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A generalized statistical convergence in intuitionistic fuzzy normed spaces

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ABSTRACT: In this paper, we introduce the notion of $\mathscr{I}$-$[V,\lambda]$-summability and $\mathscr{I}$-λ-statistical convergence with respect to the intuitionistic fuzzy norm (μ,ν), investigate their relationship, and make some observations about these classes. We mainly examine the relation between these two new methods and the relation between $\mathscr{I}$-λ-statistical convergence and $\mathscr{I}$-statistical convergence in the corresponding intuitionistic fuzzy normed space.

KEYWORDS: ideal, filter, $\mathscr{I}$-statistical convergence, $\mathscr{I}$-λ-statistical convergence, $\mathscr{I}$-$[V,\lambda]$-summability

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INTRODUCTION AND PRELIMINARIES

The idea of convergence of a real sequence has been extended to statistical convergence by Fast[1] as follows: let K be a subset of $\mathbb{N} \equiv \{1,2,\dots\}$. Then the asymptotic density of K is defined by $\delta(K) := \lim_{n\to\infty}(1/n)|\{k \leqslant n : k \in K\}|$, where $|S|$ denotes the cardinality of the set S. A number sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0, \delta(\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}) = 0$. If $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to L we write $st\text{-}\lim x_k = L$. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy[2] and Šalát[3].

In Ref. 4, Kostyrko et al introduced the concept of $\mathscr{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathscr{I}$-convergence is an interesting generalization of statistical convergence. More investigations in this direction and more applications of ideals can be found in Refs. 5–7. In another direction, the idea of λ-statistical convergence was introduced and studied by Mursaleen[8] as an extension of the $[V,\lambda]$-summability of Leindler[9]. λ-statistical convergence is a special case of the more general A-statistical convergence studied in Ref. 10.

Following the introduction of fuzzy set theory by Zadeh[11], there has been extensive research to find applications and fuzzy analogues of the classical theories. The theory of intuitionistic fuzzy sets was introduced by Atanassov[12]; it has been extensively used in decision-making problems[13]. The concept of an intuitionistic fuzzy metric space was introduced in Ref. 14. Saadati and Park[15] introduced the notion of an intuitionistic fuzzy normed space. Some work related to the convergence of sequences in several normed linear spaces in a fuzzy setting can be found in Refs. 16–19.

Here we intend to unify these two approaches and use ideals to introduce the concept of $\mathscr{I}-\lambda$-statistical convergence with respect to the intuitionistic fuzzy norm (μ,ν), and investigate some of its consequences.

Definition 1 [Ref. 20] A triangular norm (t-norm) is a continuous mapping $* : [0,1]\times[0,1] \to [0,1]$ such that $(S,*)$ is an abelian monoid with unit one and $c*d \leqslant a*b$ if $c \leqslant a$ and $d \leqslant b$ for all $a,b,c,d \in [0,1]$.

Definition 2 [Ref. 20] A binary operation $\diamond : [0,1]\times[0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:
(i) $\diamond$ is associative and commutative,
(ii) $\diamond$ is continuous,
(iii) $a\diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a\diamond b \leqslant c\diamond d$ whenever $a \leqslant c$ and $b \leqslant d$ for each $a,b,c,d \in [0,1]$.

For example, we can give $a*b = ab$, $a*b = \min\{a,b\}$, $a\diamond b = \min\{a+b,1\}$ and $a\diamond b = \max\{a,b\}$ for all $a,b \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park[15] has recently introduced the concept of intuitionistic fuzzy normed space as follows.

Definition 3 [Ref. 15] The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$, and $s, t > 0$:
(i) $\mu(x,t) + \nu(x,t) \leqslant 1$;
(ii) $\mu(x,t) > 0$;
(iii) $\mu(x,t) = 1$ if and only if $x = 0$;
(iv) $\mu(\alpha x, t) = \mu(x, t/|\alpha|)$ for each $\alpha \neq 0$;
(v) $\mu(x,t) * \mu(y,s) \leqslant \mu(x+y, t+s)$;
(vi) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$;
(viii) $\nu(x,t) < 1$;
(ix) $\nu(x,t) = 0$ if and only if $x = 0$;
(x) $\nu(\alpha x, t) = \mu(x, t/|\alpha|)$ for each $\alpha \neq 0$;
(xi) $\nu(x,t) \diamond \nu(y,s) \geqslant \nu(x+y, t+s)$;
(xii) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(xiii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm. As a standard example, we can give the following. Let $(X, \|\cdot\|)$ be a normed space, and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \frac{t}{t + \|x\|} \text{ and } \nu(x,t) = \frac{\|x\|}{t + \|x\|}.$$

Then observe that $(X, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

Definition 4 [Ref. 15] Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = \{x_k\}$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \varepsilon$ and $\nu(x_k - L, t) < \varepsilon$ for all $k \geqslant k_0$. It is denoted by

$$(\mu,\nu)\text{-}\lim x = L \text{ or } x_k \overset{(\mu,\nu)}{\to} L$$

as $k \to \infty$.

$\mathscr{I}$-λ-STATISTICAL CONVERGENCE ON IFNS

In this section we deal with the relation between these two new methods and with relations between $\mathscr{I}$-λ-statistical convergence and $\mathscr{I}$-statistical convergence introduced by the authors recently in an intuitionistic fuzzy normed space. Before proceeding further, we should recall some notation for $\mathscr{I}$-statistical convergence and ideal convergence.

The family $\mathscr{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (i) $\varnothing \notin \mathscr{I}$; (ii) $A, B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$; (iii) $A \in \mathscr{I}$, $B \subset A$ imply $B \in \mathscr{I}$, while an admissible ideal $\mathscr{I}$ of Y further satisfies $\{x\} \in \mathscr{I}$ for each $x \in Y$. If $\mathscr{I}$ is an ideal in Y then the collection $F(\mathscr{I}) = \{M \subset Y : M^c \in \mathscr{I}\}$ forms a filter in Y which is called the filter associated with $\mathscr{I}$. Let $\mathscr{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be $\mathscr{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geqslant \varepsilon\}$ belongs to $\mathscr{I}$ (see Ref. 4).

Definition 5 [Refs. 6, 7] A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathscr{I}$-statistically convergent to L or $S(I)$-convergent to L if, for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leqslant n : \|x_k - L\| \geqslant \varepsilon\}| \geqslant \delta \right\} \in \mathscr{I}$$

or equivalently if for each $\varepsilon > 0$

$$\delta_{\mathscr{I}}(A(\varepsilon)) = \mathscr{I} - \lim \delta_n(A(\varepsilon)) = 0,$$

where $A(\varepsilon) = \{k \leqslant n : \|x_k - L\| \geqslant \varepsilon\}$ and $\delta_n(A(\varepsilon)) = |A(\varepsilon)|/n$.

In this case we write $x_k \to L(S(\mathscr{I}))$. The class of all $\mathscr{I}$-statistically convergent sequences will be denoted simply by $S(\mathscr{I})$. Let $\mathscr{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathscr{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathscr{I}$-statistical convergence is the statistical convergence.

Definition 6 [Ref. 21] Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be $\mathscr{I}$-statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy normed space and is denoted by

$$x_k \overset{(\mu,\nu)}{\to} L(S^{(\mu,\nu)}(\mathscr{I})),$$

if for every $\varepsilon > 0$, and every $\delta > 0$ and $t > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leqslant n : \mu(x_k - L, t) \leqslant 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geqslant \varepsilon\}| \geqslant \delta \right\} \in \mathscr{I}.$$

Let $\mathscr{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathscr{I}_f$ is an admissible ideal in $\mathbb{N}$, and $\mathscr{I}$-statistical convergence coincides with the notion of statistical convergence introduced in Ref. 22.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leqslant \lambda_n + 1$, $\lambda_1 = 1$. The collection of such a sequence λ will be denoted by Δ.

The generalized de Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k,$$

where $I_n = [n-\lambda_n+1, n]$. We are now ready to obtain our main results.

Definition 7 Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. A sequence $x=(x_k)$ is said to be $\mathscr{I}$-$[V,\lambda]$-summable to $L\in X$ with respect to the intuitionistic fuzzy norm (μ,ν) and is denoted by $\mathscr{I}$-$[V,\lambda]^{(\mu,\nu)}$-$\lim x = L$, if for any $\delta>0$ and $t>0$,

$$\{n\in\mathbb{N} : \mu(t_n(x)-L,t)\leqslant 1-\delta \text{ or } \nu(t_n(x)-L,t)\geqslant \delta\}\in\mathscr{I}.$$

Definition 8 A sequence $x=(x_k)$ is said to be $\mathscr{I}$-λ-statistically convergent or $\mathscr{I}$-S_λ convergent to L with respect to the intuitionistic fuzzy norm (μ,ν), and denoted by $\mathscr{I}$-$S_\lambda^{(\mu,\nu)}$-$\lim x = L$ or $x_k\to L(\mathscr{I}$-$S_\lambda^{(\mu,\nu)})$, if for every $\varepsilon>0,\delta>0$ and $t>0$,

$$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}|\{k\in I_n : \mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}|\geqslant\delta\right\}\in\mathscr{I}.$$

Let $\mathscr{I}_{\mathrm{f}}$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathscr{I}_{\mathrm{f}}$ is an admissible ideal in $\mathbb{N}$ and $\mathscr{I}$-λ-statistical convergence is the λ-statistical convergence introduced in Ref. 8.

We shall denote by $S^{(\mu,\nu)}(\mathscr{I})$, $S_\lambda^{(\mu,\nu)}(\mathscr{I})$ and $[V,\lambda]^{(\mu,\nu)}(\mathscr{I})$ the collections of all $\mathscr{I}$-statistically convergent, $\mathscr{I}-S_\lambda^{(\mu,\nu)}$-convergent and $\mathscr{I}$-$[V,\lambda]^{(\mu,\nu)}$-convergent sequences, respectively.

Theorem 1 *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS, and let $\lambda=(\lambda_n)$ be a sequence in Δ.*

(i) *If $x_n\to L[V,\lambda]^{(\mu,\nu)}(\mathscr{I})$ then $x_k\to L(S_\lambda^{(\mu,\nu)}(\mathscr{I}))$.*

(ii) *If $x\in m(X)$, the space of all bounded sequences of X and $x_k\to L(S_\lambda^{(\mu,\nu)}(\mathscr{I}))$ then $x_k\to L[V,\lambda]^{(\mu,\nu)}(\mathscr{I})$.*

(iii) $S_\lambda^{(\mu,\nu)}(\mathscr{I})\cap m(X) = [V,\lambda]^{(\mu,\nu)}(\mathscr{I})\cap m(X)$.

Proof: (i) By hypothesis, for every $\varepsilon>0,\delta>0$ and $t>0$, let $x_k\to L[V,\lambda]^{(\mu,\nu)}(\mathscr{I})$. We have

$$\begin{aligned}\sum_{k\in I_n}&(\mu(x_k-L,t) \text{ or } \nu(x_k-L,t))\\ &\geqslant \sum_{\substack{k\in I_n\ \&\ \mu(x_k-L,t)<1-\varepsilon\\ \text{or } \nu(x_k-L,t)>\varepsilon}}(\mu(x_k-L,t) \text{ or } \nu(x_k-L,t))\\ &\geqslant \varepsilon\,|\{k\in I_r : \mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}|.\end{aligned}$$

Then observe that

$$\begin{aligned}&\frac{1}{\lambda_n}|\{k\in I_n : \mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}|\geqslant\delta\\ &\Rightarrow \frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k-L,t)\leqslant(1-\varepsilon)\delta \text{ or } \frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k-L,t)\geqslant\varepsilon\delta,\end{aligned}$$

which implies

$$\begin{aligned}&\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}|\{k\in I_n : \mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}|\geqslant\delta\right\}\\ &\subset\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}\left\{\sum_{k\in I_n}\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \sum_{k\in I_n}\nu(x_k-L,t)\geqslant\varepsilon\right\}\geqslant\varepsilon\delta\right\}.\end{aligned}$$

Since $x_k\to L[V,\lambda]^{(\mu,\nu)}(\mathscr{I})$, we immediately see that $x_k\to L(S_\lambda^{(\mu,\nu)})$, whence the result.

(ii) We assume that $x_k\to L(S_\lambda^{(\mu,\nu)}(\mathscr{I}))$ and $x\in l_\infty^{(\mu,\nu)}$. The inequalities $\mu(x_k-L,t)\geqslant 1-M$ or $\nu(x_k-L,t)\leqslant M$ hold for all k. Let $\varepsilon>0$ be given. Then we have

$$\begin{aligned}&\frac{1}{\lambda_n}\sum_{k\in I_n}(\mu(x_k-L,t) \text{ or } \nu(x_k-L,t))\\ &=\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\ \&\ \mu(x_k-L,t)\leqslant 1-\varepsilon\\ \nu(x_k-L,t)\geqslant\varepsilon}}(\mu(x_k-L,t)\text{or } \nu(x_k-L,t))\\ &\quad+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\ \&\ \mu(x_k-L,t)>1-\varepsilon\\ \nu(x_k-L,t)<\varepsilon}}(\mu(x_k-L,t)\text{or } \nu(x_k-L,t))\\ &\leqslant\frac{M}{\lambda_n}|\{k\in I_n : \mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}|+\varepsilon.\end{aligned}$$

Note that

$$A_{\mu,\nu}(\varepsilon,t)=\left\{n\in\mathbb{N} : \frac{1}{\lambda_n}|\{k\in I_n : \mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}|\geqslant\frac{\varepsilon}{M}\right\}$$

belong to $\mathscr{I}$. If $n \in (A_{\mu,\nu}(\varepsilon,t))^c$ then we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k-L,t) > 1-2\varepsilon \text{ or}$$
$$\frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k-L,t) < 2\varepsilon.$$

Now

$$T_{\mu,\nu}(\varepsilon,t)=\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}\mu(x_k-L,t)\leqslant 1-2\varepsilon \text{ or } \frac{1}{\lambda_n}\sum_{k\in I_n}\nu(x_k-L,t)\geqslant 2\varepsilon\right\}.$$

Hence $T_{\mu,\nu}(\varepsilon,t)\subset A_{\mu,\nu}(\varepsilon,t)$ and so, by definition of an ideal, $T_{\mu,\nu}(\varepsilon,t)\in\mathscr{I}$. Hence we conclude that $x_k\to L[V,\lambda]^{(\mu,\nu)}(\mathscr{I})$. (iii) This readily follows from (i) and (ii). □

Theorem 2
(i) $S^{(\mu,\nu)}(\mathscr{I})\subset S_\lambda^{(\mu,\nu)}(\mathscr{I})$ *if* $\liminf_{n\to\infty}\lambda_n/n>0$.
(ii) *If* $\liminf_{n\to\infty}\lambda_n/n=0$, *$\mathscr{I}$-strongly (by which we mean that $\exists$ a subsequence $(n(j))_{j=1}^{\infty}$, for which $(\lambda_{n(j)}/n(j))(1/j)\forall j$ and $\{n(j)\ :\ j\in\mathbb{N}\}\notin\mathscr{I}$) then* $S^{(\mu,\nu)}(\mathscr{I})\subsetneqq S_\lambda^{(\mu,\nu)}(\mathscr{I})$.

Proof: (i) For given $\varepsilon>0$ and every $t>0$, we have

$$\begin{aligned}
&\frac{1}{n}\left|\{k\leqslant n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\\
&\geqslant \frac{1}{n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\\
&=\frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|.
\end{aligned}$$

If $\liminf_{n\to\infty}\lambda_n/n=\alpha$ then from the definition $\{n\in\mathbb{N}:\lambda_n/n<\frac{1}{2}\alpha\}$ is finite. For every $\delta>0$,

$$\begin{aligned}
&\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\geqslant\delta\right\}\\
&\subset\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\geqslant\frac{\alpha}{2}\delta\right\}\\
&\qquad\cup\left\{n\in\mathbb{N}:\frac{\lambda_n}{n}<\frac{\alpha}{2}\right\}.
\end{aligned}$$

Since $\mathscr{I}$ is admissible, the set on the right-hand side belongs to $\mathscr{I}$ and this completed the proof of (i). (ii) The proof is standard. □

Theorem 3 *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. If $\lambda\in\Delta$ is such that $\lim_n\lambda_n/n=1$, then $S_\lambda^{(\mu,\nu)}(\mathscr{I})\subset S^{(\mu,\nu)}(\mathscr{I})$.*

Proof: Let $\delta>0$ be given. Since $\lim_n\lambda_n/n=1$, we can choose $m\in\mathbb{N}$ such that $\mu(\lambda_n/n-1,t)>1-\frac{1}{2}\delta$ or $\nu(\lambda_n/n-1,t)<\frac{1}{2}\delta$, for all $n\geqslant m$. Now observe that, for $\varepsilon>0$, every $t>0$ and $n\geqslant m$

$$\begin{aligned}
&\frac{1}{n}\left|\{k\leqslant n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\\
&=\frac{1}{n}\left|\{k\leqslant n-\lambda_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\\
&\quad+\frac{1}{n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\\
&\leqslant\frac{n-\lambda_n}{n}+\frac{1}{n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\\
&\leqslant 1-\left(1-\frac{\delta}{2}\right)+\frac{1}{n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\\
&=\frac{\delta}{2}+\frac{1}{n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leqslant n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\geqslant\delta\right\}\\
&\subset\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\in I_n:\mu(x_k-L,t)\leqslant 1-\varepsilon \text{ or } \nu(x_k-L,t)\geqslant\varepsilon\}\right|\geqslant\frac{\delta}{2}\right\}\cup\{1,2,3,\dots,m\}.
\end{aligned}$$

If $\mathscr{I}-S_\lambda^{(\mu,\nu)}$ -$\lim x=L$ then the set on the right-hand side belongs to $\mathscr{I}$ and so the set on the left-hand side also belongs to $\mathscr{I}$. This shows that $x=(x_k)$ is $\mathscr{I}$-statistically convergent to L with respect to the intuitionistic fuzzy norm (μ,ν). □

Theorem 4 *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS such that $\frac{1}{4}\varepsilon_n\diamond\frac{1}{4}\varepsilon_n<\frac{1}{2}\varepsilon_n$ and $(1-\frac{1}{4}\varepsilon_n)*(1-\frac{1}{4}\varepsilon_n)>1-\frac{1}{2}\varepsilon_n$. If X is a Banach space then $S_\lambda^{(\mu,\nu)}(\mathscr{I})\cap m(X)$ is a closed subset of $m(X)$*

Proof: We first assume that $(x^n) \subset S_\lambda^{(\mu,\nu)}(\mathscr{I}) \cap m(X)$ is a convergent sequence and it converges to $x \in m(X)$. We need to show that $x \in S_\lambda^{(\mu,\nu)}(\mathscr{I}) \cap m(X)$. Suppose that $x^n \to L_n(S_\lambda^{(\mu,\nu)}(\mathscr{I}))$ for all $n \in \mathbb{N}$. Take a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of strictly decreasing positive numbers converging to zero. We can find an $n \in \mathbb{N}$ such that $\sup_j \nu(x-x^j, t) < \frac{1}{4}\varepsilon_n$ for all $j \geqslant n$. Choose $0 < \delta < \frac{1}{5}$. Now let

$$A_{\mu,\nu}(\varepsilon_n, t) = \left\{ m \in \mathbb{N} : \frac{1}{\lambda_m} \left| \left\{ k \in I_m : \mu(x_k^n - L_n, t) \leqslant 1 - \frac{\varepsilon_n}{4} \text{ or } \nu(x_k^n - L_n, t) \geqslant \frac{\varepsilon_n}{4} \right\} \right| < \delta \right\}$$

belongs to $F(\mathscr{I})$ and

$$B_{\mu,\nu}(\varepsilon_n, t) = \left\{ m \in \mathbb{N} : \frac{1}{\lambda_m} |\{ k \in I_m : \mu(x_k^{n+1} - L_{n+1}, t) \leqslant 1 - \frac{\varepsilon_n}{4} \text{ or } \nu(x_k^{n+1} - L_{n+1}, t) \geqslant \frac{\varepsilon_n}{4} \right\} \Big| < \delta \Big\}$$

belongs to $F(\mathscr{I})$. Since $A_{\mu,\nu}(\varepsilon_n, t) \cap B_{\mu,\nu}(\varepsilon_n, t) \in F(\mathscr{I})$ and $\varnothing \notin F(\mathscr{I})$, we can choose $m \in A_{\mu,\nu}(\varepsilon_n, t) \cap B_{\mu,\nu}(\varepsilon_n, t)$. Then

$$\frac{1}{\lambda_m} \left| \left\{ k \in I_m : \mu(x_k^n - L_n, t) \leqslant 1 - \frac{\varepsilon_n}{4} \text{ or } \nu(x_k^n - L_n, t) \geqslant \frac{\varepsilon_n}{4} \vee \mu(x_k^{n+1} - L_{n+1}, t) \leqslant 1 - \frac{\varepsilon_n}{4} \text{ or } \nu(x_k^{n+1} - L_{n+1}, t) \geqslant \frac{\varepsilon_n}{4} \right\} \right| \leqslant 2\delta < 1.$$

Since $\lambda_m \to \infty$ and $A_{\mu,\nu}(\varepsilon_n, t) \cap B_{\mu,\nu}(\varepsilon_n, t) \in F(\mathscr{I})$ is infinite, we can choose the above m so that $\lambda_m > 5$. Hence there must exist a $k \in I_m$ for which we have simultaneously, $\mu(x_k^n - L_n, t) > 1 - \frac{1}{4}\varepsilon_n$ or $\nu(x_k^n - L_n, t) < \frac{1}{4}\varepsilon_n$ and $\mu(x_k^{n+1} - L_{n+1}, t) > 1 - \frac{1}{4}\varepsilon_n$ or $\nu(x_k^{n+1} - L_{n+1}, t) < \frac{1}{4}\varepsilon_n$. For a given $\varepsilon_n > 0$ choose $\frac{1}{2}\varepsilon_n$ such that $(1 - \frac{1}{2}\varepsilon_n) * (1 - \frac{1}{2}\varepsilon_n) > 1 - \varepsilon_n$ and $\frac{1}{2}\varepsilon_n \diamond \frac{1}{2}\varepsilon_n < \varepsilon_n$. Then it follows that

$$\nu\left(L_n - x_k^n, \frac{t}{2}\right) \diamond \nu\left(L_{n+1} - x_k^{n+1}, \frac{t}{2}\right) \leqslant \frac{\varepsilon_n}{4} \diamond \frac{\varepsilon_n}{4} < \frac{\varepsilon_n}{2}$$

and

$$\nu(x_k^n - x_k^{n+1}, t) \leqslant \sup_n \nu\left(x - x^n, \frac{t}{2}\right) \diamond \sup_n \nu\left(x - x^{n+1}, \frac{t}{2}\right) \leqslant \frac{\varepsilon_n}{4} \diamond \frac{\varepsilon_n}{4} < \frac{\varepsilon_n}{2}.$$

Hence we have

$$\nu(L_n - L_{n+1}, t) \leqslant \left[\nu\left(L_n - x_k^n, \frac{t}{3}\right) \diamond \nu\left(x_k^{n+1} - L_{n+1}, \frac{t}{3}\right) \right] \diamond \nu\left(x_k^n - x_k^{n+1}, \frac{t}{3}\right) \leqslant \frac{\varepsilon_n}{2} \diamond \frac{\varepsilon_n}{2} < \varepsilon_n$$

and similarly $\mu(L_n - L_{n+1}, t) > 1 - \varepsilon_n$. This implies that $\{L_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and let $L_n \to L \in X$ as $n \to \infty$. We shall prove that $x \to L(S_\lambda^{(\mu,\nu)}(\mathscr{I}))$. For any $\varepsilon > 0$ and $t > 0$, choose $n \in \mathbb{N}$ such that $\varepsilon_n < \frac{1}{4}\varepsilon$, $\sup_n \nu(x - x^n, t) < \frac{1}{4}\varepsilon$, $\mu(L_n - L, t) > 1 - \frac{1}{4}\varepsilon$ or $\nu(L_n - L, t) < \frac{1}{4}\varepsilon$. Now since

$$\begin{aligned} &\frac{1}{\lambda_n} |\{k \in I_n : \nu(x_k - L, t) \geqslant \varepsilon\}| \\ &\leqslant \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \nu\left(x_k - x_k^n, \frac{t}{3}\right) \diamond \left[\nu\left(x_k^n - L_n, \frac{t}{3}\right) \diamond \nu\left(L_n - L, \frac{t}{3}\right) \right] \geqslant \varepsilon \right\} \right| \\ &\leqslant \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \nu\left(x_k^n - L_n, \frac{t}{3}\right) \geqslant \frac{\varepsilon}{2} \right\} \right| \end{aligned}$$

and similarly

$$\frac{1}{\lambda_n} |\{k \in I_n : \mu(x_k - L, t) \leqslant 1 - \varepsilon\}| > \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \mu\left(x_k^n - L, \frac{t}{3}\right) \leqslant 1 - \frac{\varepsilon}{2} \right\} \right|.$$

It follows that

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : \mu(x_k - L, t) \leqslant 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geqslant \varepsilon\}| \geqslant \delta \right\} \subset \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \mu\left(x_k^n - L, \frac{t}{3}\right) \leqslant 1 - \frac{\varepsilon}{2} \text{or } \nu\left(x_k^n - L, \frac{t}{3}\right) \geqslant \frac{\varepsilon}{2} \right\} \right| \geqslant \delta \right\}$$

for any given $\delta > 0$. Hence we have $x \to L(S_\lambda^{(\mu,\nu)}(\mathscr{I}))$. □

CONCLUSIONS

In this paper we introduce the notions of $\mathscr{I}$-$[V,\lambda]$-summability and $\mathscr{I}$-λ-statistical convergence with respect to the intuitionistic fuzzy norm (μ,ν), investigate their relationship, and make some observations about these classes. We intend to unify these two approaches and use ideals to introduce the concept of $\mathscr{I}-\lambda$-statistical convergence with respect to the intuitionistic fuzzy norm (μ,ν). Our study of $\mathscr{I}$-statistical and $\mathscr{I}$-λ-statistical convergence convergence of sequences in intuitionistic fuzzy normed spaces also provides a tool to deal with convergence problems of sequences of fuzzy real numbers. These results can be used to study the convergence problems of sequences of fuzzy numbers having a chaotic pattern in intuitionistic fuzzy normed spaces.

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