslant A\sharp_{1-\nu}B.$$

It is the same version of the formula (5). Hence for all $\nu \notin [\frac{1}{2}, 1]$,

$$\nu A + (1-\nu)B + 2(\nu-1)(A\nabla B - A\sharp B) \leqslant A\diamondsuit_{1-\nu}B$$

holds.

Remark 2 If $A, B \in \mathbb{P}$ and $B \geqslant A$, $\nu \in (1,2)$, then by the monotonicity of $\sharp_\nu$ and $0 < \nu - 1 < 1$, $B^{-1} \leqslant A^{-1}$,

$$\begin{aligned} \nu A + (1-\nu)B + 2(\nu-1)(A\nabla B - A\sharp B) &\leqslant A\natural_{1-\nu}B \\ &= A^{1/2}(A^{-1/2}BA^{-1/2})^{1-\nu}A^{1/2} \\ &= A^{1/2}(A^{1/2}B^{-1}A^{1/2})^{\nu-1}A^{1/2} \\ &\leqslant A^{1/2}(A^{1/2}A^{-1}A^{1/2})^{\nu-1}A^{1/2} = A. \end{aligned}$$

This is the same as

$$0 \leqslant A\nabla B - A\sharp B \leqslant \frac{B-A}{2}.$$

By Lemma 2 (ii), (iii) and using the same processing technique as in Theorem 1, we can get the following theorems and the corresponding remarks.

Theorem 2 *Let $A, B \in \mathbb{P}$ and $\nu \notin [0,1]$. Then*

$$A\nabla B + 2(\nu-1)(A\nabla B - A\sharp B) \leqslant H_\nu^\natural(A,B).$$

Remark 3 In Ref. 14, the authors showed that if $\nu \in (0, \frac{1}{2})$, then

$$A\nabla B + 2(\nu-1)(A\nabla B - A\sharp B) \leqslant H_\nu(A,B).$$

Hence for all $\nu \notin [\frac{1}{2}, 1]$,

$$A\nabla B + 2(\nu-1)(A\nabla B - A\sharp B) \leqslant H_\nu^\diamondsuit(A,B)$$

holds.

Remark 4 If $A, B \in \mathbb{P}$ and $B \geqslant A$, $\nu \in (1,2)$, then

$$B + 4(\nu-1)(A\nabla B - A\sharp B) \leqslant A\natural_\nu B.$$

Theorem 3 *Let $A, B \in \mathbb{P}$ and $\nu \notin [0,1]$. Then*

$$(2\nu-1)(A + A\natural_2 B) - 4(\nu-1)B \leqslant A\natural_{2-2\nu}B + A\natural_{2\nu}B.$$

Remark 5 If $A, B \in \mathbb{P}$ and $B \geqslant A$, $\nu \in (1,2)$, then

$$2(\nu-1)(A-2B) + (2\nu-1)A\natural_2 B \leqslant A\natural_{2\nu}B.$$

A REVERSE OF THE HEINZ INEQUALITY FOR MATRICES

In this section, we present a reverse of the Heinz inequality for matrices. To obtain the result, we need the following lemma.

Lemma 3 (Ref. 17) *Let a, $b > 0$. If $0 \leqslant \nu \leqslant \frac{1}{2}$, then*

$$\nu^2 a + (1-\nu)^2 b \leqslant (1-\nu)^2(\sqrt{a} - \sqrt{b})^2 + a^\nu[(1-\nu)^2 b]^{1-\nu}. \quad (6)$$

If $\frac{1}{2} \leqslant \nu \leqslant 1$, then

$$\nu^2 a + (1-\nu)^2 b \leqslant \nu^2(\sqrt{a} - \sqrt{b})^2 + (\nu^2 a)^\nu b^{1-\nu}. \quad (7)$$

Based on Lemma 3, the following corollaries can be easily obtained.

Corollary 1 *Let a, $b > 0$. If $0 \leqslant \nu \leqslant \frac{1}{2}$, then*

$$2\nu(a+b) \leqslant 2(1-\nu)(\sqrt{a} - \sqrt{b})^2 + (1-\nu)^{1-2\nu}[a^\nu b^{1-\nu} + b^\nu a^{1-\nu}]. \quad (8)$$

If $\frac{1}{2} \leqslant \nu \leqslant 1$, then

$$2(1-\nu)(a+b) \leqslant 2\nu(\sqrt{a} - \sqrt{b})^2 + \nu^{2\nu-1}[a^\nu b^{1-\nu} + b^\nu a^{1-\nu}]. \quad (9)$$

Corollary 2 *Let $a,\ b>0$. If $0\leqslant\nu\leqslant\frac{1}{2}$, then*

$$2\nu(a+b)^2\leqslant 2(1-\nu)(a-b)^2 + (1-\nu)^{1-2\nu}(a^\nu b^{1-\nu}+b^\nu a^{1-\nu})^2. \quad (10)$$

If $\frac{1}{2}\leqslant\nu\leqslant 1$, then

$$2(1-\nu)(a+b)^2\leqslant 2\nu(a-b)^2 + \nu^{2\nu-1}(a^\nu b^{1-\nu}+b^\nu a^{1-\nu})^2. \quad (11)$$

Theorem 4 *Let $A,B,X\in\mathbb{M}_n$ with A, B are positive, and $\nu\in[0,1]$. Then*

$$2\nu\,\|AX+XB\|_2^2\leqslant 2(1-\nu)\,\|AX-XB\|_2^2 + (1-\nu)^{1-2\nu}\left\|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right\|_2^2$$

for $0\leqslant\nu\leqslant\frac{1}{2}$, and

$$2(1-\nu)\,\|AX+XB\|_2^2\leqslant 2\nu\,\|AX-XB\|_2^2 + \nu^{2\nu-1}\left\|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right\|_2^2$$

for $\frac{1}{2}\leqslant\nu\leqslant 1$.

Proof: By spectral decomposition, there are unitary matrices $U,V\in\mathbb{M}_n$ such that $A=U\Lambda_1U^*$ and $B=V\Lambda_2V^*$, where

$$\Lambda_1=\operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)$$

and

$$\Lambda_2=\operatorname{diag}(\mu_1,\mu_2,\dots,\mu_n)$$

where λ_i and μ_i for $i=1,2,\dots,n$ are the eigenvalues of A and B, respectively. Let $Y=U^*XV=[y_{ij}]$, then

$$\begin{aligned}AX+XB&=U(\Lambda_1Y+Y\Lambda_2)V^*\\&=U[(\lambda_i+\mu_i)y_{ij}]V^*,\\AX-XB&=U(\Lambda_1Y-Y\Lambda_2)V^*\\&=U[(\lambda_i-\mu_i)y_{ij}]V^*,\\A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu&\\&=U\Lambda_1^\nu U^*XV\Lambda_2^{1-\nu}V^*+U\Lambda_1^{1-\nu}U^*XV\Lambda_2^\nu V^*\\&=U\Lambda_1^\nu Y\Lambda_2^{1-\nu}V^*+U\Lambda_1^{1-\nu}Y\Lambda_2^\nu V^*\\&=U\left[\Lambda_1^\nu Y\Lambda_2^{1-\nu}+\Lambda_1^{1-\nu}Y\Lambda_2^\nu\right]V^*\\&=U\left[\left(\lambda_i^\nu\mu_i^{1-\nu}+\lambda_i^{1-\nu}\mu_i^\nu\right)y_{ij}\right]V^*.\end{aligned}$$

If $0\leqslant\nu\leqslant\frac{1}{2}$, then by (10) and the unitary invariance of the Hilbert-Schmidt norm, we have

$$\begin{aligned}2\nu\,\|AX+XB\|_2^2&=2\nu\sum_{i,j=1}^{n}(\lambda_i+\mu_i)^2\left|y_{ij}\right|^2\\&\leqslant 2(1-\nu)\sum_{i,j=1}^{n}(\lambda_i-\mu_i)^2\left|y_{ij}\right|^2\\&\quad+(1-\nu)^{1-2\nu}\sum_{i,j=1}^{n}(\lambda_i^\nu\mu_i^{1-\nu}+\lambda_i^{1-\nu}\mu_i^\nu)^2\left|y_{ij}\right|^2\\&=2(1-\nu)\,\|AX-XB\|_2^2\\&\quad+(1-\nu)^{1-2\nu}\left\|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right\|_2^2.\end{aligned}$$

If $\frac{1}{2}\leqslant\nu\leqslant 1$, then by (11) and using the same technique in the first part we get the other result. □

SOME REVERSES OF THE YOUNG-TYPE INEQUALITY FOR OPERATORS

In this section, we obtain some reverses of the Young-type inequality for two positive invertible operators.

Theorem 5 *Let $A,B\in\mathbb{P}$ and $\nu\in[0,1]$. Then*

$$\nu^2A+(1-\nu)^2B\leqslant 2(\nu-1)^2(A\nabla B-A\sharp B) + (1-\nu)^{2(1-\nu)}A\sharp_{1-\nu}B,$$

for $0\leqslant\nu\leqslant\frac{1}{2}$, and

$$\nu^2A+(1-\nu)^2B\leqslant 2\nu^2(A\nabla B-A\sharp B)+\nu^{2\nu}A\sharp_{1-\nu}B,$$

for $\frac{1}{2}\leqslant\nu\leqslant 1$.

Proof: For $0\leqslant\nu\leqslant\frac{1}{2}$, by (6) we have

$$\nu^2a+(1-\nu)^2b\leqslant(1-\nu)^2(\sqrt{a}-\sqrt{b})^2+a^\nu[(1-\nu)^2b]^{1-\nu},$$

for any $b>0$. If $X=A^{-1/2}BA^{-1/2}$ and thus $\operatorname{Sp}(X)\subseteq(0,+\infty)$, then we have

$$\nu^2+(1-\nu)^2b\leqslant(1-\nu)^2(1-\sqrt{b})^2+[(1-\nu)^2b]^{1-\nu},$$

for any $t\in\operatorname{Sp}(X)$. This is the same as

$$\nu^2I+(1-\nu)^2X\leqslant(1-\nu)^2(I-X^{1/2})^2+[(1-\nu)^2X]^{1-\nu}. \quad (12)$$

Multiplying both sides of (12) by $A^{1/2}$, we get

$$\nu^2A+(1-\nu)^2B\leqslant 2(\nu-1)^2(A\nabla B-A\sharp B) + (1-\nu)^{2(1-\nu)}A\sharp_{1-\nu}B.$$

□

Theorem 6 *Let* $A, B \in \mathbb{P}$ *and* $\nu \in [0,1]$. *Then*

$$2\nu A\nabla B \leqslant 2(1-\nu)(A\nabla B - A\sharp B) + (1-\nu)^{1-2\nu} H_\nu(A,B),$$

for $0 \leqslant \nu \leqslant \frac{1}{2}$, *and*

$$2(1-\nu)A\nabla B \leqslant 2\nu(A\nabla B - A\sharp B) + \nu^{2\nu-1} H_\nu(A,B),$$

for $\frac{1}{2} \leqslant \nu \leqslant 1$.

Proof: By Corollary 2 and the same processing technique as in Theorem 5, we can easily obtain the result. □

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