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On the non-abelian tensor square of groups of order p^4 where p is an odd prime

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ABSTRACT: In this paper, we determine the non-abelian tensor square, $G \otimes G$, for non-abelian groups of order p^4, where p is an odd prime.

KEYWORDS: p-groups

INTRODUCTION

The non-abelian tensor products have their roots in algebraic K-theory as well as in homotopy theory and were introduced by Brown and Loday[1,2]. The non-abelian tensor square is a special case of the non-abelian tensor product where G and H are the same group. The non-abelian tensor square of a group G, denoted as $G \otimes G$ is generated by $gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h)$, $g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h')$, for all $g, g', h, h' \in G$, where ${}^h g = hgh^{-1}$ denotes the conjugate of g by h.

In 1911, Burnside[3] obtained the classification of groups of order p^4. Jang Oh[4] proved that non-abelian groups of order p^4 satisfy the conditions in the following theorem.

Theorem 1 *Let G be a non-abelian group of order p^4. Then one of the following holds.*
(i) $|Z(G)| = p^2$, $|G'| = p$, *and* $G' \subseteq Z(G)$
(ii) $|Z(G)| = p$, $|G'| = p^2$, *and* $Z(G) \subseteq G'$.

In this paper, we focus on the non-abelian groups of order p^4 that satisfy the conditions in Theorem 1(i).

Theorem 2 *Let G be a group of order p^4, where p is an odd prime. Then G is isomorphic to exactly one group in the following list.*

$$G_1 = \langle x, y | x^{p^3} = y^p = 1, x^y = x^{1+p^2} \rangle,$$
$$G_2 = \langle x, y, z | x^p = y^p = z^{p^2} = 1, [x, y] = [y, z] = 1, [x, y] = z^p \rangle,$$
$$G_3 = \langle x, y | x^{p^2} = y^{p^2} = 1, x^y = x^{1+p} \rangle,$$
$$G_4 = M_p \times \langle w \rangle,$$
$$G_5 = \langle x, y, z | x^{p^2} = y^p = z^p = 1, [x, y] = [y, z] = 1, x^y = x^{1+p} \rangle,$$
$$G_6 = \langle x, y, z | x^{p^2} = y^p = z^p = 1, [x, y] = z, [x, z] = [y, z] = 1 \rangle,$$

where

$$M_p = \langle x, y, z | x^p = y^p = z^p = 1, [x, y] = [y, z] = 1, [x, y] = z \rangle,$$

$w = \langle w | w^p = 1 \rangle$, $[x, y] = [y, z] = 1$, *and* $[x, y] = z^p$.

PRELIMINARIES

This section includes some basic results on the Schur multiplier and non-abelian tensor square of groups which are used in order to prove our main theorem.

In 2001, Seon Ok[5] obtained the Schur multiplier of groups of order p^4, where p is an odd prime as stated in the following theorem.

Theorem 3 *Let G be groups of order p^4, where p is an odd prime. Then exactly one of the following holds:*

$$M(G) = \begin{cases} 1, & G \text{ is } G_1, \\ (\mathbb{Z}_p)^2, & G \text{ is } G_2, G_5 \text{ or } G_6, \\ \mathbb{Z}_p, & G \text{ is } G_3, \\ (\mathbb{Z}_p)^4, & G \text{ is } G_4. \end{cases}$$

The following five theorems stated are used to compute the non-abelian tensor square of some finite groups.

In 1987, Brown et al[6] computed the non-abelian tensor square of some groups such as quaternion groups, dihedral groups, symmetric groups and metacyclic groups. The non-abelian tensor square of metacyclic group is presented in the following theorem.

Theorem 4 *If* $G=\langle x,y|y^n=x^m=1, xyx^{-1}=y^l\rangle$ *where* $l^m\equiv 1 \bmod n$ *and* n *is an odd number, then* $G\otimes G=\mathbb{Z}_m\times\mathbb{Z}_{m_1}\times\mathbb{Z}_{m_2}\times\mathbb{Z}_{m_3}$ *where* $m_1=(n,l-1)$, $m_2=(n,l-1,1+l+\cdots+l^{m-1})$, $m_3=(n,(1+l)+\cdots+l^{m-1})$.

Brown et al[6] also computed the non-abelian tensor square of direct product of two groups. In this case, the non-abelian tensor square can be computed by the use of Theorem 5. They also determined two properties for Whitehead's universal quadratic functor Γ as stated in Theorem 6.

Theorem 5 *Let* G *and* H *be groups. Then* $(G\times H)\otimes(G\times H)\cong(G\otimes G)\times(G\otimes H)\times(H\otimes G)\times(H\otimes H)$.

Theorem 6 *Let* G *and* H *be abelian groups. Then*
(i) $\Gamma(G\times H)=\Gamma G\times\Gamma H\times\Gamma(G\otimes H)$,
(ii)

$$\Gamma\mathbb{Z}_n=\begin{cases}\mathbb{Z}_n, & n \text{ is odd},\\ \mathbb{Z}_{2n}, & n \text{ is even}.\end{cases}$$

In the following theorem, Blyth et al[7] computed the non-abelian tensor square of group G with G^{ab} which is finitely generated.

Theorem 7 *Let* G *be a group such that* G^{ab} *is finitely generated. If* G^{ab} *has no element of order two or if* G' *has no complement in* G *then* $G\otimes G\cong\Gamma(G^{ab})\times G\wedge G$.

Nakaoka[8] gives the conditions that can be used to compute the non-abelian tensor of a finite group.

Theorem 8 *Let* G *be a finite group and* $i\geqslant 0$. *Then*
(i) *there is an exact sequence*
$1\to[G_{i+1},G_i^{\varphi}]\to\tau(G_i,G_i)\to\tau(G_i^{ab},G_i^{ab})\to 1$
where $[G_{i+1},G_i^{\varphi}]\leqslant\tau(G_i,G_i)$,
(ii) $|G_i\otimes G_i|\leqslant|G_i^{ab}\otimes_{\mathbb{Z}}G_i^{ab}||G_{i+1}\otimes G_i|$.

The Schur multiplier, non-abelian tensor square and capability of groups of order p^2q have been considered by Rashid et al in Ref. 9, where p and q are distinct primes. In Ref. 10, they also computed the Schur multiplier of groups of order $8q$, where q is an odd prime.

PROOF OF MAIN THEOREM

In this paper, we focus on the non-abelian tensor square of non-abelian group of order p^4, where p is an odd prime. The non-abelian tensor square of groups of order p^4, where p is an odd prime is computed in the next theorem.

Theorem 9 *Let* G *be a group of order* p^4, *where* p *is an odd prime. Then*

$$G\otimes G=\begin{cases}\mathbb{Z}_{p^2}\times(\mathbb{Z}_p)^3, & G \text{ is } G_1,\\ (\mathbb{Z}_p)^9, & G \text{ is } G_2 \text{ or } G_5,\\ (\mathbb{Z}_{p^2})^2\times(\mathbb{Z}_p)^2, & G \text{ is } G_3,\\ (\mathbb{Z}_p)^{11}, & G \text{ is } G_4,\\ \mathbb{Z}_{p^2}\times(\mathbb{Z}_p)^5, & G \text{ is } G_6.\end{cases}$$

Proof: Let G be a non-abelian group of order p^4, where p is an odd prime. By Theorem 2 there are 6 types of these groups. First, we prove for G_1, by choosing $n=p^3$, $m=p$, $l=1+p^2$, and G_1 is a metacyclic group. Then by Theorem 4, $G\otimes G=\mathbb{Z}_m\times\mathbb{Z}_{m_1}\times\mathbb{Z}_{m_2}\times\mathbb{Z}_{m_3}$, where $m_1=(p^3,1+p^2-1)=p^2$, $m_2=(p^3,1+p^2-1,1+(1+p^2)+\cdots+(1+p^2)^{p-1})=p$, and $m_3=(p^3,(1+1+p^2)+\cdots+(1+p^2)^{p-1})=p$. Therefore $G\otimes G\cong\mathbb{Z}_{p^2}\times(\mathbb{Z}_p)^3$.

For group G_2, by Theorem 8, the following computations are considered. In this group, $G_2'=\mathbb{Z}_p$ and $G_2^{ab}=(\mathbb{Z}_p)^3$. As $G_2^{ab}\otimes G_2^{ab}\cong(\mathbb{Z}_p)^9$, $G_2'\wedge G_2'=1$ and $G_2'\otimes_{\mathbb{Z}[G_2^{ab}]}I(G_2^{ab})\cong\mathbb{Z}_p$. The exact sequence $1\to[G_2,G_2^{\varphi}]\to\tau(G_2,G_2)\to\tau(G_2^{ab},G_2^{ab})\to 1$ shows that $|\tau(G_2,G_2)|$ divides p^9 where $[G_2',G_2^{\varphi}]\leqslant\tau(G_2,G_2)$ and $\tau(G_2^{ab},G_2^{ab})\cong G_2^{ab}\otimes_{\mathbb{Z}}G_2^{ab}$. Hence $|[G_2',G_2^{\varphi}]|$ divides p. Then from the exact sequence, we obtain $\tau(G_2,G_2)\cong(\mathbb{Z}_p)^9$. Since $\tau(G_2,G_2)$ is abelian and $\lambda:\tau(G_2,G_2)\to G_2$ is the homomorphism, it follows p divides $\tau|G_2,G_2|$. Therefore $G_2\otimes G_2\cong(\mathbb{Z}_p)^9$.

Next, we consider the third case G_3, by choosing $n=p^2, m=p^2, l=1+p$, G_3 is metacylic group. By using the same proof as G_1, then $G_3\otimes G_3=(\mathbb{Z}_{p^2})^2\times(\mathbb{Z}_p)^2$.

For group $G_4=M_p\times\langle w\rangle$, where M_p is isomorphic to non-abelian group of order p^3 of exponent p. Then by Theorem 5, $G_4\otimes G_4\cong(M_p\times\langle w\rangle)\otimes(M_p\times\langle w\rangle)\cong(\mathbb{Z}_p)^{11}$.

For group G_5, we have $G_5=K\times\langle z\rangle$ where $K=\langle x,y|x^{p^2}=y^p=1, x^y=x^{1+p}\rangle$, $\langle z\rangle=\langle z|z^p=1\rangle$. We know that K is isomorphic to non-abelian group of order p^3 of exponent p^2. Again by using Theorem 5, we have $G_5\otimes G_5\cong(K\times\langle z\rangle)\otimes(K\times\langle z\rangle)\cong(\mathbb{Z}_p)^9$.

Lastly, for group G_6, we have $G_6'=\mathbb{Z}_p$ and $G_6^{ab}=\mathbb{Z}_p\times\mathbb{Z}_{p^2}$. The exact sequence

$1 \to [G_6', G_6^{\varphi}] \to \tau(G_6, G_6) \to \tau(G_6^{ab}, G_6^{ab}) \to 1$. On the other hand, $(G_6 \wedge G_6)/M(G_6) \cong G_6'$. By Theorem 3, $M(G_6) \cong (\mathbb{Z}_p)^2$, that is $(G_6 \wedge G_6) \cong (\mathbb{Z}_p)^3$ and by Theorem 6, we have $\Gamma(G_6^{ab}) = \Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2}) = (\mathbb{Z}_p)^2 \times \mathbb{Z}_{p^2}$. G_6^{ab} is a finitely generated abelian group with no element of order 2. Then by Theorem 7, we have $G_6 \otimes G_6 \cong \Gamma(G_6^{ab}) \times G_6 \wedge G_6 \cong \mathbb{Z}_{p^2} \times (\mathbb{Z}_p)^5$. □

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