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A novel design of IIR multiple notch filter based on an all-pass filter by using a pole-reposition technique

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ABSTRACT: In this paper, a novel and simple method based on an all-pass filter is presented by using the benefits of a pole re-position technique. The gains at frequencies of π of each single notch filter are adjusted by adding $N-1$ tuning variables. The proposed method ensures that the passband gains are uniformly flat, the notch frequencies exactly meet the specifications, and the realized 3 dB bandwidths are approximately the same as those specified. This technique is very useful for designers because there are only $N-1$ tuning variables required to adjust, and it does not need a complicated mathematical calculation. Although this technique is simple and easy to implement, it requires many iterations to find a suitable parameter for making the passband gain between two notch frequencies uniformly flat. Hence in this paper, three searching algorithms have been applied to reduce the number of iterations. We obtain a faster search and a closer frequency response to the ideal one.

INTRODUCTION

Notch filters are widely used for removing, eliminating, or cancelling unwanted frequencies or interferences. There are many applications of notch filters in the field of signal processing, such as removing power line interference in electrocardiograms, cancelling noise in broadcast TV, rejecting the interference in ultra-wideband radio systems, controlling howl in speaker phone systems, and eliminating hum in audio systems.

Unlike FIR notch filters, digital IIR notch filters are popularly used in practice because they can be designed to have a narrow stopband. They can also be designed in several ways, such as transforming analogue notch filter[1–3], the pole-reposition technique[4–6], and implementation based on all-pass filters[7–9] which is considered in this paper.

The frequency response of an ideal IIR notch filter can be expressed as

$$H_i(\mathrm{e}^{\mathrm{j}\omega}) = \begin{cases} 0, & \omega = \omega_{0i}, \\ 1, & \text{otherwise}, \end{cases} \tag{1}$$

where ω_{0i} denotes the notch frequency of the ith notch filter. A transfer function of IIR notch filter based on an all-pass filter[7–9] can be represented as

$$H_i(z) = \tfrac{1}{2}\{1 + A_i(z)\}, \tag{2}$$

where $A_i(z)$ is a second-order all-pass filter, which can be represented as

$$A_i(z) = \frac{k_{2i} + k_{1i}(1+k_{2i})z^{-1} + z^{-2}}{1 + k_{1i}(1+k_{2i})z^{-1} + k_{2i}z^{-2}}, \tag{3}$$

where components k_{1i} and k_{2i} are, respectively, given as

$$k_{1i} = -\cos(\omega_{0i}), \quad k_{2i} = \frac{1 - \tan(\mathrm{BW}_i/2)}{1 + \tan(\mathrm{BW}_i/2)}, \tag{4}$$

where BW_i is the bandwidth of the ith notch filter. By substituting (3) into (2), $H_i(z)$ can be re-written as

$$H_i(z) = \frac{1+k_{2i}}{2} \frac{1 + 2k_{1i}z^{-1} + z^{-2}}{1 + k_{1i}(1+k_{2i})z^{-1} + k_{2i}z^{-2}}. \tag{5}$$

Hence the transfer function of a multiple notch filter can be constructed by cascading each single notch filter[7] as

$$H_{\mathrm{M}}(z) = \prod_{i=1}^{N} \tfrac{1}{2}\{1 + A_i(z)\}, \tag{6}$$

where N specifies the number of notch frequencies. Its ideal magnitude response can be expressed as

$$H_{\mathrm{M}}(\mathrm{e}^{\mathrm{j}\omega}) = \begin{cases} 0, & \omega = \omega_{01}, \omega_{02}, \ldots, \omega_{0N}, \\ 1, & \text{otherwise}. \end{cases} \tag{7}$$

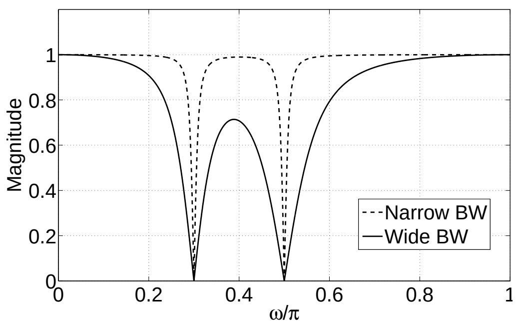

Fig. 1 Magnitude responses of narrow bandwidth case versus wide bandwidth case.

Example 1 For a double notch filter ($N = 2$) with narrow bandwidth, let $\omega_{01} = 0.3\pi$, $\omega_{02} = 0.5\pi$, $\mathrm{BW}_1 = 0.02\pi$, $\mathrm{BW}_2 = 0.02\pi$. For the wide bandwidth case, let $\omega_{01} = 0.3\pi$, $\omega_{02} = 0.5\pi$, $\mathrm{BW}_1 = 0.1\pi$, $\mathrm{BW}_2 = 0.15\pi$. The transfer functions of the narrow bandwidth case and the wide bandwidth case can be directly derived from (6), where the magnitude responses are shown in Fig. 1.

From Example 1, it is shown that the conventional all-pass based method[7] satisfies the requirements only for a narrow bandwidth case. For a wide bandwidth case, there is an ill-conditioned passband gain for frequencies between notch frequencies caused by the overlapped bandwidths (Fig. 1). In general, multiple notch filters are designed to have not-too-narrow bandwidths for stability reasons. There are two techniques for solving the overlapping problem, which are the optimal pole position technique[5], and the all-pass filter of order $2N$ technique[8].

The optimal pole position technique[5] is simple and practical, since it does not require any complicated calculation. The technique relies on solving for filter coefficients to satisfy the table of gains at specific frequencies. The optimal pole reposition algorithm[5] works only for double notch filters, but the modified version of this algorithm[6] shows that this technique also works well in the general multiple notch filter case, i.e., the case of more than two notch frequencies. This algorithm yields an acceptable result. However, an optimal method for searching the optimal values of a bandwidth factor, i.e., a parameter which is used to adjust the level of passband gains between notch frequencies, has not been proposed.

When the all-pass filter of order $2N$ technique[8] is applied, the non-uniformly-flat passband gains between notch frequencies can be solved effortlessly. However, it causes the problem of having a shift of notch frequencies. Hence a mirror image polynomial is formed to solve these notch frequency shifts. This algorithm is very efficient. However, the solving process after equating the mirror image polynomial with a polynomial formed by the numerator part of the multiple notch filter transfer function is rather difficult and it requires nonlinear polynomial equation solving, which is extremely complex, especially when the number of notch frequencies increases.

In this paper, a new approach is presented by merging the idea of the optimal pole position technique with the all-pass based scheme for solving the overlapping problem, while passband gains are ensured to be uniformly flat; the notch frequencies and the bandwidths meet the specific requirements. Moreover, a method for searching the optimal values of parameters used to control the level of passband gains between notch frequencies is also proposed.

THE NEW DESIGN

The pole position depends directly on $\cos(\omega_0)$, clearly, which is k_{1i} in (4). The pole-reposition technique is applied to the all-pass based algorithm. Hence k_{1i} in the denominator is replaced by k_{Xi}. Hence a new transfer function $\hat{H}_i(z)$ can be expressed as

$$\hat{H}_i(z) = \frac{1}{2}\frac{(1+k_{2i})(1+2k_{1i}z^{-1}+z^{-2})}{1+k_{Xi}(1+k_{2i})z^{-1}+k_{2i}z^{-2}}, \qquad (8)$$

where k_{Xi} is the modified coefficient (pole-reposition) of the ith notch filter. The tuning variables $(a, b, \ldots)$ can be considered as a specific case of p_i, where $i = 1, 2, \ldots, N$, defined as

$$\hat{H}_i(\mathrm{e}^{\mathrm{j}\pi}) = p_i\hat{H}_i(\mathrm{e}^{\mathrm{j}0}) = p_i\hat{H}_i(1)$$

with

$$\prod_{i=1}^{N} p_i = 1.$$

That is, in Table 1 ($N = 2$ case), $p_1 = 1/a$, $p_2 = a$, and in Table 2 ($N = 3$ case), $p_1 = 1/ab$, $p_2 = b$, $p_3 = a$. For simplicity, a double notch filter ($N = 2$) is derived whose desired gains are shown in Table 1.

Table 1 Desired gains for $N = 2$ case.

frequency	DC	π
notch filter 1	G_1	G_1/a
notch filter 2	G_2	$G_2 a$
double notch filter	G_1G_2	G_1G_2

Table 2 Desired gains for $N = 3$ case.

frequency	DC	π
notch filter 1	G_1	$G_1/(ab)$
notch filter 2	G_2	$G_2 b$
notch filter 3	G_3	$G_3 a$
triple notch filter	$G_1G_2G_3$	$G_1G_2G_3$

Notch filter 1

The DC gain $= G_1$. From (8), the DC gain can be derived as

$$\hat{H}_1(\mathrm{e}^{\mathrm{j}0}) = \hat{H}_1(1)G_1 = \frac{1+k_{11}}{1+k_{X1}}. \tag{9}$$

The π gain $= G_1/a$. The π gain can be derived as

$$\hat{H}_1(\mathrm{e}^{\mathrm{j}\pi}) = \hat{H}_1(-1)G_1 = a\left(\frac{1-k_{11}}{1-k_{X1}}\right). \tag{10}$$

By pairing (9) and (10), the gain of the notch filter 1 can be represented as

$$G_1 = \frac{1+k_{11}}{1+k_{X1}} = a\left(\frac{1-k_{11}}{1-k_{X1}}\right), \tag{11}$$

where a represents a real tuning variable. Thus the modified coefficient of the notch filter 1, k_{X1}, can be shown to be

$$k_{X1} = \frac{(1-a)+k_{11}(1+a)}{(1+a)+k_{11}(1-a)}. \tag{12}$$

Notch filter 2

The DC gain $= G_2$. The notch filter 2 can also be derived in a similar manner. The DC gain can be expressed as

$$\hat{H}_2(\mathrm{e}^{\mathrm{j}0}) = \hat{H}_2(1)G_2 = \frac{1+k_{12}}{1+k_{X2}}. \tag{13}$$

The π gain $= G_2 a$ The π gain can be expressed as

$$\hat{H}_2(\mathrm{e}^{\mathrm{j}\pi}) = \hat{H}_2(-1)G_2 = \frac{1}{a}\left(\frac{1-k_{12}}{1-k_{X2}}\right). \tag{14}$$

Pairing (13) and (14), the gain of notch filter 2 can be expressed as

$$G_2 = \frac{1+k_{12}}{1+k_{X2}} = \frac{1}{a}\left(\frac{1-k_{12}}{1-k_{X2}}\right). \tag{15}$$

Hence the new pole position of notch filter 2, k_{X2}, can be expressed as

$$k_{X2} = \frac{(a-1)+k_{12}(1+a)}{(1+a)+k_{12}(a-1)}. \tag{16}$$

In general, passband gains are uniformly flat and equal to one. (11) and (15) show that the passband gains at DC and frequencies of π are uniformly flat but they cannot guarantee that the passband gains will be equal to one. To fix this problem, the transfer function of a single notch filter is transformed to be:

$$H_{Xi}(z) = \frac{\hat{H}_i(z)}{G_i},$$

where G_i denotes the DC gain of the ith notch filter. Hence the transfer function of an IIR multiple notch filter of the proposed design is

$$\begin{aligned}\hat{H}_{\mathrm{M}}(z) &= \prod_{i=1}^{N} H_{Xi}(z),\\ &= \prod_{i=1}^{N} \frac{\hat{H}_i(z)}{G_i}.\end{aligned} \tag{17}$$

DESIGN EXAMPLES AND DISCUSSION

This algorithm can be done easily by tuning variables manually. However, the maximum passband gain between notch frequencies may not be unity. To overcome this problem, the tuning parameter has to minimize the following cost function.

$$\text{Error} = \int_0^{\pi} \Big| |\hat{H}_{\mathrm{M}}(\omega)| - |H_{\mathrm{I}}(\omega)| \Big| \,\mathrm{d}\omega, \tag{18}$$

where $H_{\mathrm{I}}(\omega)$ represents the transfer function of an ideal notch filter.

Since the interval of integration is in some sense small and the cost function is relatively smooth over the interval $[0, \pi]$, the composite Simpson's rule[10,11] can be used instead of an exact integration.

Example 2 As expressed in Example 1, the specifications of a multiple notch filter with $N = 2$ are given as follows:

$$H_{\mathrm{M}}(\mathrm{e}^{\mathrm{j}\omega}) = \begin{cases} 0, & \omega = 0.3\pi, 0.5\pi,\\ 1, & \text{otherwise,}\end{cases}$$

where the bandwidths of notch filters 1 and 2 are, respectively, given as

$$\mathrm{BW}_1 = 0.1\pi, \quad \mathrm{BW}_2 = 0.15\pi.$$

Hence the gains at frequencies of π can be directly used from Table 1 where (17) is also used as a new transfer function of a multiple notch filter.

Serial searching is employed. The interval $[0, \pi]$ is broken up into 100 small subintervals, with incrementing a by 0.0001 from 0 until 1. Thus the

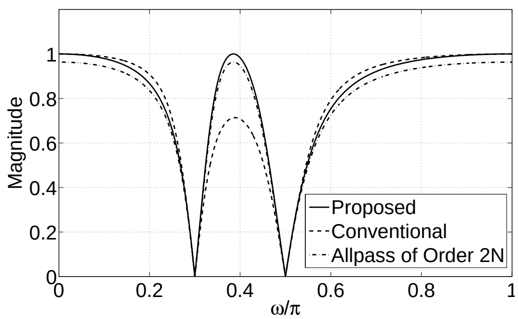

Fig. 2 Comparison of magnitude responses for $N = 2$.

Table 3 Parameters used for $N = 2$.

design	param.	H_1	H_2	error
conventional[7]	k_{1i}	-0.5878	-6.12×10^{-17}	0.36
	k_{2i}	0.7265	0.6128	
all-pass of order $2N$ [8]	k_{1i}	-0.5878	-6.12×10^{-17}	0.31
	k_{2i}	0.7265	0.6128	
	k_{1i}^*	-0.5397	-0.0705	
proposed	k_{1i}	-0.5878	-6.12×10^{-17}	0.31
	k_{2i}	0.7265	0.6128	
	k_{Xi}	-0.5397	-0.0705	
	G_i	0.8955	1.0758	
	a	0.8684	0.8684	

minimum value of a cost function takes place at $a = 0.8684$. Note that the precision of the optimum value is adequate when 4 decimal places are used.

The magnitude response of the proposed design is shown in Fig. 2, where all parameters used for each design are given in Table 3.

The corresponding transfer function of the proposed design is

$$H(z) = \frac{N(z)}{D(z)},$$

with

$$N(z) = 2.8904(1 + z^{-4}) - 3.3979(z^{-1} + z^{-3}) + 5.7808z^{-2}$$

and

$$D(z) = 4 - 4.1816z^{-1} + 5.7808z^{-2} - 2.6142z^{-3} + 1.7809z^{-4}.$$

The pole positions of the conventional design[7], all-pass of order $2N$ design[8], and the proposed design are compared in Table 4.

Fig. 2 shows that the magnitude responses of the proposed design and the all-pass filter order $2N$ design[8] are almost the same line. An ill-conditioned gain from the conventional design[7] can be solved by using both techniques, depending on the designers. But the disadvantage of Ref. 8 is that the complicated nonlinear equations for this algorithm are difficult to solve, especially, when $N \geqslant 3$. On the other hand, the proposed design can be done easily by tuning only $N-1$ variables (which is the parameter a in this case).

Table 4 Comparison of pole positions for $N = 2$.

conventional[7]	order $2N$ [8]	proposed
0.5074 ± 0.6849j	0.4659 ± 0.7138j	0.4659 ± 0.7138j
0.0000 ± 0.7828j	0.0569 ± 0.7807j	0.0568 ± 0.7808j

Example 3 The specifications of a multiple notch filter with $N = 3$ are

$$H_{\mathrm{M}}(\mathrm{e}^{\mathrm{j}\omega}) = \begin{cases} 0, & \omega = 0.1\pi, 0.2\pi, 0.6\pi, \\ 1, & \text{otherwise}, \end{cases}$$

where the bandwidths of notch filter 1, 2, and 3 are, respectively,

$$\mathrm{BW}_1 = 0.1\pi, \quad \mathrm{BW}_2 = 0.1\pi, \quad \mathrm{BW}_3 = 0.2\pi.$$

For $N = 3$, the desired gains can be expressed as in Table 2. While k_{X1}, k_{X2}, and k_{X3} can be re-adjusted directly from (12) or (16), depending on each gain at a frequency of π. Thus

$$k_{X1} = \frac{(1-ab) + k_{11}(1+ab)}{(1+ab) + k_{11}(1-ab)},$$
$$k_{X2} = \frac{(b-1) + k_{12}(1+b)}{(1+b) + k_{12}(b-1)},$$
$$k_{X3} = \frac{(a-1) + k_{13}(1+a)}{(1+a) + k_{13}(a-1)}.$$

Since there are two tuning variables, i.e., the tuning parameters a and b, then one more loop is added for tuning parameter b. Hence the parameters a and b are searched from 0 to 1 with an increment of 0.0001. Then the minimum error exists at $a = 0.8435$ and $b = 0.4040$.

Fig. 3 represents the magnitude responses for the proposed design, where all necessary parameters are given in Table 5.

The corresponding transfer function can be represented as

$$H(z) = \frac{N(z)}{D(z)},$$

with

$$N(z) = 5.0756(1 + z^{-6}) - 14.73(z^{-1} + z^{-5}) + 19.8056(z^{-2} + z^{-4}) - 19.8056z^{-3}$$

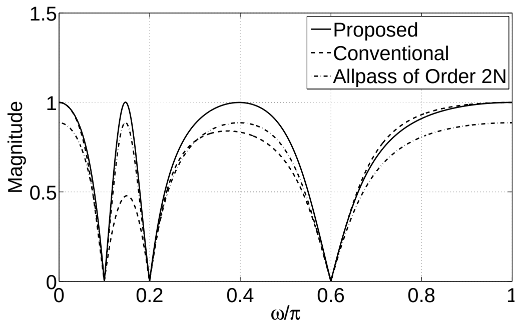

Fig. 3 Comparison of magnitude responses for $N = 3$.

Table 5 Parameters used for $N = 3$.

design	param.	H_1	H_2	H_3	error
conventional[7]	k_{1i}	−0.9511	−0.8090	0.3090	0.58
	k_{2i}	0.7265	0.7265	0.5095	
all-pass of order $2N$ [8]	k_{1i}	−0.9511	−0.8090	0.3090	0.56
	k_{2i}	0.7265	0.7265	0.5095	
	k^*_{1i}	−0.9182	−0.8629	0.2301	
proposed	k_{1i}	−0.9511	−0.8090	0.3090	0.56
	k_{2i}	0.7265	0.7265	0.5095	
	k_{Xi}	−0.8629	−0.9182	0.2302	
	G_i	0.3569	2.3344	1.0641	
	a	0.8435	–	0.8435	
	b	0.4040	0.4040	–	

and

$$D(z) = 8 - 21.8210z^{-1} + 26.0476z^{-2} - 19.8047z^{-3} + 13.5630z^{-4} - 7.6397z^{-5} + 2.1517z^{-6}$$

where the pole positions are compared in Table 6.

Table 7 shows the relationship among those parameters, i.e., bandwidths BW, tuning variable a, error calculated from cost function, and pole radius (which directly affects the stability margin), where $\mathrm{BW}_1 = \mathrm{BW}_2$ for all cases, and ω_1 and ω_2 are assumed to be 0.3π and 0.5π, respectively. Note that the pole radius directly depends on the bandwidth. Hence if the values of notch frequencies ω_1 and ω_2 are changed, it will not affect the pole radius. However, the tuning variable, a, can always changes, and the error is slightly different.

Table 6 Comparison of pole positions for $N = 3$.

conventional[7]	order $2N$ [8]	proposed
−0.2332 ± 0.6746j	−0.1737 ± 0.6924j	−0.1737 ± 0.6923j
0.6984 ± 0.4886j	0.7449 ± 0.4143j	0.7449 ± 0.4143j
0.8210 ± 0.2291j	0.7927 ± 0.3134j	0.7926 ± 0.3135j

Table 7 Case of $N = 2$ and $\mathrm{BW}_1 = \mathrm{BW}_2$.

BW	a	error	pole radius
0.01π	1	0.0439	0.9844
0.05π	0.9789	0.1326	0.9242
0.1π	0.9139	0.2563	0.8524
0.15π	0.7963	0.3660	0.7828
0.2π	0.5095	0.4603	0.7138

As shown in Example 2 and Example 3, the numbers of tuning parameters increase with the number of notch frequencies. Hence to determine tuning parameters for multiple notch filter, the serial search method can be computed intensively. Thus the new searching technique based on multidimensional search without using derivatives, which will be described later, is used instead.

FINDING THE OPTIMAL FILTER PARAMETER

When using the pole-reposition technique, the tuning parameters have to be adjusted precisely in order to make the passband gain uniformly flat and maintain the characteristic of the notch filter. According to the previous section, the weakness of this method is the long computing time when designing a multiple notch filter. In order to improve the performance of this technique, the searching algorithm has been applied to find suitable parameters that optimize the cost function. For simplicity, the cost function (18) has been revised as

$$f(\mathbf{X}) = \int_0^{\pi} |(1 - \|H_{\mathrm{M}}(\omega)\|)| \,\mathrm{d}\omega, \qquad (19)$$

where $\mathbf{X}$ is vector of tuning parameters that minimize the cost function f; $\mathbf{X} = (x_1, x_2, \ldots, x_N)$. The magnitude of an ideal notch filter is 1 since the bandwidth of ideal notch filter should be very narrow.

Note that using the square function will make the computed error larger than usual and this sometimes causes the wrong decision when choosing optimal parameters. On the other hand, using an absolute function, the result will be more accurate since it uses the actual difference to compute the error.

In this paper, we focused on the cyclic coordinate method and the Hooke and Jeeves method using lines searches as expressed below.

Algorithm 1 [Cyclic coordinate method] This method will search along directions $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_j$ where $\mathbf{d}_j$ is a zero vector size n, where n is number of tuning parameters except at the jth position, i.e., $\mathbf{d}_1 = (x_1, 0, 0, \ldots, 0)$, $\mathbf{d}_2 = (0, x_2, 0, \ldots, 0)$, etc.

Thus along each search direction $\mathbf{d}_j$, the variable x_j is changed while all other variables are kept fixed.

Step 1: Choose a scalar $\epsilon > 0$ as the termination parameter.

Step 2: Let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n$ be the coordinate direction where n is number of tuning parameters.

Step 3: Choose the point $\mathbf{x}_1$ as the searching starting point.

Step 4: Let $\mathbf{y}_1 = \mathbf{x}_1$ and $k = j = 1$.

Step 5: Let λ_j be an optimal solution to minimize the cost function $f(\mathbf{y}_j + \lambda_j \mathbf{d}_j)$ along $\mathbf{d}_j$ direction. Then let $\mathbf{y}_{j+1} = \mathbf{y}_j + \lambda_j \mathbf{d}_j$. If $j < n$, replace j by $j + 1$, and repeat Step 5. Otherwise, go to Step 6.

Step 6: Let $\mathbf{x}_{k+1} = \mathbf{y}_{n+1}$. If $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| < \epsilon$, then stop. Otherwise, let $\mathbf{y}_1 = \mathbf{x}_{k+1}$, $j = 1$, $k =\rightarrow k + 1$, and repeat Step 5.

For the general case of an N-notch filter, the number of tuning parameters and direction vectors is $N - 1$.

Algorithm 2 [Hooke and Jeeves method using line searches] The method of Hooke and Jeeves performs an explanatory search and a pattern search. The explanatory search is similar to the cyclic coordinate method (i.e., all parameters are fixed except one searching parameter). The additional pattern search is the acceleration step by searching along the line which is created by two optimal points. Then the process is repeated.

Step 1: Choose a scalar $\epsilon > 0$ as the termination parameter.

Step 2: Let $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n$ be the coordinate direction where n is number of tuning parameters.

Step 3: Choose the point $\mathbf{x}_1$ as the searching starting point.

Step 4: Let $\mathbf{y}_1 = \mathbf{x}_1$ and $k = j = 1$.

Step 5: Let λ_j be an optimal solution to minimize the cost function $f(\mathbf{y}_j + \lambda_j \mathbf{d}_j)$. Then let $\mathbf{y}_{j+1} = \mathbf{y}_j + \lambda_j \mathbf{d}_j$. If $j < n$, replace j by $j + 1$, and repeat Step 5. Otherwise, let $\mathbf{x}_{k+1} = \mathbf{y}_{n+1}$. If $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| < \epsilon$, then stop; otherwise, go to Step 6.

Step 6: Let $\mathbf{d} = \mathbf{x}_{k+1} - \mathbf{x}_k$ and let

$$\hat{\lambda} = \arg\min_{\lambda} f(\mathbf{x}_{k+1} + \lambda \mathbf{d}).$$

Let $\mathbf{y}_1 = \mathbf{x}_{k+1} + \hat{\lambda}\mathbf{d}$, $j = 1$, $k \rightarrow k+1$, and repeat Step 5.

For the general case of an N-notch filter, the number of tuning parameters and direction vectors will be $N - 1$.

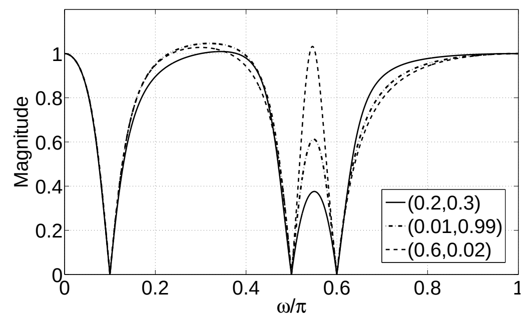

Fig. 4 Magnitude response of triple notch obtained by cyclic coordinate search with different starting points.

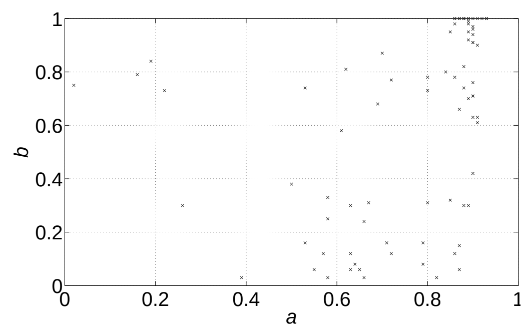

Fig. 5 Optimal tuning parameter of triple notch filter with a different set of notch frequencies and $\mathrm{BW} = 0.1\pi$.

To minimize the cost function in (19), an approximation to the cost function is computed by numerical integration using the trapezoid rule[12]. However, the accuracy of this method is quite low if there are not many sampling points. In order to improve the performance, Richardson's extrapolation is applied[12].

Since the success of the cyclic coordinate method strongly depends on the initial searching point, sometimes the best optimal solution cannot be obtained if the cost function is non-convex (Fig. 4). To improve the searching efficiency, the cyclic coordinate method has to be modified to start the search from various initial points. By plotting the cost function with respect to two adjustable parameters (Fig. 5), we found that the optimal solutions are usually located near the boundary. Hence we have to set four initial searching points at $(0.25, 0.25)$, $(0.25, 0.75)$, $(0.75, 0.25)$, $(0.75, 0.75)$ (Fig. 6).

Sometimes the optimal solution might be located in the middle region and the searching direction from 4 initial points cannot go to the middle. Thus the result will not be the best solution. To ensure that all regions have been checked, an additional searching point has been located at $(0.5, 0.5)$ (Fig. 7).

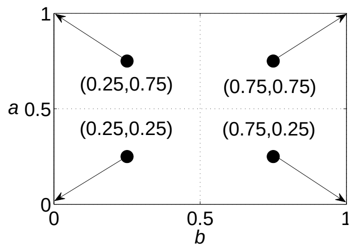

Fig. 6 Four initial searching points.

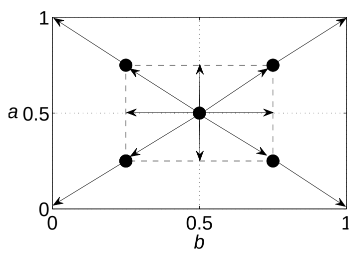

Fig. 7 Optimal searching points.

DESIGN EXAMPLES AND DISCUSSION

Example 4 Triple notch filter ($N = 3$) with $\omega_1 = 0.1\pi$, $\omega_2 = 0.5\pi$, $\omega_3 = 0.6\pi$, and $\mathrm{BW} = 0.1\pi$ per sample.

Applying the proposed algorithm, the cyclic coordinate search and Hooke and Jeeves method using line searches with starting point at $(0.6, 0.4)$ give the values of tuning parameters a and b of 0.0140 and 0.8870, 0.9990 and 0.9200, respectively. The modified cyclic coordinate search gives 0.6680 and 0.0280, respectively. As shown in Fig. 8, the modified cyclic coordinate search gives the best magnitude response, which is similar to the result obtained by serial search with an increment of 0.001.

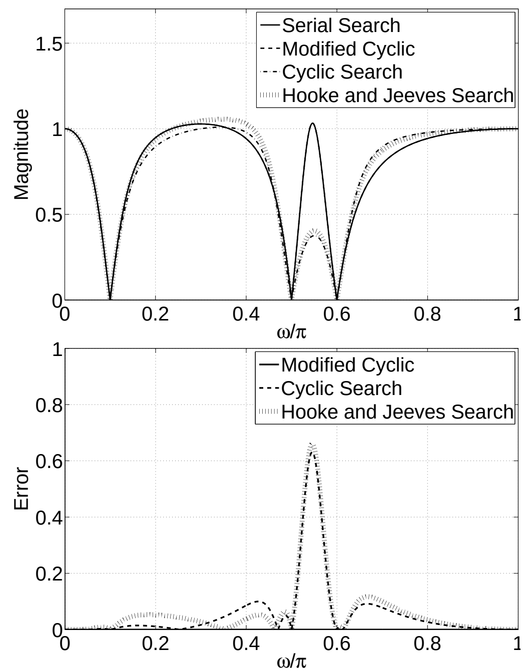

Fig. 8 Magnitude of response of serial search and proposed method for $N = 3$.

Example 5 Quad notch filter ($N = 4$) with specification $\omega_1 = 0.1\pi$, $\omega_2 = 0.2\pi$, $\omega_3 = 0.6\pi$, $\omega_4 = 0.7\pi$ and $\mathrm{BW} = 0.05\pi$ per sample.

Applying the proposed algorithm, the cyclic coordinate search and the Hooke and Jeeves method using line searches with starting point at $(0.5, 0.5, 0.5)$ give the values of tuning parameters a, b, and c, of 0.0320, 0.4990, and 0.694, 0.0320, 0.4990, and 0.694, respectively. The modified cyclic coordinate search gives 0.0230, 0.6960, and 0.6950. As shown in Fig. 9, the modified cyclic coordinate search gives the best magnitude response, which is similar to the result obtained by serial search with 0.001 increment.

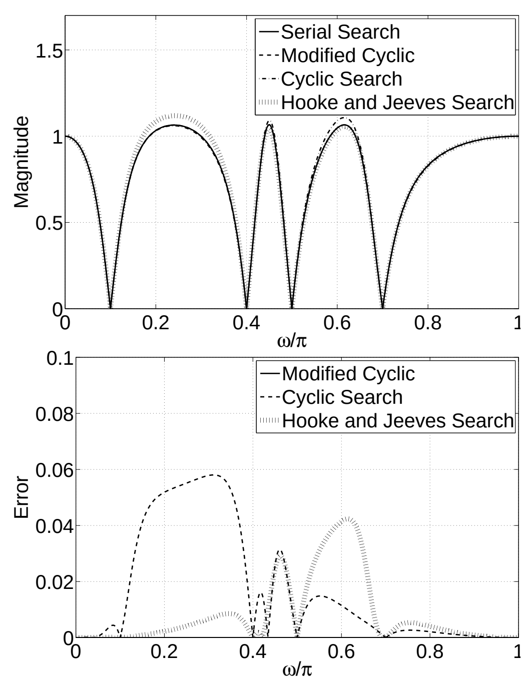

Fig. 9 Magnitude of response of serial search and proposed method for $N = 4$.

Table 8 and Table 9 show the error and optimal parameter from the cyclic coordinate and Hooke and Jeeves methods, respectively, with initial point $(0.6, 0.4)$ required to converge at various ω_1, ω_2, ω_3, and BW. The reason we use point $(0.6, 0.4)$ is this

Table 8 The error and optimal parameters from the cyclic coordinate method with initial point $(0.6, 0.4)$ required to converge at various ω_1, ω_2, ω_3, and BW. Average computing time = 9.0 s.

ω_1/π	ω_2/π	ω_3/π	BW/π	a	b	error
0.1	0.2	0.6	0.1	0.8970	0.6320	0.6515
0.2	0.4	0.6	0.1	0.8660	0.9790	0.6545
0.1	0.2	0.3	0.1	0.5040	0.3810	0.6693
0.1	0.3	0.6	0.1	0.8870	0.9210	0.6511
0.1	0.5	0.6	0.1	0.0140	0.8870	0.7164
0.1	0.5	0.9	0.1	0.8900	1.0000	0.6325
0.1	0.2	0.3	0.05	0.1100	0.9730	0.3926
0.2	0.4	0.6	0.05	0.9640	0.3901	0.3901

Table 9 The error and optimal parameters from the Hooke and Jeeves method using line searches with initial point $(0.6, 0.4)$ required to converge at various ω_1, ω_2, ω_3, and BW. Average computing time = 9.3 s.

ω_1/π	ω_2/π	ω_3/π	BW	a	b	error
0.1	0.2	0.6	0.1	0.8970	0.6320	0.6515
0.2	0.4	0.6	0.1	0.8660	0.9770	0.6545
0.1	0.2	0.3	0.1	0.5040	0.3810	0.6693
0.1	0.3	0.6	0.1	0.8870	0.9210	0.6511
0.1	0.5	0.6	0.1	0.9990	0.920	0.7046
0.1	0.5	0.9	0.1	0.8900	1	0.6325
0.1	0.2	0.3	0.05	0.2150	0.3860	0.6273
0.2	0.4	0.6	0.05	0.9640	0.9940	0.3901

point results in the lowest computational time.

According to Table 8 and Table 9, the two proposed algorithms give very similar results in terms of adjustable parameters and the number of iterations when starting at the same initial point. Hence either method can be used for finding an optimal solution.

Although most of the time, these two algorithms can find best optimal result, the obtained result might be different due to many local minimum points of cost function as shown in Fig. 10 and Fig. 11. Hence different initial searching points might lead to different optimal solutions. To overcome this problem, the cyclic coordinate method has been modified to start the search from various starting points.

Table 10 and Table 11 show the error and optimal parameters from the plane and modified cyclic coordinate methods, respectively, required to converge at various ω_1, ω_2, ω_3, and BW.

CONCLUSIONS

The advantages of the proposed design are as follows. (1) It can guarantee that the passband gains are uniformly flat while the notch frequencies and the realized bandwidths meet the specifications of the designers. (2) It does not need a complicated mathematical calculation. (3) It can be easily implemented by either serial searching or the optimal multidimensional search without using derivatives. The proposed algorithm yields higher efficiency and searching speed with the lowest possible error.

The non-unity maximum passband gains between notch frequencies always emerge when cascading ev-

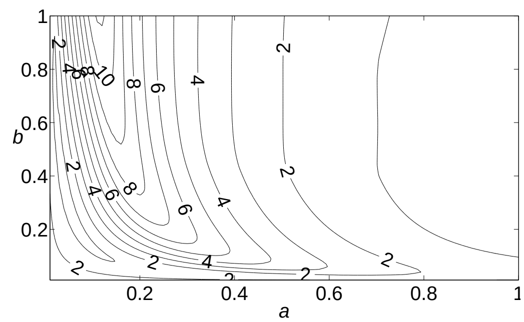

Fig. 10 Contour plot of typical cost function versus tuning parameter (a,b).

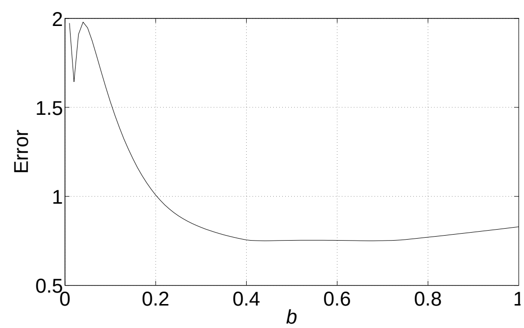

Fig. 11 Plot of typical cost function versus tuning parameter b when $a = 0.8$ (non-convex).

Table 10 The error and optimal parameters from a plane search required to converge at various ω_1, ω_2, ω_3, and BW. Average computing time = 26 s.

ω_1/π	ω_2/π	ω_3/π	BW/π	a	b	error
0.1	0.2	0.6	0.1	0.8970	0.6320	0.6515
0.1	0.3	0.6	0.1	0.8870	0.9210	0.6511
0.2	0.4	0.6	0.1	0.8660	0.9760	0.6545
0.1	0.5	0.6	0.1	0.6680	0.0280	0.6507
0.1	0.2	0.3	0.1	0.5040	0.3810	0.6693
0.1	0.5	0.9	0.1	0.8900	1.000	0.6325
0.1	0.2	0.6	0.05	0.9710	0.6650	0.3891
0.2	0.4	0.6	0.05	0.9640	0.9940	0.3901
0.1	0.2	0.3	0.05	0.9030	0.2700	0.3926

Table 11 The error and optimal parameters from a plane search required to converge at various ω_1, ω_2, ω_3, ω_4, and BW. Average computing time = 58 s.

ω_1/π	ω_2/π	ω_3/π	ω_4/π	BW/π	a	b	c
0.1	0.3	0.5	0.7	0.1	0.943	1.000	0.822
0.1	0.2	0.6	0.7	0.1	0.023	0.696	0.695
0.1	0.2	0.3	0.4	0.1	0.404	0.406	0.538
0.6	0.7	0.8	0.9	0.1	1.000	0.915	0.587
0.1	0.4	0.5	0.7	0.1	0.830	0.058	0.814
0.1	0.3	0.5	0.7	0.05	0.981	0.998	0.949
0.1	0.2	0.6	0.7	0.05	0.053	0.265	0.924
0.1	0.2	0.3	0.4	0.05	0.112	0.996	0.472
0.6	0.7	0.8	0.9	0.05	0.998	0.546	0.861
0.1	0.4	0.5	0.7	0.05	0.967	0.050	0.142

ery notch filter simultaneously. This ill-conditioning degrades the performance of a multiple notch filter as its bandwidths increases. The stability margin, i.e., the minimum distance from the pole to the perimeter of a unit circle, is directly proportional to the bandwidth. The value of the stability margin decreases as the value of bandwidth increases. Thus another advantage of the proposed technique is that the stability margin of the algorithm is based on all-pass filter design, which is adjustable and can be made greater than those of Refs. 4–6 with minimal error increment.

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