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On the minimum skew rank of graphs

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ABSTRACT: The minimum skew rank $\mathrm{mr}^-(\mathbb{F},G)$ of a graph G over a field $\mathbb{F}$ is the smallest possible rank among all skew symmetric matrices over $\mathbb{F}$ whose (i,j)th entry (for $i \neq j$) is non-zero whenever ij is an edge in G and is zero otherwise. We characterize the graphs G with cut vertices over an infinite field $\mathbb{F}$ such that $\mathrm{mr}^-(\mathbb{F},G) = 4$ determine the minimum skew rank of k-paths over a field $\mathbb{F}$, and show that $\mathrm{mr}^-(\mathbb{F},G) = 2\beta(G) = \mathrm{MR}^-(\mathbb{F},G)$ for a connected graph G with no even cycles and a field $\mathbb{F}$ where $\beta(G)$ is the matching number of G, and $\mathrm{MR}^-(\mathbb{F},G)$ is the largest possible rank among all skew symmetric matrices over $\mathbb{F}$.

KEYWORDS: skew-symmetric matrix, k-tree, k-path, zero forcing number, perfect matching

INTRODUCTION

We consider only simple graphs. Let G be a graph with vertex set V_G and edge set E_G. Let $\mathbb{F}$ be a field. An $n \times n$ matrix A over $\mathbb{F}$ is skew-symmetric (respectively, symmetric) if $A^{\mathrm{T}} = -A$ (respectively, $A^{\mathrm{T}} = A$), where A^{T} denotes the transpose of A. For an $n \times n$ symmetric or skew-symmetric matrix A, the graph of A, denoted $G(A)$, is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_iv_j : a_{ij} \neq 0, 1 \leqslant i < j \leqslant n\}$.

The minimum skew rank problem involves skew symmetric matrices and its study began recently[1]. If the characteristic of $\mathbb{F}$ is 2, then a skew-symmetric matrix over $\mathbb{F}$ is also symmetric. Thus it is assumed throughout this paper that the characteristic of $\mathbb{F}$ is not 2.

For a field $\mathbb{F}$ and a graph G, let $S^-(\mathbb{F},G) = \{A \in \mathbb{F}^{n\times n} : A^{\mathrm{T}} = -A,\ G(A) = G\}$ be the set of skew-symmetric matrices over $\mathbb{F}$ described by G. The minimum skew rank of G over $\mathbb{F}$, denoted by $\mathrm{mr}^-(\mathbb{F},G)$ is defined as the minimum rank of matrices in $S^-(\mathbb{F},G)$, and the corresponding maximum skew nullity of G, denoted by $M^-(\mathbb{F},G)$, is defined as the maximum nullity of matrices in $S^-(\mathbb{F},G)$. Obviously, $\mathrm{mr}^-(\mathbb{F},G) + M^-(\mathbb{F},G) = |V_G|$.

Let K_n be the complete graph with n vertices, and $K_{n_1,n_2,\dots,n_t}$ the complete t-partite graph with n_i vertices in the ith partite sets for $i = 1, 2, \dots, t$.

Note that the rank of a skew-symmetric matrix over $\mathbb{F}$ is always even. Thus $\mathrm{mr}^-(\mathbb{F},G)$ is even for any field $\mathbb{F}$ and any graph G. As observed in Ref. 1, $\mathrm{mr}^-(\mathbb{F},G) = 0$ if and only if G is an empty graph. If $\mathbb{F}$ is infinite and G is a connected graph with at least two vertices, then $\mathrm{mr}^-(\mathbb{F},G) = 2$ if and only if G is a complete multipartite graph $K_{n_1,n_2,\dots,n_t}$ for some $t \geqslant 2$, $n_i \geqslant 1$ for $i = 1, \dots, t$. An open question (Question 5.2) was posed in Ref. 1 to characterize the graphs G such that $\mathrm{mr}^-(\mathbb{F},G) = 4$.

The kth power G^k of a graph G is the graph whose vertex set is V_G, two distinct vertices being adjacent in G^k if and only if their distance in G is at most k. Let $P_n = v_1v_2\dots v_n$ be the path on n vertices. The minimum skew rank of the kth power of a path over the real field $\mathbb{R}$ was determined in Ref. 2.

The class of k-trees is defined recursively as follows[3]: (i) The complete graph K_{k+1} is a k-tree; (ii) a k-tree G with $n+1$ vertices ($n \geqslant k+1$) can be constructed from a k-tree H on n vertices by adding a vertex adjacent to all vertices of a k-clique of H. A k-path is a k-tree which is either K_{k+1} or has exactly two vertices of degree k.

The maximum skew rank of a graph G over a field $\mathbb{F}$, denoted by $\mathrm{MR}^-(\mathbb{F},G)$, is defined as the maximum rank of matrices in $S^-(\mathbb{F},G)$. Let $\beta(G)$ be the matching number of G. It was shown in Ref. 1 that

$$\mathrm{mr}^-(\mathbb{F},G) = 2\beta(G) = \mathrm{MR}^-(\mathbb{F},G) \qquad (1)$$

for a tree G and a field $\mathbb{F}$.

In this paper, we characterize the graphs G with cut vertices over an infinite field $\mathbb{F}$ such that $\mathrm{mr}^-(\mathbb{F},G) = 4$, determine the minimum skew rank of k-paths over a field $\mathbb{F}$, from which we also deduce the minimum skew rank of the kth power of a path over the real field $\mathbb{R}$, and show that (1) holds for a connected graph G with no even cycles and a field $\mathbb{F}$.

PRELIMINARIES

Let G be a graph. For $v \in V_G$, $G - v$ denotes the graph obtained from G by deleting vertex v (and all edges incident with v). For $X \subseteq V_G$, $G[X]$ denotes the subgraph of G induced by vertices in X. We need the following lemmas established in Ref. 1.

Lemma 1 *Let G be a connected graph with at least two vertices and let $\mathbb{F}$ be an infinite field. Then $\mathrm{mr}^-(\mathbb{F}, G) = 2$ if and only if G is a complete multipartite graph.*

For a field $\mathbb{F}$ and a graph G with $v \in V_G$, let $r_v^-(\mathbb{F}, G) = \mathrm{mr}^-(\mathbb{F}, G) - \mathrm{mr}^-(\mathbb{F}, G - v)$. The union of graphs G_i, $i = 1, 2, \ldots, h$, denoted by $\cup_{i=1}^h G_i$, is the graph with vertex set $\cup_{i=1}^h V_{G_i}$ and edge set $\cup_{i=1}^h E_{G_i}$.

Lemma 2 *Let G be a graph with cut vertex v and let $\mathbb{F}$ be a field where $G = \cup_{i=1}^h G_i$ and $\cap_{i=1}^h V_{G_i} = \{v\}$. Then $\mathrm{mr}^-(\mathbb{F}, G) = \sum_{i=1}^h \mathrm{mr}^-(\mathbb{F}, G_i - v) + \min\{\sum_{i=1}^h r_v^-(\mathbb{F}, G_i), 2\}$.*

Lemma 3 *Let G be a graph and let $\mathbb{F}$ be an infinite field. If $G = G_1 \cup G_2$ then $\mathrm{mr}^-(\mathbb{F}, G) \leqslant \mathrm{mr}^-(\mathbb{F}, G_1) + \mathrm{mr}^-(\mathbb{F}, G_2)$.*

Let G be a graph. A subset $Z \subseteq V_G$ defines an initial colouring by colouring all vertices in Z black and all the vertices outside Z white. The colour change rule says: if a black vertex u has exactly one white neighbour v, then change the colour of v to black. In this case we write $u \to v$. The derived set of an initial colouring Z is the set of vertices coloured black until no more changes are possible. A zero forcing set is a subset $Z \subseteq V_G$ such that the derived set of Z is V_G. The zero forcing number of G, denoted by $Z(G)$, is the minimum size of a zero forcing set of G.

Lemma 4 *Let G be a graph and $\mathbb{F}$ a field. Then $M^-(\mathbb{F}, G) \leqslant Z(G)$.*

Lemma 5 *Let G be a graph and $\mathbb{F}$ a field. Then $\mathrm{MR}^-(\mathbb{F}, G) = 2\beta(G)$.*

Lemma 6 *Let G be a graph and $\mathbb{F}$ a field. If H is an induced subgraph of G, $\mathrm{mr}^-(\mathbb{F}, H) \leqslant \mathrm{mr}^-(\mathbb{F}, G)$.*

Lemma 7 *Let G be a graph with a unique perfect matching and $\mathbb{F}$ a field. Then $\mathrm{mr}^-(\mathbb{F}, G) = |V_G|$.*

RESULTS

First we give a characterization of the graphs G with cut vertices over an infinite field $\mathbb{F}$ such that $\mathrm{mr}^-(\mathbb{F}, G) = 4$.

Theorem 1 *Let G be a graph with cut vertex v and $\mathbb{F}$ an infinite field. Then $\mathrm{mr}^-(\mathbb{F}, G) = 4$ if and only if one of the following conditions holds:*
(i) *$G = G_1 \cup G_2$ and $V_{G_1} \cap V_{G_2} = \{v\}$, where G_1, G_2 are complete multipartite graphs such that $G_1 - v$, $G_2 - v$ are nonempty.*
(ii) *$G - v$ consists of a complete multipartite component and isolated vertices.*

Proof: Suppose that $\mathrm{mr}^-(\mathbb{F}, G) = 4$. Let p be the number of complete multipartite components, and let q be the number of isolated vertices in $G - v$. Let m be the number of the remaining components. Note that the minimum skew rank of a graph that is neither a complete multipartite graph nor an empty graph is larger than 4.

Case 1. $q = 0$. By Lemma 2, $4 = \mathrm{mr}^-(\mathbb{F}, G) \geqslant 2p + 4m$. If $m = 1$, then $p = 0$, a contradiction to the fact that v is a cut vertex of G. Thus $m = 0$, implying that $p = 2$. Let W_1 and W_2 be the vertex sets of the two complete multipartite components of $G - v$ and let G_1 and G_2 be the subgraphs induced by $\{v\} \cup W_1$ and $\{v\} \cup W_2$, respectively. By Lemma 1, $\mathrm{mr}^-(\mathbb{F}, G_1 - v) = \mathrm{mr}^-(\mathbb{F}, G_2 - v) = 2$. By Lemma 2, $4 = \mathrm{mr}^-(\mathbb{F}, G) = \mathrm{mr}^-(\mathbb{F}, G_1 - v) + \mathrm{mr}^-(\mathbb{F}, G_2 - v) + \min\{r_v^-(\mathbb{F}, G_1) + r_v^-(\mathbb{F}, G_2), 2\} = 2 + 2 + \min\{r_v^-(\mathbb{F}, G_1) + r_v^-(\mathbb{F}, G_2), 2\}$. Then $r_v^-(\mathbb{F}, G_1) = r_v^-(\mathbb{F}, G_2) = 0$. Thus $\mathrm{mr}^-(\mathbb{F}, G_1) = \mathrm{mr}^-(\mathbb{F}, G_2) = 2$. By Lemma 1, G_1 and G_2 are complete multipartite graphs, and then (i) follows.

Case 2. $q \neq 0$. Note that $r_v^-(\mathbb{F}, K_2) = 2$. By Lemma 2, $4 = \mathrm{mr}^-(\mathbb{F}, G) \geqslant 2p + 4m + 2$. Then $m = 0$ and $p = 1$, and thus (ii) follows.

Now suppose that (i) holds. Note that $G_i - v$ is still a complete multipartite graph for $i = 1, 2$. By Lemma 1, $\mathrm{mr}^-(\mathbb{F}, G_1) = \mathrm{mr}^-(\mathbb{F}, G_2) = \mathrm{mr}^-(\mathbb{F}, G_1 - v) = \mathrm{mr}^-(\mathbb{F}, G_2 - v) = 2$. Then $r_v^-(\mathbb{F}, G_1) + r_v^-(\mathbb{F}, G_2) = 0$. Thus by Lemma 2, $\mathrm{mr}^-(\mathbb{F}, G) = \mathrm{mr}^-(\mathbb{F}, G_1 - v) + \mathrm{mr}^-(\mathbb{F}, G_2 - v) + \min\{0, 2\} = 4$.

Next suppose that (ii) holds. Let W be the unique complete multipartite component, and let a be the number of isolated vertices in $G - v$. By Lemma 1, $\mathrm{mr}^-(\mathbb{F}, W) = 2$. Note that $r_v^-(\mathbb{F}, K_2) = 2$. Then by Lemma 2, $\mathrm{mr}^-(\mathbb{F}, G) = \mathrm{mr}^-(\mathbb{F}, W) + a \cdot \mathrm{mr}^-(\mathbb{F}, K_1) + 2 = 2 + 0 + 2 = 4$. □

Now we consider the minimum skew rank of k-paths. Note that a k-tree with at least $k+2$ vertices has at least two vertices of degree k and any two vertices of degree k are not adjacent. The following lemma follows directly from the definition of a k-path.

Lemma 8 *Let G be a k-path with at least $k+2$ vertices, and let v be a vertex of G with degree k. Then $G-v$ is also a k-path.*

Let G be a k-path with $n \geqslant k+2$ vertices. By Lemma 8, the vertices of G may be labelled as follows: choose a vertex of degree k, label it as v_n, and label its unique neighbour of degree $k+1$ in G as v_{n-1}. Then v_{n-1} is a vertex of degree k in the k-path $G-v_n$. Repeating the process above, we may label $n-k+1$ vertices of G as $v_n, v_{n-1}, \dots, v_{k+2}$. Obviously, $G - v_n - v_{n-1} - \cdots - v_{k+2} = K_{k+1}$ and it contains a vertex of degree k in G, which is labelled as v_1, and the remaining vertices are labelled as v_2, $v_3, \dots, v_{k+1}$ such that v_2 is the unique neighbour of v_1 with degree $k+1$ in G. Note that in our labelling, v_i is not adjacent to $v_{j+1}, v_{j+2}, \dots, v_n$ if v_i is not adjacent to v_j for $j \geqslant \max\{i+1, k+2\}$. Recall that a k-tree is a chordal graph. The above labelling is the 'perfect elimination' labelling inherent to chordal graphs[4].

Theorem 2 *Let G be a k-path on n vertices and $\mathbb{F}$ an infinite field. Then*

$$\mathrm{mr}^-(\mathbb{F}, G) = \begin{cases} n-k, & \text{if } n-k \text{ is even}, \\ n-k+1, & \text{if } n-k \text{ is odd}. \end{cases}$$

Proof: First we show

$$\mathrm{mr}^-(\mathbb{F}, G) \leqslant \begin{cases} n-k, & \text{if } n-k \text{ is even}, \\ n-k+1, & \text{if } n-k \text{ is odd} \end{cases} \tag{2}$$

by induction on n. If $n = k+1$, then $G = K_{k+1}$, which is a complete multipartite graph, and thus by Lemma 1, $\mathrm{mr}^-(\mathbb{F}, G) = 2 = n-k+1$. If $n = k+2$, then $G = K_{k+2} - e$ is also a complete multipartite graph, where $e \in E_{K_{k+2}}$, and thus by Lemma 1, $\mathrm{mr}^-(\mathbb{F}, G) = 2 = n-k$. Thus (2) is true for $n = k+1$, $k+2$. Suppose that $n \geqslant k+3$ and for a k-path H on m vertices with $k+1 \leqslant m \leqslant n-1$ we have

$$\mathrm{mr}^-(\mathbb{F}, H) \leqslant \begin{cases} m-k, & \text{if } m-k \text{ is even}, \\ m-k+1, & \text{if } m-k \text{ is odd}. \end{cases}$$

Let G be a k-path on n vertices. Let $G_1 = G[\{v_1, v_2, \dots, v_{k+2}\}]$ and $G_2 = G[\{v_3, v_4, \dots, v_n\}]$. Then G_1 is a k-path on $k+2$ vertices, and G_2 is a k-path on $n-2$ vertices. Obviously, $\mathrm{mr}^-(\mathbb{F}, G_1) = 2$, and by the induction hypothesis,

$$\mathrm{mr}^-(\mathbb{F}, G_2) \leqslant \begin{cases} n-k-2, & \text{if } n-k-2 \text{ is even}, \\ n-k-1, & \text{if } n-k-2 \text{ is odd}, \end{cases}$$

i.e.,

$$\mathrm{mr}^-(\mathbb{F}, G_2) \leqslant \begin{cases} n-k-2, & \text{if } n-k \text{ is even}, \\ n-k-1, & \text{if } n-k \text{ is odd}. \end{cases}$$

Note that $G = G_1 \cup G_2$. By Lemma 3,

$$\begin{aligned} \mathrm{mr}^-(\mathbb{F}, G) &\leqslant \mathrm{mr}^-(\mathbb{F}, G_1) + \mathrm{mr}^-(\mathbb{F}, G_2) \\ &\leqslant 2 + \begin{cases} n-k-2, & \text{if } n-k \text{ is even}, \\ n-k-1, & \text{if } n-k \text{ is odd} \end{cases} \\ &= \begin{cases} n-k, & \text{if } n-k \text{ is even}, \\ n-k+1, & \text{if } n-k \text{ is odd}. \end{cases} \end{aligned}$$

This proves (2).

Next we show the reverse of (2) holds. Let $Z = \{v_1, v_2, \dots, v_k\}$. Colour all vertices in Z black and all the vertices outside Z white. We will show that Z is a zero forcing set of G. Since all neighbours of v_1 that differ from v_{k+1} are black, we have $v_1 \to v_{k+1}$. Note that v_2 is adjacent to v_{k+2} but not adjacent to $v_{k+3}, v_{k+4}, \dots, v_n$. Since all neighbours of v_2 which differ from v_{k+2} are black, we have $v_2 \to v_{k+2}$. Let $G_1 = G[\{v_1, v_2, \dots, v_{k+3}\}]$ and $G_2 = G[\{v_1, v_2, \dots, v_{k+4}\}]$. If each neighbour of v_{k+3} in G_1 is adjacent to v_{k+4} in G, then v_{k+4} is of degree $k+1$ in G_2, a contradiction. Thus there is a neighbour, say w, of v_{k+3} in G_1 such that $wv_{k+4} \notin E_G$, and then $wv_i \notin E_G$ for $i \geqslant k+5$, implying that $w \to v_{k+3}$. Repeating the process above, we may finally colour all vertices of G black. Thus Z is a zero forcing set of G. By Lemma 4, $M^-(\mathbb{F}, G) \leqslant Z(G) \leqslant k$, and then $\mathrm{mr}^-(\mathbb{F}, G) = n - M^-(\mathbb{F}, G) \geqslant n-k$. Note that the rank of a skew-symmetric matrix is even. It follows that

$$\mathrm{mr}^-(\mathbb{F}, G) \geqslant \begin{cases} n-k, & \text{if } n-k \text{ is even}, \\ n-k+1, & \text{if } n-k \text{ is odd}, \end{cases}$$

as desired. □

Obviously, P_n^k is a complete graph if $k \geqslant n$. Suppose that $k \leqslant n-1$. Obviously, $P_n^k[\{v_1, v_2, \dots, v_{k+1}\}] = K_{k+1}$, and if $k \leqslant n-2$, then for $j = 2, 3, \dots, n-k$, $P_n^k[\{v_j, v_{j+1}, \dots, v_{k+j-1}\}] = K_k$, and v_{k+j} is adjacent to $v_j, v_{j+1}, \dots, v_{k+j-1}$. Thus P_n^k is a k-path. Now by Lemma 1 and Theorem 2, we have the following result, which has been proved in Ref. 2 (when $\mathbb{F}$ is the real field $\mathbb{R}$).

Corollary 1 *Let $\mathbb{F}$ be an infinite field. Then*

$$\operatorname{mr}^-(\mathbb{F}, P_n^k) = \begin{cases} n-k, & 1 \leqslant k \leqslant n-1 \text{ and } n-k \text{ is even}, \\ n-k+1, & 1 \leqslant k \leqslant n-1 \text{ and } n-k \text{ is odd}, \\ 2, & k \geqslant n. \end{cases}$$

From Ref. 1, (1) holds if G is a tree (a connected graph with no cycles). Now we make a minor extension.

Theorem 3 *Let G be a connected graph with no even cycles and let $\mathbb{F}$ be a field. Then (1) holds.*

Proof: By Lemma 5, $\operatorname{mr}^-(\mathbb{F}, G) \leqslant \operatorname{MR}^-(\mathbb{F}, G) = 2\beta(G)$. Let M be a maximum matching of G and $\{v_1, \dots, v_k\}$, the vertices in M. Then M is a perfect matching of $H = G[\{v_1, \dots, v_k\}]$. This perfect matching is unique. Otherwise, the graph induced by the vertices of the symmetric difference of two (different) perfect matchings of H consists of even cycles, which is impossible because G contains no even cycles. By Lemma 6 and Lemma 7, $\operatorname{mr}^-(\mathbb{F}, G) \geqslant \operatorname{mr}^-(\mathbb{F}, H) = 2\beta(G)$. Then the result follows. □

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