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Characterization of regular medial algebras

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ABSTRACT: In this paper we consider the algebra (A, f, g), where f and g are m-ary and n-ary operations on the set A. We define the concept of regular algebra and we prove that for a regular medial algebra (A, f, g) there exists a commutative semigroup $(A, +)$ such that the operations f and g have linear representations on $(A, +)$. As a corollary, we show that for a regular medial algebra (A, F) there exists a commutative semigroup $(A, +)$ such that any operation $f \in F$ has a linear representation on $(A, +)$.

KEYWORDS: mode, medial groupoid, medial hyperidentity

INTRODUCTION

An algebra $A = (A, F)$ (without nullary operations) is called medial if it satisfies the identity of mediality,

$$g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm})) = f(g(x_{11}, \dots, x_{1m}), \dots, g(x_{n1}, \dots, x_{nm})), \quad (1)$$

for every n-ary $f \in F$ and m-ary $g \in F$ [1]. The n-ary operation f is called idempotent if $f(x, \dots, x) = x$ for every $x \in A$. The algebra $A = (A, F)$ is called idempotent if every operation $f \in F$ is idempotent. An idempotent medial algebra is a mode[2]. In other words, the algebra A is medial if it satisfies the hyperidentity of mediality[3]. The medial property was studied initially in Refs. 4, 5. Note that a groupoid is medial iff it satisfies the identity of mediality[6], $xy.uv \approx xu.yv$. Medial groupoids were studied by many authors. Important results on some regular medial groupoids can be found in Refs. 7–9.

Let g and f be m-ary and n-ary operations on the set A. We say that the pair of operations (f, g) is medial (entropic), if identity (1) holds in the algebra $A = (A, f, g)$ [10]. Characterization of a medial pair of binary quasigroup operations is done in Ref. 11 (also see Ref. 12).

Let $A = (A, F)$ be an algebra and let $f \in F$. We say that the element e is the unit for the operation $f \in F$ if

$$f(x, e, \dots, e) = f(e, x, e, \dots, e) = \dots = f(e, \dots, e, x) = x$$

for every $x \in A$. The element e is a unit for the algebra (A, F) if it is a unit for every operation $f \in F$.

The element e is idempotent for the operation f if $f(e, \dots, e) = e$. We say that the element e is idempotent for the algebra (A, F) if it is idempotent for the every operation $f \in F$.

The sequence $x_i, \dots, x_j$ will be denoted by x_i^j, where i and j are natural numbers. For $j < i$, x_i^j is the empty symbol. If $x_{i+1} = \dots = x_{i+k} = x$ then instead of x_{i+1}^{i+k} we will write $(\bar{x})^k$. For $k \leqslant 0$, $(\bar{x})^k$ is the empty symbol.

Definition 1 Let (f, g) be a pair of m-ary and n-ary operations of the algebra (A, F). For any element e of A, let $\alpha_1, \dots, \alpha_m$ be mappings of A into A defined by

$$\alpha_i : x \mapsto f((\bar{e})^{i-1}, x, (\bar{e})^{m-i}). \quad (2)$$

We call α_i the *ith translation by e with respect to f*. An element e is called *i-regular with respect to f* if α_i is a bijection. An element e is called i-regular for the pair operation (f, g) if it is i-regular with respect to both the operations f and g. The element e is called i-regular for the algebra (A, F) if it is an i-regular element for every operation $f \in F$.

There exist various algebraic characterizations of different classes of n-ary operations (see for example Refs. 13, 14). In this paper we consider the medial algebra (A, f, g) with m-ary operation f and n-ary operation g where $f \neq g$. If $f = g$ then the algebra (A, f, g) is an n-ary medial groupoid. Characterization of an n-ary medial groupoid is done by Evans in Ref. 15. As a generalization of Evans's results, we have the following representation of a medial algebra, which was obtained by Cho in Ref. 16.

Theorem 1 *Let (A, f, g) be a medial algebra with the idempotent element e which is an i- and j-regular element of (A, f, g) for fixed i and j $(i \neq j)$. Then there exists a commutative semigroup $(A, +)$ with the unit element e such that operations f, $g \in F$ have the following linear representation*

$$f(x_1^m) = \alpha_1 x_1 + \cdots + \alpha_m x_m,$$
$$g(x_1^n) = \beta_1 x_1 + \cdots + \beta_n x_n,$$

where α_1^m, β_1^n are pairwise commuting endomorphisms of $(A, +)$, $n \geqslant m \geqslant 2$. Furthermore, $\alpha_i, \alpha_j, \beta_i, \beta_j$ are automorphisms of $(A, +)$.

Corollary 1 *Let (A, F) be a medial algebra with the idempotent element e which is an i- and j-regular element of (A, F) for fixed i and j $(i \neq j)$. Then there exists a commutative semigroup $(A, +)$ with the unit element e such that every operation $f \in F$ has the linear representation*

$$f(x_1^m) = \gamma_1 x_1 + \cdots + \gamma_m x_m,$$

where γ_1^m are pairwise commuting endomorphisms of $(A, +)$, $m \geqslant 2$. Furthermore, γ_i, γ_j are automorphisms.

Proof: Let $f_0 \in F$ be an m-ary fixed operation. Then (A, f_0) is a medial algebra. Hence by Theorem 3.2 in Ref. 15, there exists a commutative semigroup $(A, +)$ with the unit element e such that

$$f_0(x_1^m) = \alpha_1 x_1 + \cdots + \alpha_m x_m$$

where α_1^m are translations defined in (2), and

$$x + y = f_0((\bar{e})^{i-1}, \alpha_i^{-1} x, (\bar{e})^{j-1-i}, \alpha_j^{-1} y, (\bar{e})^{m-j}).$$

Now, for any $f \in F$ the algebra (A, f_0, f) is a medial algebra with the idempotent element e which is an i- and j-regular element of the algebra (A, f_0, f) for fixed i and j $(i \neq j)$. Hence by the previous theorem and mediality of the pair operation (f_0, f) we have

$$f((\bar{e})^{i-1}, \gamma_i^{-1} x, (\bar{e})^{j-1-i}, \gamma_j^{-1} y, (\bar{e})^{m-j}) \\ = f_0((\bar{e})^{i-1}, \alpha_i^{-1} x, (\bar{e})^{j-1-i}, \alpha_j^{-1} y, (\bar{e})^{n-j}).$$

Hence there exists a commutative semigroup $(A, +)$ with the unit element e such that every operation $f \in F$ has the linear representation on the commutative semigroup $(A, +)$. □

PRELIMINARY RESULTS

In Corollary 1 we have described the structure of the medial algebra (A, F) containing an idempotent element. The purpose of this section is to obtain sufficient properties of finite medial algebras to enable us to weaken considerably, in this finite case, the assumptions we need for characterizing regular medial algebras which do not contain an idempotent element.

Definition 2 Let f be an m-ary operation and J be a non-empty subset of $\{1, 2, \ldots, m\}$. We will say that the element e is J-*regular* with respect to the operation f if e is a j-regular element with respect to f for all $j \in J$. The element e is a J-regular element for the algebra (A, F) if e is a j-regular element with respect to every $f \in F$ for all $j \in J$.

Lemma 1 *Let (A, f, g) be a finite medial algebra with the m-ary operation f and the n-ary operation g and a J-regular element e ($J \subseteq \{1, 2, \ldots, m\}$ and $m \leqslant n$). If*

$$f(a_1^m) = g(a_1^n) = e,$$

then for each $i \in J$, a_i is a J-regular element of (A, f, g).

Proof: Suppose $i, j \in J$ and $x, y \in A$ such that

$$f((\bar{a}_j)^{i-1}, x, (\bar{a}_j)^{m-i}) = f((\bar{a}_j)^{i-1}, y, (\bar{a}_j)^{m-i}).$$

Then we have

$$g(f((\bar{a}_1)^{i-1}, e, (\bar{a}_1)^{m-i}), \ldots, f((\bar{a}_j)^{i-1}, x, (\bar{a}_j)^{m-i}), \\ \ldots, f((\bar{a}_n)^{i-1}, e, (\bar{a}_n)^{m-i})) = \\ g(f((\bar{a}_1)^{i-1}, e, (\bar{a}_1)^{m-i}), \ldots, f((\bar{a}_j)^{i-1}, y, (\bar{a}_j)^{m-i}), \\ \ldots, f((\bar{a}_n)^{i-1}, e, (\bar{a}_n)^{m-i})),$$

with $f((\bar{a}_j)^{i-1}, x, (\bar{a}_j)^{m-i})$, $f((\bar{a}_j)^{i-1}, y, (\bar{a}_j)^{m-i})$ at the jth places. Hence by mediality we have

$$f(g(a_1^n), \ldots, g((\bar{e})^{j-1}, x, (\bar{e})^{n-j}), \ldots, g(a_1^n)) \\ = f(g(a_1^n), \ldots, g((\bar{e})^{j-1}, y, (\bar{e})^{n-j}), \ldots, g(a_1^n)).$$

Hence

$$f((\bar{e})^{i-1}, g((\bar{e})^{j-1}, x, (\bar{e})^{n-j}), (\bar{e})^{m-i}) \\ = f((\bar{e})^{i-1}, g((\bar{e})^{j-1}, y, (\bar{e})^{n-j}), (\bar{e})^{m-i}).$$

Thus by considering the regularity of the element e we have $x = y$. Similarly, if

$$g((\bar{a}_j)^{i-1}, x, (\bar{a}_j)^{n-i}) = g((\bar{a}_j)^{i-1}, y, (\bar{a}_j)^{n-i}),$$

then $x = y$. Since A is finite, this concludes the proof. □

Lemma 2 *Let (A, f, g) be a finite medial algebra with the m-ary operation f and the n-ary operation g ($m \leqslant n$). If e is an i-regular ($1 \leqslant i \leqslant m$) element of the algebra (A, f, g), then so are $f((\bar{e})^m)$ and $g((\bar{e})^n)$.*

Proof: Suppose $x, y \in A$ such that

$$g((\overline{f((\bar{e})^m)})^{i-1}, x, (\overline{f((\bar{e})^m)})^{n-i}) = g((\overline{f((\bar{e})^m)})^{i-1}, y, (\overline{f((\bar{e})^m)})^{n-i}),$$

and t_1, t_2 are elements of A satisfying

$$f((\bar{e})^{i-1}, t_1, (\bar{e})^{m-i}) = g((\bar{e})^{i-1}, t_2, (\bar{e})^{n-i}) = e.$$

By Lemma 1 t_1, t_2 are i-regular elements for the algebra (A, f, g). Hence if

$$f((\bar{t}_2)^{i-1}, x_1, (\bar{t}_2)^{m-i}) = x,$$
$$f((\bar{t}_2)^{i-1}, y_1, (\bar{t}_2)^{m-i}) = y,$$

then we have

$$g((\overline{f((\bar{e})^m)})^{i-1}, f((\bar{t}_2)^{i-1}, x_1, (\bar{t}_2)^{m-i}), (\overline{f((\bar{e})^m)})^{n-i}) = g((\overline{f((\bar{e})^m)})^{i-1}, f((\bar{t}_2)^{i-1}, y_1, (\bar{t}_2)^{m-i}), (\overline{f((\bar{e})^m)})^{n-i}).$$

Hence by mediality we have

$$f((\overline{g((\bar{e})^{i-1}, t_2, (\bar{e})^{n-i})})^{i-1}, g((\bar{e})^{i-1}, x_1, (\bar{e})^{n-i}), (\overline{g((\bar{e})^{i-1}, t_2, (\bar{e})^{n-i})})^{m-i}) = f((\overline{g((\bar{e})^{i-1}, t_2, (\bar{e})^{n-i})})^{i-1}, g((\bar{e})^{i-1}, y_1, (\bar{e})^{n-i}), (\overline{g((\bar{e})^{i-1}, t_2, (\bar{e})^{n-i})})^{m-i}).$$

Since e is an i-regular element of (A, f, g), $x_1 = y_1$. Hence $x = y$. Since A is finite, this implies $f((\bar{e})^m)$ is an i-regular element with respect to the operation g. Similarly, $f((\bar{e})^m)$ is an i-regular element with respect to the operation f, and $g((\bar{e})^n)$ is an i-regular element with respect to operations f and g. □

Lemma 3 *Let (A, f, g) be a finite medial algebra with the m-ary operation f and the n-ary operation g ($m \leqslant n$) and let*

$$a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m, \dots, a_n$$

be J-regular ($J \subseteq \{1, 2, \dots, m\}$) elements of the algebra (A, f, g) where J contains at least two elements. Then, for every $b \in A$, there are unique $x_1, x_2 \in A$ such that

$$f(a_1^{i-1}, x_1, a_{i+1}^m) = b, \quad g(a_1^{i-1}, x_2, a_{i+1}^n) = b.$$

Proof: Suppose $x_1, y_1 \in A$ such that

$$f(a_1^{i-1}, x_1, a_{i+1}^m) = f(a_1^{i-1}, y_1, a_{i+1}^m).$$

We will prove that $x_1 = y_1$. Let

$$g((\bar{a}_k)^{j-1}, t_k, (\bar{a}_k)^{n-j}) = a_1,$$

for $k = 1, \dots, i-1, i+1, \dots, m$. Then we have

$$g(f(a_1^{i-1}, a_1, a_{i+1}^m), \dots, f(a_1^{i-1}, x_1, a_{i+1}^m), \dots, f(t_1^{i-1}, a_1, t_{i+1}^m), \dots, f(a_1^{i-1}, a_1, a_{i+1}^m)) = g(f(a_1^{i-1}, a_1, a_{i+1}^m), \dots, f(a_1^{i-1}, y_1, a_{i+1}^m), \dots, f(t_1^{i-1}, a_1, t_{i+1}^m), \dots, f(a_1^{i-1}, a_1, a_{i+1}^m)).$$

Hence by mediality, we have

$$f(g((\bar{a}_1)^{j-1}, t_1, (\bar{a}_1)^{n-j}), \dots, g((\bar{a}_1)^{j-1}, x_1, (\bar{a}_1)^{n-j}), \dots, g((\bar{a}_m)^{j-1}, t_m, (\bar{a}_m)^{n-j})) = f(g((\bar{a}_1)^{j-1}, t_1, (\bar{a}_1)^{n-j}), \dots, g((\bar{a}_1)^{j-1}, y_1, (\bar{a}_1)^{n-j}), \dots, g((\bar{a}_m)^{j-1}, t_m, (\bar{a}_m)^{n-j})).$$

Hence

$$f((\bar{a}_1)^{i-1}, g((\bar{a}_1)^{i-1}, x_1, (\bar{a}_1)^{n-i}), (\bar{a}_1)^{m-i}) = f((\bar{a}_1)^{i-1}, g((\bar{a}_1)^{i-1}, y_1, (\bar{a}_1)^{n-i}), (\bar{a}_1)^{m-i}).$$

Two applications of the i-regularity of a_1 yield $x_1 = y_1$. Similarly, if

$$g(a_1^{i-1}, x_2, a_{i+1}^n) = g(a_1^{i-1}, y_2, a_{i+1}^n)$$

then $x_2 = y_2$. □

From the above lemmas we know that if the finite medial algebra (A, f, g) (f is m-ary and g are n-ary operations) contains an element e which is an i- and j-regular element of (A, f, g), then:

(i) $f((\bar{e})^m)$ and $g((\bar{e})^n)$ are also i- and j-regular elements for (A, f, g);

(ii) there are unique elements $t_1, t_2 \in A$ which are i- and j-regular for (A, f, g) such that (for $i < j$)

$$f((\bar{e})^{i-1}, f((\bar{e})^m), (\bar{e})^{j-1-i}, t_1, (\bar{e})^{m-j}) = e,$$
$$g((\bar{e})^{i-1}, g((\bar{e})^n), (\bar{e})^{j-1-i}, t_2, (\bar{e})^{n-j}) = e;$$

(iii) for every $b \in A$, there are unique elements $x_1, x_2 \in A$ such that

$$f((\bar{e})^{i-1}, x_1, (\bar{e})^{j-1-i}, t_1, (\bar{e})^{m-j}) = b,$$
$$g((\bar{e})^{i-1}, x_2, (\bar{e})^{j-1-i}, t_2, (\bar{e})^{n-j}) = b,$$

where t_1, t_2 are the elements described in (ii).

It is easy to prove that in the finite medial algebra (A, f, g) the set of J-regular elements (where $J \subseteq \{1, 2, \dots, m\}$ contains at least two elements) is closed under the operations f, g. If the finite medial algebra (A, f, g) contains at least one J-regular element, then the algebra (A, f, g) contains a J-regular subset of J-regular elements of the algebra (A, f, g) which is closed under the operations f, g.

CHARACTERIZATION OF REGULAR MEDIAL ALGEBRAS

We now discuss the structure of a medial algebra (A, f, g) which does not contain an idempotent element. We construct new operations f^*, g^* on A in terms of f, g, such that (f^*, g^*) is a medial pair operation and the medial algebra (A, f^*, g^*) contains an idempotent element. If certain regularity conditions are assumed for (A, f, g) then this idempotent element is also a J-regular element in (A, f^*, g^*) and hence we are able to use Theorem 1 to describe the structure of the pair operation (f^*, g^*).

Definition 3 Let $f, g \in F$ be m-ary and n-ary operations ($m \leqslant n$), $J \subseteq \{1, 2, \dots, m\}$ (where J contains at least two elements) and

$$a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m, \dots, a_n$$

are fixed J-regular elements of the algebra (A, f, g). The pair operation (f, g) is an (i, J)-*regular pair operation* (where $i \in J$) if for every $x \in A$ we have

$$f(a_1^{i-1}, x, a_{i+1}^m) = g(a_1^{i-1}, x, a_{i+1}^n). \tag{3}$$

The pair operation (f, g) is a J-*regular pair operation* if (f, g) is (i, J)-regular for every $i \in J$. The pair operation (f, g) is a *regular pair operation* if a (f, g) is J-regular pair operation for some $J \subseteq \{1, 2, \dots, m\}$ (where J contains at least two elements).

An algebra (A, F) is called a *regular algebra* if every pair operation of (A, F) is a regular pair operation. The equality (3) is a *co-identity* in the sense of Ref. 17.

Lemma 4 *Let (A, f, g) be a medial algebra with m-ary operation f and n-ary operation g, let π be a permutation of $\{1, \dots, m\}$, and let ρ be permutations of $\{1, 2, \dots, n\}$. Then the algebra (A, f^*, g^*) defined by the operations*

$$f^*(x_1^m) = f(x_{\pi 1}^{\pi m}),$$
$$g^*(x_1^n) = g(x_{\rho 1}^{\rho n})$$

is a medial algebra.

Proof: For $x_{ij} \in A$, since (A, f, g) is a medial algebra, we have

$$\begin{aligned} &g^*(f^*(x_{11}^{m1}), \dots, f^*(x_{1n}^{mn})) \\ &\qquad = g(f(x_{\pi 1 \rho 1}^{\pi m \rho 1}), \dots, f(x_{\pi 1 \rho n}^{\pi m \rho n})) \\ &\qquad = g(f(x_{\pi 1 \rho 1}^{\pi 1 \rho n}), \dots, f(x_{\pi m \rho 1}^{\pi m \rho n})) \\ &\qquad = g^*(f^*(x_{11}^{1n}), \dots, f(x_{m1}^{mn})). \end{aligned}$$

Similarly, f^* and g^* commute with themselves. □

Lemma 5 *Let (A, f, g) be a regular medial algebra with m-ary operation f and n-ary operation g and let $e, t \in A$ be i- and j-regular elements of (A, f, g) ($i, j \in J \subseteq \{1, 2, \dots, m\}$ and $i < j$) such that*

$$f((\bar{e})^{i-1}, f((\bar{e})^m), (\bar{e})^{j-1-i}, t, (\bar{e})^{m-j}) = e,$$
$$g((\bar{e})^{i-1}, g((\bar{e})^n), (\bar{e})^{j-1-i}, t, (\bar{e})^{n-j}) = e.$$

Then the algebra (A, f^*, g^*) with operations on A, defined by

$$f^*(x_1^m) = f((\bar{e})^{i-1}, f(x_1^m), (\bar{e})^{j-1-i}, t, (\bar{e})^{m-j}),$$
$$g^*(x_1^n) = g((\bar{e})^{i-1}, g(x_1^n), (\bar{e})^{j-1-i}, t, (\bar{e})^{n-j})$$

is a medial algebra with e as an idempotent element.

Proof: In view of Lemma 4, it is sufficient to prove this for $i = 1$ and $j = 2$.

$$\begin{aligned} &g^*(f^*(x_{11}^{1m}), \dots, f^*(x_{n1}^{nm})) \\ &\qquad = g(g(f^*(x_{11}^{1m}), \dots, f^*(x_{n1}^{nm})), t, (\bar{e})^{n-2}). \end{aligned}$$

But

$$\begin{aligned} &g(f^*(x_{11}^{1m}), \dots, f^*(x_{n1}^{nm})) \\ &= g(f(f(x_{11}^{1m}), t, (\bar{e})^{m-2}), \dots, f(f(x_{n1}^{nm}), t, (\bar{e})^{m-2})) \\ &= f(g(f(x_{11}^{1m}), \dots, f(x_{n1}^{nm})), g((\bar{t})^n), (\overline{g((\bar{e})^n)})^{m-2}) \\ &= f(f(g(x_{11}^{n1}), \dots, g(x_{1m}^{nm})), g((\bar{e})^n), (\overline{g((\bar{e})^n)})^{m-2}). \end{aligned}$$

Since (A, f, g) is a regular algebra, we have

$$g((\bar{t})^n) = f((\bar{t})^m),$$
$$g((\bar{e})^n) = f((\bar{e})^m).$$

Hence by mediality we have

$$\begin{aligned} &g(f^*(x_{11}^{1m}), \dots, f^*(x_{n1}^{nm})) \\ &= f(f(g(x_{11}^{n1}), \dots, g(x_{1m}^{nm})), f((\bar{t})^m), (\overline{f((\bar{e})^m)})^{m-2}) \\ &= f(f(g(x_{11}^{n1}), t, (\bar{e})^{m-2}), \dots, f(g(x_{1m}^{nm}), t, (\bar{e})^{m-2})). \end{aligned}$$

Since (A, f, g) is a regular algebra, we have

$$f(g(x_{11}^{n1}), t, (\bar{e})^{m-2}) = g(g(x_{11}^{n1}), t, (\bar{e})^{n-2}),$$
$$f(g(x_{1m}^{nm}), t, (\bar{e})^{m-2}) = g(g(x_{1m}^{nm}), t, (\bar{e})^{n-2}).$$

Hence by mediality we have

$$\begin{aligned} &g(f^*(x_{11}^{1m}), \dots, f^*(x_{n1}^{nm})) \\ &= f(g(g(x_{11}^{n1}), t, (\bar{e})^{n-2}), \dots, g(g(x_{1m}^{nm}), t, (\bar{e})^{n-2})) \\ &= f(g^*(x_{11}^{n1}), \dots, g^*(x_{1m}^{nm})). \end{aligned}$$

Hence

$$\begin{aligned} g^*(f^*(x_{11}^{1m}), \dots, f^*(x_{n1}^{nm})) &= g(g(f^*(x_{11}^{1m}), \dots, f^*(x_{n1}^{nm})), t, (\bar{e})^{n-2}) \\ &= g(f(g^*(x_{11}^{n1}), \dots, g^*(x_{1m}^{nm})), t, (\bar{e})^{n-2}) \\ &= f(f(g^*(x_{11}^{n1}), \dots, g^*(x_{1m}^{nm})), t, (\bar{e})^{m-2}) \\ &= f^*(g^*(x_{11}^{n1}), \dots, g^*(x_{1m}^{nm})) \end{aligned}$$

since (A, f, g) is a regular algebra. Hence (f^*, g^*) is a medial pair operation. Similarly, f^* and g^* commute with themselves. Hence (A, f^*, g^*) is a medial algebra. It is clear that $f^*((\bar{e})^m) = g^*((\bar{e})^n) = e$. □

Lemma 6 *Let (A, f, g) be a regular medial algebra with m-ary operation f and n-ary operation g. Then there is a commutative semigroup $(A, +)$ such that*

$$f(x_1^m) = \varphi(\alpha_1 x_1 + \ldots + \alpha_m x_m),$$

$$g(x_1^n) = \psi(\alpha_1 x_1 + \ldots + \alpha_m x_m + \beta_{m+1} x_{m+1} + \ldots + \beta_n x_n),$$

where α_1^m, β_1^n are pairwise commuting endomorphisms of $(A, +)$ and φ, ψ are bijections on the set A for $2 \leqslant m \leqslant n$.

Proof: Let $J \subseteq \{1, 2, \dots, m\}$, $i, j \in J$ and e be an i- and j-regular element in (A, f, g). Then by the results of the preceding section, there is an i-, j-regular element t such that

$$f((\bar{e})^{i-1}, f((\bar{e})^m), (\bar{e})^{j-1-i}, t, (\bar{e})^{m-j}) = e,$$
$$g((\bar{e})^{i-1}, g((\bar{e})^n), (\bar{e})^{j-1-i}, t, (\bar{e})^{n-j}) = e,$$

since (A, f, g) is a regular algebra. Furthermore, for k either i or j, and any $b \in A$, the equations

$$f((\bar{e})^{i-1}, f((\bar{e})^{k-1}, x, (\bar{e})^{m-k}), (\bar{e})^{j-1-i}, t, (\bar{e})^{m-j}) = b,$$
$$g((\bar{e})^{i-1}, g((\bar{e})^{k-1}, x, (\bar{e})^{n-k}), (\bar{e})^{j-1-i}, t, (\bar{e})^{n-j}) = b,$$

have unique solutions. Hence e is an i-, j-regular element with respect to the pair operation (f^*, g^*) on A defined by

$$f^*(x_1^m) = f((\bar{e})^{i-1}, f(x_1^m), (\bar{e})^{j-1-i}, t, (\bar{e})^{m-j}),$$
$$g^*(x_1^n) = g((\bar{e})^{i-1}, g(x_1^n), (\bar{e})^{j-1-i}, t, (\bar{e})^{n-j}).$$

Hence by Lemma 5 the pair operation (f^*, g^*) is medial with e as an idempotent i-, j-regular element. Thus by Theorem 1 there is a commutative semigroup $(A, +)$ with the unit element e such that

$$f^*(x_1^m) = \alpha_1 x_1 + \dots + \alpha_m x_m,$$
$$g^*(x_1^n) = \beta_1 x_1 + \dots + \beta_n x_n,$$

where α_1^m, β_1^n are commuting endomorphisms of $(A, +)$. But (A, f, g) is a regular algebra $\alpha_i = \beta_i$ for $i = 1, 2, \dots m$. Hence

$$g^*(x_1^n) = \alpha_1 x_1 + \dots + \alpha_m x_m + \beta_{m+1} x_{m+1} + \dots + \beta_n x_n.$$

Again, by the results of the previous section, the mappings

$$\varphi^{-1} : x \to f((\bar{e})^{i-1}, x, (\bar{e})^{j-1-i}, t, (\bar{e})^{m-j}),$$
$$\psi^{-1} : x \to g((\bar{e})^{i-1}, x, (\bar{e})^{j-1-i}, t, (\bar{e})^{n-j}),$$

are bijections on the set A. Thus $f(x_1^m) = \varphi f^*(x_1^m)$ and $g(x_1^n) = \psi g^*(x_1^n)$. □

Lemma 7 *Let $(A, +)$ be a commutative semigroup with a unit element and let $\varphi_1, \dots, \varphi_m$ be bijections on the set A such that*

$$\begin{aligned} &\varphi_1(x_{11} + \dots + x_{1m}) + \dots + \varphi_m(x_{m1} + \dots + x_{mm}) \\ &= \varphi_1(x_{11} + \dots + x_{m1}) + \dots + \varphi_m(x_{1m} + \dots + x_{mm}). \end{aligned} \quad (4)$$

Then there is an automorphism η of $(A, +)$ and fixed elements c_1^n such that for each i, we have

$$\varphi_i x = \eta x + c_i,$$

for all $x \in A$.

Proof: Let $(A, +)$ be a commutative semigroup with a unit element e. In (4), for fixed i and all j except $j = 1$, put $x_{ij} = \varphi_i^{-1} e$ and let all other x_{pq} be unit elements except x_{1i} and x_{i1}. Then we have

$$\varphi_1 x_{1i} + \varphi_i x_{i1} = \varphi_1 x_{i1} + \varphi_i x_{1i}.$$

Hence if $x_{1i} = \varphi_1^{-1} e$ then

$$\varphi_i x_{i1} = \varphi_1 x_{i1} + \varphi_i \varphi_1^{-1} e$$

for all $x_{i1} \in A$. Since φ_1, φ_i are permutations on $(A, +)$ for all $x \in A$ we have

$$\varphi_i x = \varphi_1 x + k_i,$$

where k_i is a fixed regular element of $(A, +)$. Substituting for the φ_i in the equation (4) and cancelling the k_i, which we may do since they are regular elements, we get

$$\varphi_1(x_{11}+\cdots+x_{1m})+\cdots+\varphi_1(x_{m1}+\cdots+x_{mm}) \\ = \varphi_1(x_{11}+\cdots+x_{m1})+\cdots+\varphi_1(x_{1m}+\cdots+x_{mm}). \tag{5}$$

In (5) let $x_{ii} = \varphi_1^{-1}e$ where $i \neq 1, 2$ and let all other x_{ij} be the unit element e except x_{11}, x_{12}. Then, we have

$$\varphi_1(x_{11} + x_{12}) + \varphi_1 e = \varphi_1 x_{11} + \varphi_1 x_{12},$$

for all x_{11}, x_{12}. Hence if $x_{11} = x_{12} = \varphi_1^{-1}e$, then

$$\varphi_1(\varphi_1^{-1}e + \varphi_1^{-1}e) + \varphi_1 e = e.$$

It means that $\varphi_1 e$ has an additive inverse and hence is a regular element. Now we define a bijection η on A by

$$\varphi_1 x = \eta x + \varphi_1 e,$$

for all $x \in A$. It follows immediately that η is an automorphism of $(A, +)$. Hence

$$\varphi_i x = \varphi_1 x + k_i = \eta x + \varphi_1 e + k_i = \eta x + c_i,$$

where $c_i = \varphi_1 e + k_i$ as the sum of two regular elements is a regular element. □

Theorem 2 *Let (A, f, g) be a regular medial algebra with m-ary operation f and n-ary operation g ($m \leqslant n$). Then there is a commutative semigroup $(A, +)$ with a unit element such that*

$$f(x_1^m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d_1, \\ g(x_1^n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + d_2$$

where d_1, d_2 are fixed regular elements in $(A, +)$ and γ_1^m, λ_1^n are commuting automorphisms of the semigroup $(A, +)$ ($J \subseteq \{1, 2, \ldots, m\}$).

Proof: Let (A, f, g) be a regular medial algebra. By Lemma 6 we know that there is a commutative semigroup with a unit element e such that

$$f(x_1^m) = \varphi(\alpha_1 x_1 + \cdots + \alpha_m x_m), \\ g(x_1^n) = \psi(\beta_1 x_1 + \cdots + \beta_n x_n)$$

where α_1^m, β_1^n are pairwise commuting endomorphisms of $(A, +)$, and φ, ψ are bijections on the set A, and $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, m$. Since the operation f is medial we have

$$\varphi(\alpha_1\varphi(\alpha_1 x_{11} + \cdots + \alpha_m x_{1m}) \\ + \cdots + \alpha_m\varphi(\alpha_1 x_{m1} + \cdots + \alpha_m x_{mm})) \\ = \varphi(\alpha_1\varphi(\alpha_1 x_{11} + \cdots + \alpha_m x_{m1}) \\ + \cdots + \alpha_m\varphi(\alpha_1 x_{1m} + \cdots + \alpha_m x_{mm})).$$

Hence

$$\alpha_1\varphi\alpha_1^{-1}(\alpha_1\alpha_1 x_{11} + \cdots + \alpha_1\alpha_m x_{1n}) \\ + \cdots + \alpha_m\varphi\alpha_n^{-1}(\alpha_m\alpha_1 x_{m1} + \cdots + \alpha_m\alpha_m x_{mm}) \\ = \alpha_1\varphi\alpha_1^{-1}(\alpha_1\alpha_1 x_{11} + \cdots + \alpha_1\alpha_m x_{n1}) \\ + \cdots + \alpha_m\varphi\alpha_n^{-1}(\alpha_m\alpha_1 x_{1m} + \cdots + \alpha_m\alpha_m x_{mm}),$$

since φ is a bijection. Let $\chi_i = \alpha_i\varphi\alpha_i^{-1}$ and $\alpha_i\alpha_j x_{ij} = y_{ij}$. Then by substitution and since $\alpha_1, \ldots, \alpha_m$ are commuting endomorphisms of the commutative semigroup $(A, +)$, we have

$$\chi_1(y_{11} + \cdots + y_{1m}) + \cdots + \chi_m(y_{m1} + \cdots + y_{mm}) \\ = \chi_1(y_{11}+\cdots+y_{m1})+\cdots+\chi_m(y_{1m}+\cdots+y_{mm}).$$

Hence by the preceding lemma, there is an automorphism η of the semigroup $(A, +)$ and regular elements $c_1, \ldots, c_m$ such that

$$\chi_i x = \alpha_i\varphi\alpha_i^{-1}x = \eta x + c_i, \\ \varphi x = \alpha_i^{-1}\eta\alpha_i x + \alpha_i^{-1}\alpha_i c_i, \\ \varphi x = \sigma x + d_1,$$

where $\sigma = \alpha_i^{-1}\eta\alpha_i$ is an automorphism of the semigroup $(A, +)$ and $d_1 = \alpha_i^{-1}\alpha_i c_i$ is a fixed regular element in $(A, +)$. Hence

$$f(x_1^m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d_1$$

where $\gamma_i = \sigma\alpha_i$ is an automorphism of the semigroup $(A, +)$. Similarly,

$$g(x_1^n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + d_2.$$

It is easy to check that γ_1^m, λ_1^n are commuting automorphisms of the semigroup $(A, +)$. □

Corollary 2 *Let (A, F) be a regular medial algebra. Then there exists a commutative semigroup $(A, +)$ such that every operation $f \in F$ has the representation*

$$f(x_1^m) = \gamma_1 x_1 + \cdots + \gamma_m x_m + d,$$

where d is a fixed regular element in $(A, +)$ and $\gamma_1, \ldots, \gamma_m$ are commuting automorphisms of the semigroup $(A, +)$.

Corollary 3 *If (Q, f) is a medial n-ary quasigroup, then there exists an abelian group $(Q, +)$ such that*

$$f(x_1^m) = \alpha_1 x_1 + \cdots + \alpha_m x_m + d,$$

where the α_i's are pairwise commuting automorphisms of the group $(Q, +)$ and $d \in Q$ [15].

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