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Some remarks on cardinal arithmetic without choice

Supakun Panasawatwong*, **Pimpen Vejjajiva**

Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330 Thailand

*Corresponding author, e-mail: supakun.p@gmail.com

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ABSTRACT: One important consequence of the Axiom of Choice is the absorption law of cardinal arithmetic. It states that for any cardinals $\mathfrak{m}$ and $\mathfrak{n}$, if $\mathfrak{m} \leqslant \mathfrak{n}$ and $\mathfrak{n}$ is infinite, then $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$ and if $\mathfrak{m} \neq 0$, $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{n}$. In this paper, we investigate some conditions that make this property hold as well as an instance when such a property cannot be proved in the absence of the Axiom of Choice. We further find some conditions that preserve orderings in cardinal arithmetic. These results also lead to the conditions that make the cancellation law of cardinal arithmetic hold in set theory without choice.

INTRODUCTION

The Axiom of Choice (AC) plays many important roles in cardinal numbers. Many important theorems cannot be proved without it. We cannot prove that every infinite set has a denumerable subset. We also cannot calculate the sums or products of some cardinals. Things are more awkward since any two cardinal numbers may not be compared to each other. However, its non-constructiveness causes some paradoxical situations, for example, the existence of a non-measurable subset of $\mathbb{R}$ and the Banach-Tarski paradox [1]. Some mathematicians do not accept this axiom, so their works must be done without it. Therefore it is interesting to know which theorems can and which theorems cannot be proved without AC.

One important consequence of AC is the absorption law of cardinal arithmetic. It states that for any cardinals $\mathfrak{m}$ and $\mathfrak{n}$, if $\mathfrak{m} \leqslant \mathfrak{n}$ and $\mathfrak{n}$ is infinite, then $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$ and if $\mathfrak{m} \neq 0$, $\mathfrak{m} \cdot \mathfrak{n} = \mathfrak{n}$. This makes it easy to compute the sum and product of any two cardinals. Without AC, this property is no longer true. In this paper, we will first investigate some conditions that make this property hold as well as an instance when such a property cannot be proved in the absence of AC.

As mentioned earlier, without AC, some cardinals are not comparable. As a result of the lack of the absorption law of cardinal arithmetic, we sometimes cannot calculate cardinal sums or products. Thus it is not easy to decide whether the inequality $\mathfrak{m} + \mathfrak{p} < \mathfrak{n} + \mathfrak{q}$ is true or false when cardinals $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$, and $\mathfrak{q}$ are given and sometimes we are not able to know the answer. In set theory with AC, by the absorption law together with the fact that any two cardinals are comparable, we can answer the question immediately. Thus without AC, it is interesting to know what conditions would allow us to preserve orderings in cardinal arithmetic. These results also lead to the conditions that make the cancellation law of cardinal arithmetic hold in set theory without choice.

We first give some background in set theory followed by our main results.

PRELIMINARIES

All basic concepts in set theory used in this paper are defined in the usual way. We use $a, b, c, \ldots, A, B, C, \ldots$ for sets. We write $\langle A, B \rangle$ for the ordered pair of A and B and $f \restriction A$ for the restriction of a function f to A. We write $A \approx B$ and say A and B have the same *cardinality* if they are *equinumerous*, i.e., there is a one-to-one function from A onto B and $A \preceq B$ if there is a one-to-one function from A into B. ZF denotes the Zermelo-Fraenkel set theory and ZFC denotes ZF with the Axiom of Choice.

We write $|A|$ for the cardinality of A. An equivalent of AC is the Well-ordering Theorem which states that every set can be well-ordered. Thus in ZFC, every set is equinumerous to some ordinal. We let such least ordinal be the cardinality of that set. In the absence of AC, we cannot guarantee that a set is equinumerous to some ordinal. Thus the actual definition of the cardinality of a set is quite complicated and will be omitted since we will not use it directly. All we need to know is that it is defined so that the following property holds: $|A| = |B| \leftrightarrow A \approx B$ for any sets A and B.

$\mathfrak{m}$ is a *cardinal number* or simply a *cardinal* if $\mathfrak{m} = |M|$ for some set M. For any cardinals $\mathfrak{m}$ and $\mathfrak{n}$ where $\mathfrak{m} = |M|$ and $\mathfrak{n} = |N|$, we say $\mathfrak{m}$ is *less* *than or equal to* $\mathfrak{n}$, written $\mathfrak{m} \leqslant \mathfrak{n}$, if $M \preceq N$ and write $\mathfrak{m} < \mathfrak{n}$ if $\mathfrak{m} \leqslant \mathfrak{n}$ and $\mathfrak{m} \neq \mathfrak{n}$. Cardinal arithmetic can be defined as follows. For any cardinals $\mathfrak{m}$ and $\mathfrak{n}$ where $\mathfrak{m} = |M|$ and $\mathfrak{n} = |N|$, $\mathfrak{m}+\mathfrak{n} = |M \cup N|$ where $M \cap N = \emptyset$, $\mathfrak{m} \cdot \mathfrak{n} = |M \times N|$, and $\mathfrak{m}^{\mathfrak{n}} = |^N M|$ where $^N M = \{f \mid f : N \to M\}$.

Addition and multiplication of cardinals are commutative and associative. Distributive property of multiplication over addition also holds.

The following theorems[2] are basic properties of cardinals needed for later work.

Theorem 1 (Schröder-Bernstein Theorem) *If* $\mathfrak{m} \leqslant \mathfrak{n}$ *and* $\mathfrak{n} \leqslant \mathfrak{m}$*, then* $\mathfrak{m} = \mathfrak{n}$ *for all cardinals* $\mathfrak{m}$ *and* $\mathfrak{n}$.

Theorem 2 *If* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are cardinals such that* $\mathfrak{m} \leqslant \mathfrak{n}$*, then there exists a cardinal* $\mathfrak{p}$ *such that* $\mathfrak{m} + \mathfrak{p} = \mathfrak{n}$.

Theorem 3 *If* $\mathfrak{m}$*,* $\mathfrak{n}$*, and* $\mathfrak{p}$ *are cardinals such that* $\mathfrak{m} < \mathfrak{n}$*, then*

$$\mathfrak{m} + \mathfrak{p} \leqslant \mathfrak{n} + \mathfrak{p},$$
$$\mathfrak{m} \cdot \mathfrak{p} \leqslant \mathfrak{n} \cdot \mathfrak{p},$$
$$\mathfrak{p}^{\mathfrak{m}} \leqslant \mathfrak{p}^{\mathfrak{n}} \quad \textit{where } \mathfrak{p} > 0.$$

Each natural number is constructed so that it is the set of all smaller natural numbers, namely, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}, \dots$ and so on. The basic properties of natural numbers will be omitted and will be used in the ordinary way. Full details can be found in any elementary Set Theory textbooks.

Let ω denote the set of all natural numbers.

A set is *finite* if it is equinumerous to some natural number. Otherwise it is *infinite*. It is a basic property that every finite set is equinumerous to a unique natural number. We let such natural number be the cardinality of that set. A cardinal is finite if it is the cardinality of a finite set. Otherwise it is infinite. Note that every natural number is a finite cardinal and vice versa.

The cardinality of an infinite well-ordered set is called an *aleph*. Thus an aleph is the cardinality of some ordinal. $\aleph_0$ is $|\omega|$. It is easy to see that two alephs can be compared and every non-empty class of alephs has a least element.

A cardinal $\mathfrak{m}$ is *countable* if $\mathfrak{m} \leqslant \aleph_0$.

It is a consequence of AC that $\omega \preceq M$ for all infinite set M[3]. We call such an infinite set *Dedekind infinite*. Otherwise it is *Dedekind finite*. Note that a Dedekind finite set could be infinite. A cardinal $\mathfrak{m}$ is called *Dedekind infinite* if $\aleph_0 \leqslant \mathfrak{m}$ or equivalently M has a bijection to some of its proper subsets[1]. Otherwise $\mathfrak{m}$ is *Dedekind finite*. Thus a Dedekind finite cardinal cannot be compared with $\aleph_0$ unless it is finite.

Throughout this paper, we use letters from the German alphabet $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \mathfrak{q}, \dots$ for cardinal numbers, letters from the Greek alphabet $\alpha, \beta, \gamma, \dots$ for ordinal numbers, $k, l, m, n, \dots$ for natural numbers, and a letter from the Hebrew alphabet $\aleph$ for an aleph, unless otherwise stated.

THE MAIN THEOREMS

Throughout this section we shall work in ZF.

Absorption law of cardinal arithmetic

As mentioned earlier, the absorption law of cardinal arithmetic is a consequence of AC. Without AC, what conditions would allow us to obtain that property. It is obvious that the property holds for natural numbers and alephs since they are cardinals of well-ordered sets and in ZFC (in which the property holds) every set can be well-ordered, so every cardinal is either a natural number or an aleph. What about cardinals apart from those? It has been shown in ZF that if $\mathfrak{m} \leqslant \mathfrak{c} \leqslant \mathfrak{n}$, then $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$ for all cardinals $\mathfrak{m}$ and $\mathfrak{n}$ where $\mathfrak{c} = |\mathbb{R}|$[4]. We generalize the theorem as follows.

Theorem 4 *If* $\mathfrak{m} \leqslant \mathfrak{q} \leqslant \mathfrak{n}$ *where* $\mathfrak{q} = 2\mathfrak{q}$*, then* $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$ *for all cardinals* $\mathfrak{q}$*,* $\mathfrak{m}$*, and* $\mathfrak{n}$.

Proof: Let $\mathfrak{q}$, $\mathfrak{m}$, and $\mathfrak{n}$ be cardinals such that $\mathfrak{q} = 2\mathfrak{q}$ and $\mathfrak{m} \leqslant \mathfrak{q} \leqslant \mathfrak{n}$. By Theorem 2, there exists a cardinal $\mathfrak{p}$ such that $\mathfrak{n} = \mathfrak{q} + \mathfrak{p}$. Then $\mathfrak{m} + \mathfrak{n} \leqslant \mathfrak{q} + \mathfrak{n} = \mathfrak{q} + (\mathfrak{q} + \mathfrak{p}) = 2\mathfrak{q} + \mathfrak{p} = \mathfrak{q} + \mathfrak{p} = \mathfrak{n} \leqslant \mathfrak{m} + \mathfrak{n}$. Thus $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$. □

Since $2\aleph = \aleph$ for all alephs $\aleph$, by the above theorem, we have the following corollaries.

Corollary 1 *If* $\mathfrak{m}$ *is an aleph and* $\mathfrak{m} \leqslant \mathfrak{n}$*, then* $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$.

Corollary 2 *If* $\mathfrak{m}$ *is countable and* $\mathfrak{n}$ *is Dedekind infinite, then* $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$.

The theorem below shows that the above result cannot be obtained for an arbitrary infinite $\mathfrak{n}$.

Theorem 5 *If* $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$ *for all natural numbers* $\mathfrak{m}$ *and all infinite cardinals* $\mathfrak{n}$*, then every infinite cardinal is Dedekind infinite.*

Proof: Let $\mathfrak{n}$ be an infinite cardinal, say $\mathfrak{n} = |N|$ where $0 \notin N$. By the assumption, $1 + \mathfrak{n} = \mathfrak{n}$. Thus $1 \cup N = \{0\} \cup N \approx N$ where N is a proper subset of $1 \cup N$. Thus $1 \cup N$ is Dedekind infinite, and so is $\mathfrak{n}$. □

Since it cannot be proved, without AC, that every infinite cardinal is Dedekind infinite[3], Theorem 5 tells us that it cannot be proved from ZF that $\mathfrak{m} + \mathfrak{n} = \mathfrak{n}$ for all natural numbers $\mathfrak{m}$ and all infinite cardinals $\mathfrak{n}$.

Order preserving in cardinal arithmetic

In general, $\leqslant$ in Theorem 3 cannot be replaced by $<$. For example, $1 < 2$ but $1 + \aleph_0 = \aleph_0 = 2 + \aleph_0$[2] (this also follows immediately from Corollary 2). Similarly for multiplication and exponentiation.

With AC, by the absorption law of cardinal arithmetic, we have that for cardinals $\mathfrak{m}$ and $\mathfrak{n}$ such that $\mathfrak{m} < \mathfrak{n}$, $\mathfrak{m} + \mathfrak{p} < \mathfrak{n} + \mathfrak{p}$ if and only if $\mathfrak{p}$ is finite or $\mathfrak{p} < \mathfrak{n}$. Without AC, things are more complicated since we may not be able to compute the sum of some cardinals.

Now we consider the conditions, in the absence of AC, that make $\mathfrak{m} + \mathfrak{p} < \mathfrak{n} + \mathfrak{p}$ when $\mathfrak{m} < \mathfrak{n}$. This is trivial if all $\mathfrak{m}, \mathfrak{n}$, and $\mathfrak{p}$ are finite. By Corollary 2, this also holds if $\mathfrak{m}$ and $\mathfrak{n}$ are Dedekind infinite and $\mathfrak{p}$ is countable. What more can be proved without AC? The following are our results where we can assert that $\mathfrak{m} + \mathfrak{p} < \mathfrak{n} + \mathfrak{p}$ when $\mathfrak{m} < \mathfrak{n}$ despite not being able to compute the sums $\mathfrak{m} + \mathfrak{p}$ and $\mathfrak{n} + \mathfrak{p}$. First, we need the following lemma.

Lemma 1 *If $\mathfrak{p}$ and $\mathfrak{q}$ are Dedekind finite cardinals, then so are $\mathfrak{p} + \mathfrak{q}$ and $\mathfrak{p} \cdot \mathfrak{q}$.*

Proof: Let $\mathfrak{p} = |P|$ and $\mathfrak{q} = |Q|$ where P and Q are disjoint.

Suppose $\mathfrak{p} + \mathfrak{q}$ is Dedekind infinite. Then there is an injection $f : \omega \to P \cup Q$. Without loss of generality, we can assume that $\operatorname{ran}(f) \cap P$ is infinite. Thus $f^{-1}[\operatorname{ran}(f) \cap P]$ is an infinite subset of ω, so $\omega \approx \operatorname{ran}(f) \cap P \preceq P$. Hence P is Dedekind infinite.

Now, suppose $\mathfrak{p} \cdot \mathfrak{q}$ is Dedekind infinite. Then there is an injection $g : \omega \to P \times Q$.

Case 1. $\operatorname{ran}(g) \cap (\{p\} \times Q)$ is finite for all $p \in P$.

Since g is an injection with an infinite domain, the set $D := \operatorname{dom}(\operatorname{ran}(g))$ is an infinite subset of P. Hence the set $N := \{n \in \omega \mid n \text{ is the least element in} g^{-1}[\{p\} \times Q] \text{ for some} p \in D\}$ is an infinite subset of ω. Thus $\omega \approx N \approx D \preceq P$, so P is Dedekind infinite.

Case 2. $\operatorname{ran}(g) \cap (\{p\} \times Q)$ is infinite for some $p \in P$.

Then $g^{-1}[\operatorname{ran}(g) \cap (\{p\} \times Q)]$ is an infinite subset of ω, so $\omega \approx \operatorname{ran}(g) \cap (\{p\} \times Q)$. Define $h : \operatorname{ran}(g) \cap (\{p\} \times Q) \to Q$ by $h(\langle p, q \rangle) = q$. Then h is one-to-one, so $\omega \preceq Q$, i.e., Q is Dedekind infinite. □

Theorem 6 *For all cardinals $\mathfrak{n}$ and Dedekind finite cardinals $\mathfrak{m}$ and $\mathfrak{p}$, if $\mathfrak{m} < \mathfrak{n}$, then*

$$\mathfrak{m} + \mathfrak{p} < \mathfrak{n} + \mathfrak{p} \text{ and}$$
$$\mathfrak{m} \cdot \mathfrak{p} < \mathfrak{n} \cdot \mathfrak{p} \quad \text{whenever } \mathfrak{p} \neq 0.$$

Proof: Let $\mathfrak{n}$ be a cardinal and $\mathfrak{m}$ and $\mathfrak{p}$ be Dedekind finite cardinals such that $\mathfrak{m} < \mathfrak{n}$, say $\mathfrak{m} = |M|$, $\mathfrak{n} = |N|$, and $\mathfrak{p} = |P|$. We may assume $M \subset N$ and $P \cap N = \emptyset$. Clearly, $\mathfrak{m} + \mathfrak{p} \leqslant \mathfrak{n} + \mathfrak{p}$.

By Lemma 1, $\mathfrak{m} + \mathfrak{p}$ is Dedekind finite. If $\mathfrak{n}$ is Dedekind infinite, then so is $\mathfrak{n} + \mathfrak{p}$ and thus $\mathfrak{m} + \mathfrak{p} \neq \mathfrak{n} + \mathfrak{p}$. Suppose $\mathfrak{n}$ is Dedekind finite. By Lemma 1, $\mathfrak{n} + \mathfrak{p}$ is Dedekind finite. Since $M \cup P \subset N \cup P$ which is Dedekind finite, $M \cup P \not\approx N \cup P$. Thus $\mathfrak{m} + \mathfrak{p} \neq \mathfrak{n} + \mathfrak{p}$.

Similarly for multiplication. □

The proofs of Theorems 7 and 8 in the following are modified from the work of Halbeisen and Shelah[5].

Theorem 7 *For all Dedekind finite cardinals $\mathfrak{p}$, if $\mathfrak{m} < \mathfrak{n}$ where each of $\mathfrak{m}$ and $\mathfrak{n}$ is either a natural number or an aleph, then*

$$\mathfrak{m} + \mathfrak{p} < \mathfrak{n} + \mathfrak{p} \text{ and}$$
$$\mathfrak{m} \cdot \mathfrak{p} < \mathfrak{n} \cdot \mathfrak{p} \quad \text{whenever } \mathfrak{p} \neq 0.$$

Proof: Let $\mathfrak{m}$ and $\mathfrak{n}$ be natural numbers or alephs such that $\mathfrak{m} < \mathfrak{n}$. Let α and β be the least ordinals such that $|\alpha| = \mathfrak{m}$ and $|\beta| = \mathfrak{n}$. Let $\mathfrak{p}$ be a Dedekind finite cardinal, say $\mathfrak{p} = |P|$ where P is disjoint from α and β. It is trivial if $\mathfrak{p} = 0$. Assume $\mathfrak{p} \neq 0$. Clearly, $\mathfrak{m} + \mathfrak{p} \leqslant \mathfrak{n} + \mathfrak{p}$. Suppose there exists a bijection $f : \beta \cup P \to \alpha \cup P$. Since P is Dedekind finite, we will get a contradiction from this assumption by constructing a 1-1-ω-sequence of P.

Let $p_0 \in P$. For any $0 \neq k \in \omega$, assume there exists a 1-1-k-sequence $\langle p_0, p_1, \dots, p_{k-1} \rangle_k$ of P. Let $U_k = \{p_i \mid i < k\}$. Define $\lhd := (U_k \times \beta) \cup \{\langle p_i, p_j \rangle \mid i < j < k\} \cup \{\langle x, y \rangle \mid x < y < \beta\}$. It is easy to see that $\lhd$ well-orders $\beta \cup U_k$. If $\mathfrak{m}$ is finite, then $\mathfrak{m} + k < \mathfrak{n} + k$. Assume $\mathfrak{m}$ is infinite. Then $\mathfrak{m}$ and $\mathfrak{n}$ are alephs and so they are Dedekind infinite. Thus by Corollary 2, $|\alpha \cup U_k| = \mathfrak{m} + k = \mathfrak{m} < \mathfrak{n} = \mathfrak{n} + k = |\beta \cup U_k|$. Then there exists $x \in \beta \cup U_k$ such that $f(x) \notin \alpha \cup U_k$. Let x be such $\lhd$-least element and $p_k = f(x)$. Then $p_k \in P - U_k$. Hence we have a 1-1-$(k+1)$-sequence of P. By recursion on ω, we have a 1-1-ω-sequence of P, so $\omega \preceq P$, i.e., P is Dedekind infinite.

For multiplication, it is clear that $\mathfrak{m} \cdot \mathfrak{p} \leqslant \mathfrak{n} \cdot \mathfrak{p}$. Suppose there exists a bijection $f : \beta \times P \to \alpha \times P$. As in the proof of addition, we have a 1-1-k-sequence $\langle p_0, p_1, \dots, p_{k-1} \rangle_k$ of P and $U_k = \{p_i \mid i < k\}$ where $0 \neq k \in \omega$. Define a relation $\lhd$ on $\beta \times U_k$ by

$$\langle\delta, p_i\rangle \lhd \langle\rho, p_j\rangle \Leftrightarrow (i < j) \vee (i = j \wedge \delta < \rho).$$

It is easy to see that $\lhd$ well-orders $\beta \times U_k$. As in the proof of addition, we can show that $|\alpha \times U_k| < |\beta \times U_k|$. Then there exists $\langle\delta, p_i\rangle \in \beta \times U_k$ such that $f(\langle\delta, p_i\rangle) \notin \alpha \times U_k$. Let $\langle\delta, p_i\rangle$ be such $\lhd$-least element. Then $f(\langle\delta, p_i\rangle) \in (\alpha \times P) - (\alpha \times U_k)$. Let p_k be the second component of $f(\langle\delta, p_i\rangle)$. We can see that p_k is distinct from every element of U_k. Hence we have a 1-1-$(k+1)$-sequence of P. By recursion on ω, we have a 1-1-ω-sequence of P, so $\omega \preceq P$. □

From the previous example, we can see that $1 + \aleph_0 = 2 + \aleph_0$ whereas $1 \neq 2$. This means that the cancellation law for addition of cardinals fails in general and in particular when the cancelled part is infinite. It has been shown that the law cannot be obtained without AC even the cancelled cardinal is Dedekind finite[3]. It follows immediately from the above theorem that this can be done if each of $\mathfrak{m}$ and $\mathfrak{n}$ is either a natural number or an aleph.

Corollary 3 *For all Dedekind finite cardinals* $\mathfrak{p}$*, if each of* $\mathfrak{m}$ *and* $\mathfrak{n}$ *is either a natural number or an aleph such that*

(i) $\mathfrak{m} + \mathfrak{p} = \mathfrak{n} + \mathfrak{p}$*, then* $\mathfrak{m} = \mathfrak{n}$.

(ii) $\mathfrak{m} \cdot \mathfrak{p} = \mathfrak{n} \cdot \mathfrak{p}$ *where* $\mathfrak{p} > 0$*, then* $\mathfrak{m} = \mathfrak{n}$.

The rest of this section shows our results about cardinal exponentiation.

Theorem 8 *For all Dedekind finite cardinals* $\mathfrak{p} > 1$*, if* $m < n$ *where* m *and* n *are natural numbers, then* $\mathfrak{p}^m < \mathfrak{p}^n$.

Proof: Let $\mathfrak{p} > 1$ be a Dedekind finite cardinal, say $\mathfrak{p} = |P|$, and let m and n be natural numbers such that $m < n$. Clearly, $\mathfrak{p}^m \leqslant \mathfrak{p}^n$.

Suppose there is a bijection $F : {}^nP \to {}^mP$. As in the proof of Theorem 7, we will show that $\omega \preceq P$ by constructing a 1-1-ω-sequence of P. Let p_0 and p_1 be distinct elements in P. For any $1 < k \in \omega$, assume there is a 1-1-k-sequence $\langle p_0, p_1, \dots, p_{k-1}\rangle_k$ of P and let $U_k = \{p_i \mid i < k\}$. Define a relation $\lhd$ on ${}^nU_k$ by

$$f \lhd g \Leftrightarrow \exists l < n(f \restriction l = g \restriction l \wedge f(l) = p_i \wedge g(l) = p_j \text{ where } i < j < k).$$

It is easy to see that $\lhd$ well-orders ${}^nU_k$. Since $1 < k \in \omega$, $k^m < k^n$. Then there exists $f \in {}^nU_k$ such that $F(f) \notin {}^mU_k$. We may assume f is such $\lhd$-least element. Since $F(f) \in {}^mP$, there exists a least natural number $l < m$ such that $F(f)(l) \in P - U_k$. Let $p_k = F(f)(l)$. Hence we have a 1-1-$(k+1)$-sequence of P. By recursion on ω, we have a 1-1-ω-sequence of P and so $\omega \preceq P$. □

Thus we have the following result.

Corollary 4 *For all Dedekind finite cardinals* $\mathfrak{p} > 1$*, if* $\mathfrak{p}^m = \mathfrak{p}^n$ *where* m *and* n *are natural numbers, then* $m = n$.

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