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Multiple solutions for $p(x)$-Laplacian type equations

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ABSTRACT: We prove the existence of at least three weak solutions for the Dirichlet problem when the nonlinear term f is sublinear and $p(x)$ is greater than n. This Dirichlet problem involves a general elliptic operator in divergence form (in particular, a $p(x)$-Laplace operator). Our method relies upon a recent critical point theorem obtained by Bonanno and Marano, and is combined with the theory of variable exponent Lebesgue-Sobolev spaces.

KEYWORDS: variational method, variable exponent, divergence type operator, three critical points theorem

INTRODUCTION

In this paper, we are concerned with the following problem involving an elliptic operator in divergence form:

$$\begin{aligned} -\operatorname{div}(a(x,\nabla u)) = \lambda f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ is a bounded domain with smooth boundary, $p \in C(\overline{\Omega})$, with $\inf_{x\in\Omega} p(x) > n$, $\lambda > 0$ is a real parameter, and f is a Carathéodory function. The potential a satisfies a set of assumptions (see below). The operator includes the $p(x)$-Laplace operator and other important cases such as the generalized mean curvature operator. We point out that the extension from the p-Laplace operator to the $p(x)$-Laplace operator is not trivial, since the $p(x)$-Laplace operator possesses a more complicated structure than the p-Laplace operator. For example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its spectrum is zero. This causes many problems. For instance, some classical theories and methods, such as the Lagrange multiplier theorem and the theory of Sobolev space, cannot be applied.

Recently, in the framework of the Sobolev space $W^{1,p}(\Omega)$, De Nápoli and Mariani[1] studied problem (1), where

$$|a(x,\xi)| \leqslant C(1+|\xi|^{p-1}) \tag{2}$$

for all $\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$. When the nonlinear term f satisfies the Ambrosetti-Rabinowitz type condition, the authors showed the existence of at least one solution for problem (1) by using the arguments of classical mountain pass type. It is well known that by the Ambrosetti-Rabinowitz type condition, one can deduce that f is $(p-1)$-*superlinear* at infinity. In Ref. 2, the authors handled the counterpart of the above case, i.e., assumed that f is $(p-1)$-*sublinear* at infinity and autonomous, that is to say, $f(x,t) = f(t)$. Using a three critical points theorem by Bonanno[3], the authors proved that problem (1) with condition (2) has at least three weak solutions. Duc and Thanh Vu[4] studied problem (1) in the 'nonuniform' case when

$$|a(x,\xi)| \leqslant C(\theta(x)+\sigma(x)|\xi|^{p-1}) \tag{3}$$

for all $\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$, in which θ and σ are two non-negative measurable functions satisfy $\theta \in L^{p/(p-1)}(\Omega)$, $\sigma \in L^1_{\text{loc}}(\Omega)$, and $\sigma(x) \geqslant 1$ for a.e. $x \in \Omega$. It is clear that condition (3) is weaker than (2), so Ref. 4 extended the result of Ref. 1. They showed in Ref. 4 that problem (1) has at least one weak solution provided that the functions a and f satisfy some further suitable conditions. Regarding some extensions of Ref. 4, the readers may consult Refs. 5–8, in which the authors studied the existence of solutions for problem (1) with condition (3) and subcritical nonlinearities. On the other hand, in the framework of the generalized variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, Bonanno and Chinnì[9] obtained the existence of at least three solutions for problem (1) via a recent critical point theorem of Bonanno and Marano[10] where it is assumed that $a(x,\xi) = |\xi|^{p(x)-2}\xi$ and the nonlinear term f satisfies the sublinear case. With the aid of adequate variational methods and a variant of the Mountain Pass lemma, Mihăilescu and Rădulescu[11] proved that if λ is large enough, there exist at least two distinct non-negative, nontrivial weak solutions for problem (1) in the general divergence form a and the particular nonlinear term $f(x,t) = t^{\gamma-1} - t^{\beta-1}$ with $1 < \beta < \gamma < \inf_{x\in\overline{\Omega}} p(x)$ and $t \geqslant 0$. Motivated by the studies in Refs. 2, 9, 11, we consider the existence of three nontrivial solutions for problem (1) in the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ under the following assumptions:

$$n < p^- := \inf_{x\in\Omega} p(x) \leqslant p(x) \leqslant p^+ := \sup_{x\in\Omega} p(x) < \infty$$

for general nonlinear term f.

PRELIMINARIES

We review some definitions and basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$. For more properties of the variable exponent Lebesgue-Sobolev spaces, refer to Refs. 12–15.

Set $C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}) : h(x) > 1$ for all $x \in \overline{\Omega}\}$. For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For $p \in C_+(\overline{\Omega})$, we introduce the *variable exponent Lebesgue space*,

$$L^{p(x)}(\Omega) = \left\{ u \,\middle|\, u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty \right\},$$

endowed with the so-called *Luxemburg norm* on this space by the formula

$$|u|_{p(x)} = |u|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\right\},$$

which is a separable and reflexive Banach space.

Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$ (see Corollary 2.7 in Ref. 14). For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the following Hölder-type inequality is valid:

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \leqslant \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)}. \tag{4}$$

An important role in manipulating the Lebesgue-Sobolev spaces is played by the $p(x)$-*modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

If $u \in L^{p(x)}(\Omega)$, the following relations hold (see Theorem 1.3 in Ref. 15):

$$|u|_{p(x)} < 1 (= 1; > 1) \;\Leftrightarrow\; \rho(u) < 1 (= 1; > 1); \tag{5}$$

$$|u|_{p(x)} > 1 \;\Rightarrow\; |u|_{p(x)}^{p^-} \leqslant \rho(u) \leqslant |u|_{p(x)}^{p^+}; \tag{6}$$

$$|u|_{p(x)} < 1 \;\Rightarrow\; |u|_{p(x)}^{p^+} \leqslant \rho(u) \leqslant |u|_{p(x)}^{p^-}. \tag{7}$$

Next, the *variable exponent Sobolev space* $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

Finally, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ under the norm $\|u\| = |\nabla u|_{p(x)}$. Then $(W^{1,p(x)}(\Omega), \|\cdot\|_{1,p(x)})$ and $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ are separable and reflexive Banach spaces. Since the Poincaré-type inequality

$$|u|_{p(x)} \leqslant C |\nabla u|_{p(x)}, \quad \forall\, u \in W_0^{1,p(x)}(\Omega)$$

where C is a positive constant holds, we know that $\|u\|$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $\|u\|$ instead of $\|u\|_{1,p(x)}$ in the following (see Proposition 2.5 in Ref. 16).

As pointed out in Theorem 2.2 in Ref. 15, we deduce that $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ since $n < p^- \leqslant p(x)$ for any $x \in \overline{\Omega}$. We define the norm

$$\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|.$$

Then there exists a constant $c_0 > 0$ such that

$$\|u\|_\infty \leqslant c_0 \|u\|, \quad \forall\, u \in W_0^{1,p(x)}(\Omega). \tag{8}$$

Our main tool is a recent result obtained in Ref. 10, which we recall in a convenient form.

Theorem 1 *Let X be a reflexive real Banach space. Let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*. Let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

(i)
$$\frac{\sup_{\Phi(x)\leqslant r}\Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})};$$

(ii) *for each*
$$\lambda \in \Lambda_r := \left(\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leqslant r}\Psi(x)}\right),$$

the functional $\Phi - \lambda\Psi$ *is coercive.*
Then, for each $\lambda \in \Lambda_r$*, the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

ASSUMPTIONS AND MAIN RESULT

Assume that $a(x,\xi) : \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ is the continuous derivative with respect to ξ of the mapping $A(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, that is, $a(x,\xi) := \frac{\partial}{\partial\xi}A(x,\xi)$. Suppose that $a(x,\xi)$ and $A(x,\xi)$ satisfy the following hypotheses:

(i) $A(x,0) = 0$ for all $x \in \Omega$

(ii) There exists a positive constant c_1 such that $|a(x,\xi)| \leqslant c_1(1 + |\xi|^{p(x)-1})$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$

(iii) $0 \leqslant (a(x,\xi) - a(x,\psi)) \cdot (\xi - \psi)$ for all $x \in \Omega$ and $\xi,\ \psi \in \mathbb{R}^n$, with equality if and only if $\xi = \psi$

(iv) There exists $k > 0$ such that
$$A\left(x, \frac{\xi+\psi}{2}\right) \leqslant \tfrac{1}{2}A(x,\xi) + \tfrac{1}{2}A(x,\psi) - k|\xi-\psi|^{p(x)}$$
for all $x \in \Omega$ and $\xi,\ \psi \in \mathbb{R}^n$

(v) $|\xi|^{p(x)} \leqslant a(x,\xi)\xi \leqslant p(x)A(x,\xi)$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

We point out that the elliptic operator in divergence form $-\operatorname{div}(a(x,\xi))$ includes the $p(x)$-Laplace operator and other important cases, such as the generalized mean curvature operator.

Example 1 Set $A(x,\xi) = (1/p(x))|\xi|^{p(x)}$, $a(x,\xi) = |\xi|^{p(x)-2}\xi$, where $p(x) \geqslant 2$. Then we get the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$.

Example 2 Set $A(x,\xi) = (1/p(x))[(1+|\xi|^2)^{p(x)/2} - 1]$, $a(x,\xi) = (1 + |\xi|^2)^{(p(x)-2)/2}\xi$, where $p(x) \geqslant 2$. Then we obtain the generalized mean curvature operator $\operatorname{div}((1 + |\nabla u|^2)^{(p(x)-2)/2}\nabla u)$.

In order to state our main result of this paper, we assume that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $1 \leqslant s^- \leqslant s^+ < p^-$ and $c > 0$ such that
$$|f(x,t)| \leqslant c(1 + |t|^{s(x)-1}) \tag{9}$$
for all $x \in \Omega$ and $t \in \mathbb{R}$.

Let us denote the n-dimensional closed Euclidean ball with centre $x \in \Omega$ and radius $\delta > 0$ by $B(x,\delta)$. Before introducing our result we observe that, putting $\delta(x) = \sup\{\delta > 0 : B(x,\delta) \subseteq \Omega\}$ for all $x \in \Omega$, one can prove that there exists $x_0 \in \Omega$ such that $B(x_0,D) \subseteq \Omega$, where $D = \sup_{x\in\Omega}\delta(x)$. Finally, for each $r > 0$, let
$$\gamma_r := \max\left\{(p^+r)^{1/(p^-)}, (p^+r)^{1/(p^+)}\right\}$$
and $\omega := \pi^{n/2}/[\frac{n}{2}\Gamma(\frac{n}{2})]$ the measure of the n-dimensional unit ball.

The main result of this paper is given by the following theorem.

Theorem 2 *Let* $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ *be a potential which fulfils the hypotheses* (i)–(v)*, and let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a function satisfies* (9) *and such that* $\operatorname{ess\,inf}_{x\in\Omega} F(x,t) := \operatorname{ess\,inf}_{x\in\Omega} \int_0^t f(x,s)\,\mathrm{d}s \geqslant 0$ *for all* $t \in \mathbb{R}$*. Assume also that there exist two positive constants* r *and* h*, with*
$$r < \frac{1}{p^+}\omega D^n\left(1 - \frac{1}{2^n}\right)\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}, \tag{10}$$
such that
$$\frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}{c_1\left[\frac{2h}{D} + \frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}\right](2^n - 1)} := \beta_h > \alpha_r := \frac{1}{r}\int_\Omega \sup_{|t|\leqslant c_0\gamma_r} F(x,t)\,\mathrm{d}x. \tag{11}$$

Then, for each $\lambda \in (1/\beta_h, 1/\alpha_r)$*, problem* (1) *admits at least three weak solutions.*

In the sublinear context, our main result extends in a natural way not only Refs. 2, 11, but also some other work where $a(x,\xi) = |\xi|^{p(x)-2}\xi$ (e.g., Ref. 9 and Theorem 4.3 in Ref. 16).

PROOF OF MAIN RESULT

Let X and X^* denote the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$ and the dual space of $W_0^{1,p(x)}(\Omega)$, respectively. To apply Theorem 1, we set the functionals $\Phi,\ \Psi : X \to \mathbb{R}$ as
$$\Phi(u) = \int_\Omega A(x,\nabla u)\,\mathrm{d}x$$
and
$$\Psi(u) = \int_\Omega F(x,u)\,\mathrm{d}x.$$

It is easy to see that $\Phi \in C^1(X, \mathbb{R})$, sequentially weakly lower semicontinuous with the derivative given by

$$\langle \Phi'(u), v \rangle = \int_\Omega a(x, \nabla u) \nabla v \, \mathrm{d}x$$

for any $u, v \in X$ (see Ref. 11). A simple calculation based on hypothesis (9) and relations (6), (7) shows that $\Psi \in C^1(X, \mathbb{R})$ and Ψ' is compact with the derivative given by

$$\langle \Psi'(u), v \rangle = \int_\Omega f(x, u) v \, \mathrm{d}x$$

for any $u, v \in X$. Hence, if there exists $\lambda > 0$ such that u is a critical point of the operator $\Phi - \lambda\Psi$, we deduce that $u \in X$ is the weak solution for problem (1).

In order to apply Theorem 1 to search for the weak solutions for problem (1), we first establish the following basic property of the functional Φ.

Lemma 1 *The functional Φ is coercive and $\Phi' : X \to X^*$ is a homeomorphism.*

Proof: (i) From (6) and (v), it is clear that for any $u \in X$ with $\|u\| > 1$, we have

$$\Phi(u) \geqslant \frac{1}{p^+} \|u\|^{p^-},$$

and thus Φ is coercive.

(ii) Firstly, we show that Φ' is uniformly monotone. Using (iii) and integrating over Ω, we obtain for all u, $v \in X$ with $u \neq v$ such that

$$\begin{aligned}\int_\Omega (a(x, \nabla u) - a(x, \nabla v)) \cdot (\nabla u - \nabla v) \, \mathrm{d}x \\ = \langle \Phi'(u) - \Phi'(v), u - v \rangle > 0,\end{aligned}$$

which means that Φ' is uniformly monotone. Note that the uniform monotonicity of Φ' implies that Φ' is an injection.

From (6) and (v), it is clear that for any $u \in X$ with $\|u\| > 1$, we obtain

$$\frac{\langle \Phi'(u), u \rangle}{\|u\|} \geqslant \frac{\|u\|^{p^-}}{\|u\|} = \|u\|^{p^- - 1}$$

and thus $\lim_{n\to\infty}(\langle \Phi'(u), u \rangle / \|u\|) = \infty$, i.e., Φ' is coercive. Thus it is a surjection in view of the Minty-Browder Theorem (see Theorem 26 A(d), p. 557 in Ref. 17). Hence Φ' has an inverse mapping $(\Phi')^{-1} : X^* \to X$. Hence the continuity of $(\Phi')^{-1}$ is sufficient to ensure that Φ' is a homeomorphism. Let $\{f_n\}$ be a sequence of X^* and there exists a $f \in X^*$ such that $f_n \to f$ in X^*. Let u_n and u in X be such that $(\Phi')^{-1}(f_n) = u_n$ and $(\Phi')^{-1}(f) = u$. From the coercivity of Φ', it follows that the sequence $\{u_n\}$ is bounded in reflexive Banach space X. Without loss of generality, we can deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $\hat{u} \in X$ such that $u_n \rightharpoonup \hat{u}$ weakly in X, which implies

$$\begin{aligned}\lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(\hat{u}), u_n - \hat{u} \rangle \\ = \lim_{n\to\infty} \langle f_n - f, u_n - \hat{u} \rangle = 0.\end{aligned}$$

It follows by the assertion and the continuity of Φ' that $u_n \to \hat{u}$ in X and $\Phi'(u_n) \to \Phi'(\hat{u}) = \Phi'(u)$ in X^*. Moreover, since Φ' is an injection, we conclude that $u = \hat{u}$. So the continuity of $(\Phi')^{-1}$ is obtained. □

Proof of Theorem 2: As seen before, Φ and Ψ satisfy the regularity assumptions of Theorem 1. Firstly, let us introduce a piecewise function $\bar{v} \in X$:

$$\bar{v}(x) = \begin{cases} 0, & x \in \Omega \setminus B(x_0, D), \\ h, & x \in B(x_0, \frac{D}{2}), \\ \frac{2h}{D}(D - |x - x_0|), & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}) \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. The hypotheses (ii) and (v) yield

$$\begin{aligned}\frac{1}{p^+} \min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} \omega D^n \left(1 - \frac{1}{2^n}\right) \\ \leqslant \Phi(\bar{v}) \leqslant c_1 \omega D^n \left(1 - \frac{1}{2^n}\right) \\ \times \left[\frac{2h}{D} + \frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}\right]\end{aligned}$$

and

$$\begin{aligned}\Psi(\bar{v}) \geqslant \int_{B(x_0, \frac{D}{2})} F(x, \bar{v}) \, \mathrm{d}x \\ \geqslant \operatorname{ess\,inf}_{x\in\Omega} F(x, h) \omega \left(\frac{D}{2}\right)^n.\end{aligned}$$

From (10), it follows that $r < \Phi(\bar{v})$. Moreover, since the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is continuous, for each $u \in X$ with $\Phi(u) \leqslant r$,

$$\begin{aligned}\max_{x\in\Omega} |u(x)| \\ \leqslant c_0 \max\left\{(p^+ r)^{1/(p^-)}, (p^- r)^{1/(p^+)}\right\} = c_0 \gamma_r\end{aligned}$$

and so

$$\sup_{\Phi(u)\leqslant r} \Psi(u) \leqslant \int_\Omega \sup_{|t|\leqslant c_0\gamma_r} F(x,t)\,\mathrm{d}x.$$

From (11), we obtain

$$\frac{\sup_{\Phi(u)\leqslant r}\Psi(u)}{r} \leqslant \alpha_r < \beta_h \leqslant \frac{\Psi(\bar v)}{\Phi(\bar v)}$$

and so condition (i) of Theorem 1 is verified.

Next, we prove that for each $\lambda > 0$, the functional $\Phi - \lambda\Psi$ is coercive. For any $s \in C_+(\overline{\Omega})$, we have

$$|u(x)|^{s(x)} \leqslant |u(x)|^{s^+} + |u(x)|^{s^-}. \tag{12}$$

The hypothesis (9) and relations (8), (12) imply

$$\begin{aligned}\Psi(u) &\leqslant \int_\Omega c\big(|u| + \frac{1}{s^-}|u|^{s(x)}\big)\,\mathrm{d}x\\ &\leqslant \int_\Omega c\Big(|u| + \frac{1}{s^-}\big(|u(x)|^{s^+} + |u(x)|^{s^-}\big)\Big)\,\mathrm{d}x\\ &\leqslant c\|u\|_\infty|\Omega| + \frac{c}{s^-}\big(\|u\|_\infty^{s^+} + \|u\|_\infty^{s^-}\big)|\Omega|\\ &\leqslant cc_0\|u\||\Omega| + \frac{c}{s^-}\big((c_0\|u\|)^{s^+} + (c_0\|u\|)^{s^-}\big)|\Omega|.\end{aligned}$$

The above inequality and relation (6) give

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) \geqslant \frac{1}{p^+}\|u\|^{p^-} - \lambda cc_0\|u\||\Omega|\\ - \frac{\lambda c}{s^-}\big((c_0\|u\|)^{s^+} + (c_0\|u\|)^{s^-}\big)|\Omega|\end{aligned}$$

for any $u \in X$ with $\|u\| > 1$. Since $1 \leqslant s^- \leqslant s^+ < p^-$, the coercivity of $\Phi - \lambda\Psi$ is obtained. Finally, taking into account that

$$\bar\Lambda := \Big(\frac{1}{\beta_h}, \frac{1}{\alpha_r}\Big) \subseteq \Big(\frac{\Phi(\bar v)}{\Psi(\bar v)}, \frac{r}{\sup_{\Phi(u)\leqslant r}\Psi(u)}\Big),$$

Theorem 1 ensures that for each $\lambda \in \bar\Lambda$, the functional $\Phi - \lambda\Psi$ admits at least three critical points in X that are weak solutions for problem (1). □

Remark 1 Actually, in Theorem 2, it is enough to require $\operatorname{ess\,inf}_{x\in\Omega} F(x,t) \geqslant 0$ for all $t \in [0,h]$.

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