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A characterization of $L_3(4)$

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ABSTRACT: Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\mathrm{nse}(G) = \{s_k \mid k \in \omega(G)\}$. $L_3(2) \cong L_2(7)$ is uniquely determined by $\mathrm{nse}(G)$. In this paper, we prove that if G is a group such that $\mathrm{nse}(G) = \mathrm{nse}(L_3(4))$, then $G \cong L_3(4)$.

KEYWORDS: element order, linear group, Thompson's problem, number of elements of the same order

INTRODUCTION

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a group. The set of element orders of G is denoted by $\omega(G)$. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\mathrm{nse}(G) = \{s_k | k \in \omega(G)\}$. Let $\pi(G)$ denote the set of primes p such that G contains an element of order p. A finite group G is called a simple K_n-group if G is a simple group with $|\pi(G)| = n$. Thompson posed a very interesting problem related to algebraic number fields as follows[1].

Thompson's Problem. Let $T(G) = \{(n, s_n) | n \in \omega(G)$ and $s_n \in \mathrm{nse}(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

It was proved that if G is a group and M some simple K_i-group, $i = 3, 4$, then $G \cong M$ if and only if $|G| = |M|$ and $\mathrm{nse}(G) = \mathrm{nse}(M)$ (see Refs. 2, 3). And the groups A_{12}, A_{13} and $L_5(2)$ are characterizable by order and nse (see Refs. 4–6).

We only consider the sizes of elements of the same order but disregard the actual orders of elements in $T(G)$ of Thompson's Problem. In other words, can $\mathrm{nse}(G)$ characterize finite simple groups? Some groups for $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$, have been characterized by only the set $\mathrm{nse}(G)$ (see Refs. 7, 8). In this paper it is shown that the projective special linear group $L_3(4)$ can also be characterized by $\mathrm{nse}(L_3(4))$.

SOME LEMMAS

Lemma 1 (Ref. 9) *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2 (Ref. 10) *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Lemma 3 (Ref. 8) *Let G be a group containing more than two elements. If the maximum number s of elements of the same order in G is finite, then G is finite and $|G| \leqslant s(s^2 - 1)$.*

Lemma 4 (Theorem 9.3.1 of Ref. 11) *Let G be a finite soluble group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $(m, n) = 1$. Let $\pi = \{p_1, \dots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \dots, s\}$:*
(i) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ *for some* p_j.
(ii) *The order of some chief factor of G is divisible by $q_i^{\beta_i}$*

PROOF OF THEOREM

Let G be a group such that $\mathrm{nse}(G) = \mathrm{nse}(L_3(4))$, and s_n be the number of elements of order n. By Lemma 3 we have G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 1 and the above discussion, we have

$$\begin{cases} \phi(m) \mid s_m, \\ m \mid \sum_{d \mid m} s_d. \end{cases} \tag{1}$$

Theorem 1 *Let G be a group with $\mathrm{nse}(G)\,\mathrm{nse}(L_3(4)) = \{1, 315, 2240, 3780, 5760, 8064\}$, where $L_3(4)$ is the projective special linear group of degree 3 over the finite field of order 4. Then $G \cong L_3(4)$.*

Proof: We prove the theorem by first proving that $\pi(G) \subsetneqq \{2, 3, 5, 7\}$, second showing that $|G| = |L_3(4)|$, and so $G \cong L_3(4)$.

By (i), $\pi(G) \subsetneqq \{2,3,5,7,19\}$. If $m > 2$, then $\phi(m)$ is even, then $s_2 = 315$, $2 \in \pi(G)$. If $3,5,7 \in \pi(G)$, then $s_3 = 2240$, $s_5 = 8064$, and $s_7 = 5760$. In the following, we prove that $19 \notin \pi(G)$. If $19 \in \pi(G)$, then by (i), $s_{19} = 3780$. If $2 \cdot 19 \in \omega(G)$, then $s_{38} \notin \mathrm{nse}(G)$. Therefore $38 \notin \omega(G)$. It follows that the Sylow 19-subgroup P_{19} acts fixed point freely on the set of elements of order 2, then $|P_{19}| \mid s_2 (= 315)$, a contradiction. Hence $\pi(G) \subsetneqq \{2,3,5,7\}$.

If $2^i \in \omega(G)$, then $\phi(2^i) = 2^{i-1} \mid s_{2^i}$ and so $1 \leqslant i \leqslant 8$. If $3^j \in \omega(G)$, then $\phi(3^j) \mid s_{3^j}$ and so $1 \leqslant j \leqslant 4$. If $5^k \in \omega(G)$, then $1 \leqslant k \leqslant 2$. If $5^2 \in \omega(G)$, then $s_{25} \notin \mathrm{nse}(G)$, a contradiction. Hence $k = 1$. If $7^l \in \omega(G)$, then $1 \leqslant l \leqslant 2$. If $7^2 \in \omega(G)$, then $s_{49} \notin \mathrm{nse}(G)$, a contradiction. Therefore $l = 1$.

If $2^m \cdot 3^n \in \omega(G)$, then $1 \leqslant m \leqslant 7$ and $1 \leqslant n \leqslant 4$. If $2^a \cdot 5 \in \omega(G)$, then $1 \leqslant a \leqslant 6$. If $2^b \cdot 7 \in \omega(G)$, then $1 \leqslant b \leqslant 7$.

If $3^c \cdot 5 \in \omega(G)$, then $1 \leqslant c \leqslant 3$. By (i), $s_{15} = s_{45} = 5760$. If $3^d \cdot 7 \in \omega(G)$, then $1 \leqslant d \leqslant 3$.

If $5 \cdot 7 \in \omega(G)$, the $s_{35} \notin \mathrm{nse}(G)$, a contradiction. Hence $5 \cdot 7 \notin \omega(G)$.

If $2^e \cdot 3^f \cdot 5$, then $1 \leqslant e \leqslant 5$ and $1 \leqslant f \leqslant 4$. If $2^g \cdot 3^h \cdot 7$, then $1 \leqslant g \leqslant 6$ and $1 \leqslant h \leqslant 3$.

Hence we have $\omega(G) \subsetneqq \{1,2,2^2,2^3,2^4,2^5,2^6,2^7,2^8\} \cup \{3,3^2,3^3,3^4\} \cup \{5\} \cup \{7\} \cup \{2\cdot3,2^2\cdot3,2^3\cdot3,2^4\cdot3,2^5\cdot3,2^6\cdot3,2^7\cdot3,2\cdot3^2,2^2\cdot3^2,2^3\cdot3^2,2^4\cdot3^2,2^5\cdot3^3,2^6\cdot3^2,2^7\cdot3^2,2\cdot3^3,2^2\cdot3^3,2^3\cdot3^3,2^4\cdot3^3,2^5\cdot3^3,2^6\cdot3^3,2^7\cdot3^3,2\cdot3^4,2^2\cdot3^4,2^3\cdot3^4,2^4\cdot3^4,2^5\cdot3^4,2^6\cdot3^4,2^7\cdot3^4\} \cup \{2\cdot5,2^2\cdot5,2^3\cdot5,2^3\cdot5,2^4\cdot5,2^5\cdot5,2^6\cdot5\} \cup \{2\cdot7,2^2\cdot7,2^3\cdot7,2^4\cdot7,2^5\cdot7,2^6\cdot7,2^7\cdot7\} \cup \{3\cdot5,3^3\cdot5,3^3\cdot5\} \cup \{2\cdot3\cdot5,2^2\cdot3\cdot5,2^3\cdot3\cdot5,2^4\cdot3\cdot5,2^5\cdot3\cdot5,2\cdot3^2\cdot5,2^2\cdot3^2\cdot5,2^3\cdot3^2\cdot5,2^4\cdot3^2\cdot5,2^5\cdot3^2\cdot5,2\cdot3^3\cdot5,2^2\cdot3^3\cdot5,2^3\cdot3^3\cdot5,2^4\cdot3^3\cdot5,2^5\cdot3^3\cdot5,2\cdot3^4\cdot5,2^2\cdot3^4\cdot5,2^3\cdot3^4\cdot5,2^4\cdot3^4\cdot5,2^5\cdot3^4\cdot5\} \cup \{2\cdot3\cdot7,2^2\cdot3\cdot7,2^3\cdot3\cdot7,2^4\cdot3\cdot7,2^5\cdot3\cdot7,2^6\cdot3\cdot7,2\cdot3^2\cdot7,2^2\cdot3^2\cdot7,2^3\cdot3^2\cdot7,2^4\cdot3^2\cdot7,2^5\cdot3^2\cdot7,2^6\cdot3^2\cdot7,2\cdot3^3\cdot7,2^2\cdot3^3\cdot7,2^3\cdot3^3\cdot7,2^4\cdot3^3\cdot7,2^5\cdot3^3\cdot7,2^6\cdot3^3\cdot7,2\cdot3^4\cdot7,2^2\cdot3^4\cdot7,2^3\cdot3^4\cdot7,2^4\cdot3^4\cdot7,2^5\cdot3^4\cdot7,2^6\cdot3^4\cdot7\}$

Hence $|G| = 20160 + 2240k_1 + 3780k_2 + 5760k_3 + 8064k_4 = 2^l \cdot 3^m \cdot 5^n \cdot 7^p$, where k_1, k_2, k_3, k_4, l, m, n and p are non-negative integers. So $5040 + 560k_1 + 945k_2 + 1440k_3 + 2016k_4 = 2^{l-2} \cdot 3^m \cdot 5^n \cdot 7^p$. Now we consider the cases.

Case (a). $\pi(G) = \{2\}$. In this case, $5040 + 560k_1 + 945k_2 + 1440k_3 + 2016k_4 = 2^{l-2}$ and $0 \leqslant k_1 + k_2 + k_3 + k_4 \leqslant 3$. Since $2^4(336 + 35k_1 + 90k_3 + 126k_4) + 945k_2 = 2^{l-2}$, it follows that $2^4 \mid k_2$, Hence $k_2 = 0$. So $336 + 35k_1 + 90k_3 + 126k_4 = 2^{l-6}$. It is easy to get that $k_1 = 0$ or 3, which means $3 \mid 2^{l-6}$. So $l = 6$, and $336 + 35k_1 + 90k_3 + 126k_4 = 1$, a contradiction.

Case (b). $\pi(G) = \{2,3\}$. We know that $\exp(P_3) = 3, 9, 27$, or 81. By (i), $s_9, s_{27} \in \{3780, 5760, 8064\}$ and so we get that $s_{81} \notin \mathrm{nse}(G)$. Hence $\exp(P_3) = 3$, 9, or 27.

Subcase b.1. $\exp(P_3) = 3$. By Lemma 1, $|P_3| \mid 1 + s_3$, and so $|P_3| \leqslant 9$. If $|P_3| = 3$, then P_3 is cyclic and by (i), $n_3 = s_3/\phi(3) = 2240/2 = 1120$, which means $5 \mid |G|$, a contradiction. If $|P_3| = 9$, then $5040 + 560k_1 + 945k_2 + 1440k_3 + 2016k_4 = 2^{l-2} \cdot 3^2$, where k_1, k_2, k_3, k_4 and m are non-negative integers and $0 \leqslant k_1 + k_2 + k_3 + k_4 \leqslant 11$. Hence we have $5040 \leqslant 2^{l-2} \cdot 3^2 \leqslant 5040 + 2016 \cdot 11$ and so $l = 10$ or 11.

Case $l = 10$. Therefore $9(560 + 105k_2 + 224k_4) + 560k_1 + 1440k_3 = 2^8 \cdot 9$. Thus $9 \mid k_1$, and so $k_1 = 0$ or $k_1 = 9$

Case $k_1 = 0$. If $k_3 = 0$, then $16(35 + 14k_4) + 105k_2 = 2^{10}$ and so $k_2 = 0$. But the equation $35 + 14k_4 = 2^6$ has no solution. If $k_3 = 3$, then $4(140 + 95k_3 + 6k_4) + 105k_2 = 2^{10}$, and so $4 \mid k_2$. Hence $k_2 = 0, 4, 8$. If $k_2 = 0$, then the equation $560 + 480 + 224k_4 = 2^{10}$ has no solution in $\mathbb{N}$. If $k_2 = 4$, then the equation $1360 + 224k_4 = 1024$ has no solution in $\mathbb{N}$. Then $k_2 = 8$ and so $k_4 = 0$, we also get a contradiction. If $k_3 = 9$, then $4 \mid k_2$ and so $k_2 = 0$. Therefore we have that $1700 + 224k_4 = 2^{10}$, but the equation has no solution in $\mathbb{N}$.

Case $k_1 = 9$. We can get the same result as $k_1 = 0$.

Case $l = 11$. We also get same the results as $l = 10$.

Subcase b.2. $\exp(P_3) = 9$. By Lemma 1, $|P_3| \mid 1 + s_3 + s_9$ and so $|P_3| \leqslant 27$. If $|P_3| = 9$, then P_3 is cyclic. By (i) $s_9 = 3780, 5760$, 8064, and so $n_3 = s_9/\phi(9) = 945, 1440, 2016$. It follows that $5 \mid |G|$ or $7 \mid |G|$, a contradiction. If $|P_3| = 27$, then $5040 + 560k_1 + 945k_2 + 1440k_3 + 2016k_4 = 2^{l-2} \cdot 3^3$, where k_1, k_2, k_3, k_4 and m are non-negative integers and $0 \leqslant k_1 + k_2 + k_3 + k_4 \leqslant 16$. Hence we have $5040 \leqslant 2^{l-2} \cdot 3^3 \leqslant 5040 + 2016 \cdot 16$ and so $l = 8, 9$, or 10.

Case $l = 8$. We have that $9 \mid k_1$, and so $k_1 = 0$ or 9. If $k_1 = 0$, then $16(35 + 10k_3 + 14k_4) + 105k_2 = 2^6 \cdot 3$, it follows that $16 \mid k_2$. Hence $k_2 = 0$, and so $35 + 10k_3 + 14k_4 = 12$, but the equation has no solution in $\mathbb{N}$.

Case $l = 9, 10$. As with the case $l = 8$, we get a contradiction.

Subcase b.3. $\exp(P_3) = 27$. By Lemma 1, $|P_3| \mid 1 + s_3 + s_9 + s_{27}$, and so $|P_3| \leqslant 81$. If $|P_3| = 27$, then P_3 is cyclic. By (i), $s_{27} = 3780, 5760$ or 8064 and so $n_3 = s_{27}/\phi(27) = 210, 320$, 448, it follows that $5 \mid |G|$ or $7 \mid |G|$, a contradiction. If $|P_3| = 81$, then by Lemma 2, $s_{81} = 27t$ for some integer t and so $s_{27} = 3780$. But by Lemma 1, $27 \mid 1+s_3+s_9+s_{27}(= 9800, 11780)$, a contradiction.

Case (c). $\pi(G) = \{2, 5\}$. Since $s_5 = 8064$, by Lemma 1, $5 \mid 1+s_5$, so we have $|P_5| = 5$. Then by (i) $n_5(G) = s_5/\phi(5) = 2016$, which means that $7 \mid |G|$, a contradiction.

Case (d). $\pi(G) = \{2, 7\}$. Since $s_7 = 5760$, by Lemma 1, $7 \mid 1+s_7$, so we have $|P_7| = 7$. Then by (i) $n_7(G) = s_7/\phi(7) = 960$, which means that $5 \mid |G|$, a contradiction.

Case (e). $\pi(G) = \{2, 3, 5\}$. The proof is similar to Case (c).

Case (f). $\pi(G) = \{2, 3, 7\}$. The proof is similar to Case (d).

Case (g). $\pi(G) = \{2, 5, 7\}$. The proof is similar to Case (c) or (d).

Case (h). $\pi(G) = \{2, 3, 5, 7\}$. In the following, we first show that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ or $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$, then prove that there is no group such that $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$ and $\mathrm{nse}(G) = \mathrm{nse}(L_3(4))$, and show that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and $\mathrm{nse}(G) = \mathrm{nse}(L_3(4))$, we have $G \cong L_3(4)$.

Step 1. $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ or $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$ We know that $|P_5| = 5$, $|P_7| = 7$. We will show that $15 \notin \omega(G)$. If $15 \in \omega(G)$, set P and Q are Sylow 5-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G. Therefore we have $s_{15} = \phi(15) \cdot n_5 \cdot k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_5)$. As $n_5 = s_5/\phi(5) = 8064/4 = 2016$, $2016 \mid s_{15}$ and so $s_{15} = 8064$. But $15 \mid 1 + s_3 + s_5 + s_{15}(= 18369)$, a contradiction. We conclude that $15 \notin \omega(G)$. It follows that the group P_3 acts fixed point freely on the set of elements of order 19 and so $|P_3| \mid s_5(= 8064)$. So we have $|P_3| \mid 3^2$.

We will show that $14 \notin \omega(G)$. If $14 \in \omega(G)$, set P and Q are Sylow 7-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G. Therefore we have $s_{14} = \phi(14) \cdot n_7 \cdot k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_7)$. As $n_7 = s_7/\phi(7) = 5760/6 = 960$, $960 \mid s_{14}$ and so $s_{14} = 5760$. But $14 \mid 1 + s_2 + s_7 + s_{14}(= 11836)$, we get a contradiction. We have that $14 \notin \omega(G)$. It follows that the group P_2 acts fixed point freely on the set of elements of order 7 and so $|P_2| \mid s_7(= 8064)$. So we have $|P_2| \mid 2^7$. Therefore we have $|G| = 2^l \cdot 3^m \cdot 5 \cdot 7$. But

$$\sum_{s_k \in \mathrm{nse}(G)} s_k = 20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \leqslant 2^l \cdot 3^m \cdot 5 \cdot 7.$$

So we have the results.

Step 2. $G \cong L_3(4)$. First show that there is no group such that $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$ and $\mathrm{nse}(G) = \mathrm{nse}(L_3(4))$. Then get the result in Ref. 3. Since $s_7 = 5760$, $n_7 = s_7/\phi(7) = 5760/6 = 2^6.3.5$. Since G is soluble, then by Lemma 4, $5 \equiv 1 \pmod 7$, a contradiction. So G is insoluble. Therefore we can suppose that G has a normal series $1 \lhd K \lhd L \lhd G$ such that L/K is isomorphic to a simple K_i-group with $i = 3, 4$ as 25 and 49 do not divide order of G. If L/K is isomorphic to a K_3-simple group, then from Ref. 12 $L/K \cong A_5$, A_6, $L_2(7)$, $L_2(8)$, $U_3(3)$, or $U_4(2)$. From Ref. 13, $n_5(L/K) = n_5(A_5) = 6$, and so $n_5(G) = 6t$ and $5 \nmid t$ for some integer t. Hence the number of elements of order 5 in G is $s_5 = 6t \cdot 4 = 24t$. Since $s_5 \in \mathrm{nse}(G)$, then $s_5 = 8064$ and so $t = 2016$. Therefore $2^5 \cdot 3^2 \cdot 7 \mid |K| \mid 2^4 \cdot 3 \cdot 7$, which is a contradiction. For the other cases, similarly we can rule out these. If L/K is isomorphic to a K_4-simple group, then from Ref. 14, we have the following. L/K is isomorphic to one of the following groups: A_7, A_8, A_9, A_{10}; $L_2(49)$, $L_3(4)$, $S_4(7)$, $S_6(2)$, $U_3(5)$, $U_4(3)$, J_2, or $O_8^+(2)$.

If $L/K \cong A_7$, then from Ref. 13, $n_7(L/K) = 120$, and so $n_7(G) = 120t$ with $7 \nmid t$. Hence the number of elements of order 7 in G is $s_7 = 120t \cdot 6 = 720t$. Thus $s_7 = 5760$ and $t = 8$. Now $n_7(G) = 960$. On the other hand, by Sylow's Theorem $n_7(G) = 1, 8, 64$ or 288, a contradiction. For the remaining groups except $L_3(4)$, we can also rule out by the methods as A_7.

In the following we show that $G \cong 2.L_3(4)$. From above, $L/K \cong L_3(4)$. Let $\bar{G} = G/K$ and $\bar{L} = L/K$. Then $L_3(4) \leqslant \bar{L} \cong \bar{L}C_{\bar{G}}(\bar{L})/C_{\bar{G}}(\bar{L}) \leqslant \bar{G}/C_{\bar{G}}(\bar{L}) = N_{\bar{G}}(\bar{L})/C_{\bar{G}}(\bar{L}) \leqslant \mathrm{Aut}(\bar{L})$

Set $M = \{xK \mid xK \in C_{\bar{G}}(\bar{L})\}$, then $G/M \cong \bar{G}/C_{\bar{G}}(\bar{L})$ and so $L_3(4) \leqslant G/M \leqslant \mathrm{Aut}(L_3(4))$. Therefore $G/M \cong L_3(4)$, $G/M \cong 2.L_3(4)$ or G/M is isomorphic to $3.L_3(4)$, $S_3.L_3(4)$, $2.S_3.L_3(4)$, or $2.2.L_3(4)$.

If G/M is isomorphic to $3.L_3(4)$, $S_3.L_3(4)$, $2.S_3.L_3(4)$ or $2.2.L_3(4)$, then order consideration rules out this case.

If $G/M \cong L_3(4)$, $|M| = 2$ and so G has a normal subgroup of order 2, which is generated by a central involution. Thus G has an element of order 14, which is a contradiction.

If $G/M \cong 2.L_3(4)$, then $|M| = 1$. By Sylow's theorem, $n_7(G) = 1, 8, 36, 64, 288$. On the other hand, since $s_7 = 5760$ and $\exp(P_7) = 7$, we have $n_7 = s_7/\phi(7) = 5760/6 = 960$, a contradiction.

Second, since $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and $\mathrm{nse}(G) = \mathrm{nse}(L_3(4))$, we have from Ref. 3, $G \cong L_3(4)$. This completes the proof. □

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