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New estimates of the remainder term in Simpson's quadrature formula for functions of bounded variation

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ABSTRACT: In this paper we establish a refinement of Simpson's inequality for functions of bounded variation. As an application, we obtain some new estimates of the remainder term in Simpson's quadrature formula.

KEYWORDS: Simpson's inequality, generalized trapezoid and Ostrowski inequalities

INTRODUCTION

The following inequality is well known in the literature as Simpson's inequality:

$$\left|\int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant \tfrac{1}{2880}\|f^{(4)}\|_\infty (b-a)^5, \quad (1)$$

where the mapping $f : [a,b] \to \mathbb{R}$ is supposed to be four times differentiable on the interval (a,b) and have the fourth derivative bounded on (a,b), that is, $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$. This inequality gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications.

It is well known that if either the mapping f is not four times differentiable or the fourth derivative $f^{(4)}$ is unbounded on (a,b) then we cannot apply the classical Simpson quadrature formula. This disadvantage was overcome in the result of Dragomir et al [1,2] where the following result was proved.

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Denote by $V_a^b(f)$ its total variation on $[a,b]$. Then one has the inequality*

$$\left|\int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant \tfrac{1}{3}(b-a)V_a^b(f). \quad (2)$$

The constant $\frac{1}{3}$ is sharp in the sense that it cannot be replaced by a smaller one. To simplify notation, we will write V_a^b instead of $V_a^b(f)$ from now on.

Pečarić and Varošanec [3] generalized the above result to functions whose nth derivative, $n \in \{0,1,2,3\}$ is of bounded variation. The corresponding version for Ostrowski's inequality and the generalized trapezoid inequality was obtained in Refs. 4–6, from which one can derive the midpoint inequality [7] and the trapezoid inequality. Recently, by using a critical lemma, Dragomir [8] proved refinement of the generalized trapezoid and Ostrowski inequalities for functions of bounded variation, the particular cases of which provide refinements of the trapezoid and midpoint inequalities. For other related results, the reader may refer to Refs. 9–13 and the references therein.

The main aim of this paper is to establish a refinement of Simpson's inequality (2) for functions of bounded variation. As an application, we obtain some new estimates of the remainder term in Simpson's quadrature formula.

A REFINEMENT OF SIMPSON'S INEQUALITY

We need the following lemma, which is a slight improvement of Lemma 2.1 of Ref. 8.

Lemma 1 *Let $u, f : [a,b] \to \mathbb{R}$. If u is continuous on $[a,b]$ and f is of bounded variation on $[c,b] \supseteq [a,b]$ then*

$$\left|\int_a^b u(t)\,\mathrm{d}f(t)\right| \leqslant \int_a^b |u(t)|\,\mathrm{d}(V_c^t) \leqslant [V_a^b]^{\frac{1}{q}}\left[\int_a^b |u(t)|^p\,\mathrm{d}(V_c^t)\right]^{\frac{1}{p}} \leqslant \max_{t\in[a,b]}|u(t)|V_a^b, \quad (3)$$

if $p > 1$ *and* $1/p + 1/q = 1$.

Proof: See Lemma 2.1 of Ref. 8 with a slight improvement. □

The following result may be stated as a refinement of Simpson's inequality (2).

Theorem 2 *Assume that the function* $f : [a, b] \to \mathbb{R}$ *is of bounded variation on* $[a, b]$. *Then*

$$\left|\int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant (b-a)Q, \quad (4)$$

where

$$\begin{aligned} Q &:= \tfrac{1}{6}V_a^b + \frac{1}{b-a}\left(\int_a^{(5a+b)/6} - \int_{(5a+b)/6}^{(a+b)/2} + \int_{(a+b)/2}^{(a+5b)/6} - \int_{(a+5b)/6}^{b}\right) V_a^t\,\mathrm{d}t \\ &\leqslant \tfrac{1}{6}V_a^b + \tfrac{1}{6}V_{(5a+b)/6}^{(a+5b)/6} \\ &\leqslant \tfrac{1}{3}V_a^b. \end{aligned} \quad (5)$$

We also have

$$\begin{aligned} Q &\leqslant \frac{1}{b-a}[V_a^b]^{1/q}\left[\left(\frac{b-a}{6}\right)^p V_a^b + p\int_a^b r_p(t)(V_a^t)\,\mathrm{d}t\right]^{\frac{1}{p}} \\ &\leqslant \frac{1}{b-a}[V_a^b]^{1/q}\left\{\left(\frac{b-a}{6}\right)^p V_a^b + \left[\left(\frac{b-a}{3}\right)^p - \left(\frac{b-a}{6}\right)^p\right] V_{(5a+b)/6}^{(a+5b)/6}\right\}^{1/p} \\ &\leqslant \tfrac{1}{3}V_a^b, \end{aligned} \quad (6)$$

where $p > 1,\ 1/p + 1/q = 1$ *and* $r_p : [a, b] \to \mathbb{R}$ *with*

$$r_p(t) := \begin{cases} \left(\frac{5a+b}{6} - t\right)^{p-1}, t \in \left[a, \frac{5a+b}{6}\right], \\ -\left(t - \frac{5a+b}{6}\right)^{p-1}, t \in \left(\frac{5a+b}{6}, \frac{a+b}{2}\right], \\ \left(\frac{a+5b}{6} - t\right)^{p-1}, t \in \left(\frac{a+b}{2}, \frac{a+5b}{6}\right], \\ -\left(t - \frac{a+5b}{6}\right)^{p-1}, t \in \left(\frac{a+5b}{6}, b\right]. \end{cases} \quad (7)$$

Proof: Define the kernel $S(t)$ by

$$S(t) := \begin{cases} t - \frac{5a+b}{6}, & t \in \left[a, \frac{a+b}{2}\right], \\ t - \frac{a+5b}{6}, & t \in \left(\frac{a+b}{2}, b\right]. \end{cases} \quad (8)$$

Using integration by parts for Riemann-Stieltjes integrals, we obtain (see Ref. 5)

$$\int_a^b S(t)\,\mathrm{d}f(t) = \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,\mathrm{d}t. \quad (9)$$

If f is of bounded variation on $[a, b]$, then on taking the modulus and applying the first inequality in (3) we obtain

$$\begin{aligned} \left|\int_a^b S(t)\,\mathrm{d}f(t)\right| &\leqslant \left|\int_a^{(a+b)/2}\left(t - \frac{5a+b}{6}\right)\mathrm{d}f(t)\right| + \left|\int_{(a+b)/2}^{b}\left(t - \frac{a+5b}{6}\right)\mathrm{d}f(t)\right| \\ &\leqslant \int_a^{(a+b)/2}\left|t - \frac{5a+b}{6}\right|\mathrm{d}(V_a^t) + \int_{(a+b)/2}^{b}\left|t - \frac{a+5b}{6}\right|\mathrm{d}(V_a^t) \\ &= \int_a^{(5a+b)/6}\left(\frac{5a+b}{6} - t\right)\mathrm{d}(V_a^t) + \int_{(5a+b)/6}^{(a+b)/2}\left(t - \frac{5a+b}{6}\right)\mathrm{d}(V_a^t) \\ &\quad + \int_{(a+b)/2}^{(a+5b)/6}\left(\frac{a+5b}{6} - t\right)\mathrm{d}(V_a^t) + \int_{(a+5b)/6}^{b}\left(t - \frac{a+5b}{6}\right)\mathrm{d}(V_a^t) \\ &= \frac{b-a}{6}V_a^b + \left(\int_a^{(5a+b)/6} - \int_{(5a+b)/6}^{(a+b)/2} + \int_{(a+b)/2}^{(a+5b)/6} - \int_{(a+5b)/6}^{b}\right) V_a^t\,\mathrm{d}t \\ &= (b-a)Q, \end{aligned}$$

and (4) is proved.

Since $V_a^{\cdot}$ is monotonic nondecreasing on $[a, b]$,

$$\int_a^{(5a+b)/6}(V_a^t)\,\mathrm{d}t \leqslant \frac{b-a}{6}V_a^{(5a+b)/6},$$

$$\int_{(5a+b)/6}^{(a+b)/2}(V_a^t)\,\mathrm{d}t \geqslant \frac{b-a}{3}V_a^{(5a+b)/6},$$

$$\int_{(a+b)/2}^{(a+5b)/6}(V_a^t)\,\mathrm{d}t \leqslant \frac{b-a}{3}V_a^{(a+5b)/6},$$

$$\int_{(a+5b)/6}^{b} (V_a^t)\,\mathrm{d}t \geqslant \frac{b-a}{6} V_a^{(a+5b)/6},$$

which gives

$$\begin{aligned}(b-a)Q &\leqslant \frac{b-a}{6}V_a^b + \frac{b-a}{6}V_a^{(5a+b)/6} - \frac{b-a}{3}V_a^{(5a+b)/6} \\ &\quad + \frac{b-a}{3}V_a^{(a+5b)/6} - \frac{b-a}{6}V_a^{(a+5b)/6} \\ &= \frac{b-a}{6}V_a^b + \frac{b-a}{6}V_a^{(a+5b)/6} - \frac{b-a}{6}V_a^{(5a+b)/6} \\ &= \frac{b-a}{6}V_a^b + \frac{b-a}{6}V_{(5a+b)/6}^{(a+5b)/6}\end{aligned}$$

and (5) is proved.

Using the second part of the second inequality in (3) and Hölder's inequality, we deduce that

$$\begin{aligned}(b-a)Q &\leqslant [V_a^{(a+b)/2}]^{1/q}\left[\int_a^{(a+b)/2}\left|t-\frac{5a+b}{6}\right|^p \mathrm{d}(V_a^t)\right]^{\frac{1}{p}} \\ &\quad + [V_{(a+b)/2}^b]^{1/q}\left[\int_{(a+b)/2}^b\left|t-\frac{a+5b}{6}\right|^p \mathrm{d}(V_a^t)\right]^{\frac{1}{p}} \\ &\leqslant [V_a^b]^{1/q}\left[\int_a^{(a+b)/2}\left|t-\frac{5a+b}{6}\right|^p \mathrm{d}(V_a^t)\right. \\ &\quad \left. + \int_{(a+b)/2}^b\left|t-\frac{a+5b}{6}\right|^p \mathrm{d}(V_a^t)\right]^{1/p}. \end{aligned} \tag{10}$$

Now observe that

$$\begin{aligned}R &:= \int_a^{(a+b)/2}\left|t-\frac{5a+b}{6}\right|^p \mathrm{d}(V_a^t) \\ &\quad + \int_{(a+b)/2}^b\left|t-\frac{a+5b}{6}\right|^p \mathrm{d}(V_a^t) \\ &= \int_a^{(5a+b)/6}\left(\frac{5a+b}{6}-t\right)^p \mathrm{d}(V_a^t) \\ &\quad + \int_{(5a+b)/6}^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)^p \mathrm{d}(V_a^t) \\ &\quad + \int_{(a+b)/2}^{(a+5b)/6}\left(\frac{a+5b}{6}-t\right)^p \mathrm{d}(V_a^t) \\ &\quad + \int_{(a+5b)/6}^{b}\left(t-\frac{a+5b}{6}\right)^p \mathrm{d}(V_a^t) \\ &= p\int_a^{(5a+b)/6}(V_a^t)\left(\frac{5a+b}{6}-t\right)^{p-1} \mathrm{d}t \\ &\quad + \left(\frac{b-a}{3}\right)^p V_a^{(a+b)/2} \\ &\quad - p\int_{(5a+b)/6}^{(a+b)/2}(V_a^t)\left(t-\frac{5a+b}{6}\right)^{p-1} \mathrm{d}t \\ &\quad - \left(\frac{b-a}{3}\right)^p V_a^{(a+b)/2} \\ &\quad + p\int_{(a+b)/2}^{(a+5b)/6}(V_a^t)\left(\frac{a+5b}{6}-t\right)^{p-1} \mathrm{d}t \\ &\quad + \left(\frac{b-a}{6}\right)^p V_a^b \\ &\quad - p\int_{(a+5b)/6}^{b}(V_a^t)\left(t-\frac{a+5b}{6}\right)^{p-1} \mathrm{d}t \\ &= \left(\frac{b-a}{6}\right)^p V_a^b + p\int_a^b r_p(t)(V_a^t)\,\mathrm{d}t,\end{aligned}$$

where r_p is given in (7). Using (10), we obtain the first part of (6).

Since $V_a^{\cdot}$ is monotonic nondecreasing on $[a,b]$, we have

$$\begin{aligned}R &\leqslant \left(\frac{b-a}{6}\right)^p V_a^b \\ &\quad + p\left[\int_a^{(5a+b)/6}\left(\frac{5a+b}{6}-t\right)^{p-1}\mathrm{d}t\right]V_a^{(5a+b)/6} \\ &\quad - p\left[\int_{(5a+b)/6}^{(a+b)/2}\left(t-\frac{5a+b}{6}\right)^{p-1}\mathrm{d}t\right]V_a^{(5a+b)/6} \\ &\quad + p\left[\int_{(a+b)/2}^{(a+5b)/6}\left(\frac{a+5b}{6}-t\right)^{p-1}\mathrm{d}t\right]V_a^{(a+5b)/6} \\ &\quad - p\left[\int_{(a+5b)/6}^{b}\left(t-\frac{a+5b}{6}\right)^{p-1}\mathrm{d}t\right]V_a^{(a+5b)/6} \\ &= \left(\frac{b-a}{6}\right)^p V_a^b \\ &\quad + \left[\left(\frac{b-a}{3}\right)^p - \left(\frac{b-a}{6}\right)^p\right]V_{(5a+b)/6}^{(a+5b)/6},\end{aligned}$$

which proves (6). □

NEW ESTIMATES OF THE REMAINDER TERM IN SIMPSON'S FORMULA

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of the interval $[a,b]$, $h_i := x_{i+1} - x_i$ for $i = 0, \ldots, n-1$ and consider Simpson's quadrature formula

$$\int_a^b f(x)\,\mathrm{d}x = A_S(f, I_n) + R_S(f, I_n), \tag{11}$$

where $A_S(f, I_n)$ is Simpson's rule

$$A_S(f,I_n) := \frac{1}{6}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i. \quad (12)$$

It is well known that if the function $f : [a,b] \to \mathbb{R}$ is four times differentiable on the interval (a,b) and has the fourth derivative bounded on (a,b), then the remainder term $R_S(f,I_n)$ satisfies the estimate

$$|R_S(f,I_n)| \leqslant \tfrac{1}{2880}\|f^{(4)}\|_\infty h_i^5. \quad (13)$$

Dragomir et al[1,2] improved the above estimate for functions of bounded variation as follows:

$$|R_S(f,I_n)| \leqslant \tfrac{1}{3}\gamma(h)V_a^b, \quad (14)$$

where $\gamma(h) := \max\{h_i, i=0,\dots,n-1\}$.

In the following we give new estimates of the remainder term $R_S(f,I_n)$ for functions of bounded variation.

Theorem 3 *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$ and I_n be a partition of $[a,b]$. Then we have Simpson's quadrature formula (11) and the remainder term $R_S(f,I_n)$ satisfies the estimates*

$$\begin{aligned} &|R_S(f,I_n)| \\ &\leqslant \tfrac{1}{6}\gamma(h)V_a^b + \sum_{i=0}^{n-1}\left(\int_{x_i}^{(5x_i+x_{i+1})/6} - \int_{(5x_i+x_{i+1})/6}^{(x_i+x_{i+1})/2} + \int_{(x_i+x_{i+1})/2}^{(x_i+5x_{i+1})/6} - \int_{(x_i+5x_{i+1})/6}^{x_{i+1}}\right) V_{x_i}^t \,\mathrm{d}t \\ &\leqslant \tfrac{1}{6}\gamma(h)V_a^b + \tfrac{1}{6}\gamma(h)V_{(5a+b)/6}^{(a+5b)/6} \\ &\leqslant \tfrac{1}{3}\gamma(h)V_a^b \end{aligned}$$

and

$$\begin{aligned} &|R_S(f,I_n)| \\ &\leqslant \sum_{i=0}^{n-1}[V_{x_i}^{x_{i+1}}]^{1/q}\left[\left(\frac{x_{i+1}-x_i}{6}\right)^p V_{x_i}^{x_{i+1}} + p\int_{x_i}^{x_{i+1}} r_p(t;x_i,x_{i+1})(V_{x_i}^t)\,\mathrm{d}t\right]^{1/p} \\ &\leqslant \sum_{i=0}^{n-1}[V_{x_i}^{x_{i+1}}]^{1/q}\left\{\left(\frac{x_{i+1}-x_i}{6}\right)^p V_{x_i}^{x_{i+1}} + \left[\left(\frac{x_{i+1}-x_i}{3}\right)^p - \left(\frac{x_{i+1}-x_i}{6}\right)^p\right] V_{(5x_i+x_{i+1})/6}^{(x_i+5x_{i+1})/6}\right\}^{\frac{1}{p}} \\ &\leqslant \tfrac{1}{3}\gamma(h)V_{x_i}^{x_{i+1}}, \end{aligned}$$

where $p > 1$, $1/p+1/q = 1$. and $r_p(t;x_i,x_{i+1}) : [x_i,x_{i+1}] \to \mathbb{R}$ with

$$r_p(t) := \begin{cases} \left(\frac{5x_i+x_{i+1}}{6}-t\right)^{p-1}, & t \in \left[x_i,\ \frac{5x_i+x_{i+1}}{6}\right], \\ -\left(t-\frac{5x_i+x_{i+1}}{6}\right)^{p-1}, & t \in \left(\frac{5x_i+x_{i+1}}{6}, \frac{x_i+x_{i+1}}{2}\right], \\ \left(\frac{x_i+5x_{i+1}}{6}-t\right)^{p-1}, & t \in \left(\frac{x_i+x_{i+1}}{2}, \frac{x_i+5x_{i+1}}{6}\right], \\ -\left(t-\frac{x_i+5x_{i+1}}{6}\right)^{p-1}, & t \in \left(\frac{x_i+5x_{i+1}}{6},\ x_{i+1}\right]. \end{cases}$$

Proof: Apply Theorem 2 to the interval $[x_i,x_{i+1}]$, $i = 0,1,2,\dots,n-1$ and sum. Then use the triangle inequality to obtain the desired result. □

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