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# A logarithmically improved blow-up criterion of smooth solutions for nematic liquid crystal flows with partial viscosity

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**ABSTRACT**: In this paper we investigate the Cauchy problem for nematic liquid crystal flows with partial viscosity in $\mathbb{R}^3$. A logarithmically improved blow-up criterion of smooth solutions is established. The result is analogous to the celebrated Beale-Kato-Majda type criterion for the inviscid Euler equations of incompressible fluids.

**KEYWORDS**: BMO space, energy estimate

## INTRODUCTION

The nematic liquid crystal flow equations in three space dimensions are

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = -\Delta\phi\cdot\nabla\phi, \\ \partial_t\phi - \Delta\phi + u\cdot\nabla\phi = |\nabla\phi|^2\phi, \\ \nabla\cdot u = 0, \end{cases} \tag{1}$$

where $u(t,x)$ represents the velocity field of the incompressible fluid, $\nu$ is the kinematic viscosity, $p(t,x)$ is the pressure, and $\phi$ denotes the macroscopic average of the nematic liquid crystal orientation field.

The hydrodynamic theory of liquid crystals was established by Ericksen[1] and Leslie[2] in the 1960s. Since the general Ericksen-Leslie system is very complicated, we only consider a simplified model (1) of the Ericksen-Leslie system[3–5], which is probably the simplest mathematical system one can derive for modelling nematic liquid crystal flow without destroying the basic nonlinear structure. (1) is a system of nonhomogeneous Navier-Stokes equations coupled with harmonic map flow. Huang and Wang[6] established a blow-up criterion for the short time classical solutions to (1) in 2 and 3 dimensions. More precisely, $0 < T_* < +\infty$ is the maximal time interval iff (i) for $n=3$, $\int_0^{T_*}(\|\nabla\times u\|_{L^\infty} + \|\nabla\phi\|_{L^\infty}^2)\,\mathrm{d}t = \infty$, and (ii) for $n=2$, $\int_0^{T_*}\|\nabla\phi\|_{L^\infty}^2\,\mathrm{d}t = \infty$.

In this paper, we consider the nematic liquid crystal flow equations equations (1) with partial viscosity, i.e., $\nu = 0$. The corresponding nematic liquid crystal flow equations are thus

$$\partial_t u + u\cdot\nabla u + \nabla p = -\Delta\phi\cdot\nabla\phi, \tag{2a}$$
$$\partial_t\phi - \Delta\phi + u\cdot\nabla\phi = |\nabla\phi|^2\phi, \tag{2b}$$
$$\nabla\cdot u = 0. \tag{2c}$$

We investigate (2) with the initial conditions

$$t=0: u = u_0(x), \quad \phi = \phi_0(x). \tag{3}$$

Recall that when $\phi$ is a constant vector, (1) reduces to the incompressible Navier-Stokes equations. Regularity criteria for the Navier-Stokes equations have been obtained[7–9]. Logarithmically improved regularity criteria for the Navier-Stokes equations have also been established[10–12]. For the incompressible Euler equations, the well-known Beale-Kato-Majda's criterion[13] says that any solution $u$ is smooth up to time $T$ under the assumption that $\int_0^T\|\nabla\times u(t)\|_{L^\infty}\,\mathrm{d}t < \infty$. Beale-Kato-Majda's criterion was slightly improved by Kozono-Taniuchi[14] under the assumption $\int_0^T\|\nabla\times u(t)\|_{\mathrm{BMO}}\,\mathrm{d}t < \infty$. In the absence of global well-posedness, the development of blow-up/non blow-up theory is of major importance for both theoretical and practical purposes. In this paper, we obtain a Beale-Kato-Majda type blow-up criterion of smooth solutions to the Cauchy problem (2), (3).

**Theorem 1** *Assume that* $u_0 \in H^m(\mathbb{R}^3,\mathbb{R}^3)$, $m \geqslant 3$ *with* $\nabla\cdot u_0 = 0$ *and* $\phi_0 \in H^{m+1}(\mathbb{R}^3, S^2)$. *Let* $(u,\phi)$

*be a smooth solution to* (2), (3) *for* $0 \leqslant t < T$. *If*

$$\int_0^T \frac{\|\nabla \times u(t)\|_{\mathrm{BMO}}}{\sqrt{\ln(e + \|\nabla \times u(t)\|_{\mathrm{BMO}})}} + \|\nabla\phi(t)\|_{L^\infty}^2 \,\mathrm{d}t < \infty, \quad (4)$$

*then the solution* $(u, \phi)$ *can be extended beyond* $t = T$.

We have the following corollary immediately.

**Corollary 1** *Assume that* $u_0 \in H^m(\mathbb{R}^3, \mathbb{R}^3)$, $m \geqslant 3$ *with* $\nabla \cdot u_0 = 0$ *and* $\phi_0 \in H^{m+1}(\mathbb{R}^3, S^2)$. *Let* $(u, \phi)$ *be a smooth solution to* (2), (3) *for* $0 \leqslant t < T$. *Suppose that* $T$ *is the maximal existence time, then*

$$\int_0^T \frac{\|\nabla \times u(t)\|_{\mathrm{BMO}}}{\sqrt{\ln(e + \|\nabla \times u(t)\|_{\mathrm{BMO}})}} + \|\nabla\phi(t)\|_{L^\infty}^2 \,\mathrm{d}t = \infty. \quad (5)$$

Here BMO denotes the homogenous space of bounded mean oscillations associated with the norm

$$\|f\|_{\mathrm{BMO}} =: \sup_{x\in\mathbb{R}^n, R>0} \frac{1}{|B_R(x)|} \times \int_{B_R(x)} \left| f(y) - \frac{1}{|B_R(y)|}\int_{B_R(y)} f(z)\,\mathrm{d}z \right| \mathrm{d}y.$$

The paper is arranged as follows. We first state some preliminary results on functional settings and some important inequalities and finally prove the blow-up criterion of smooth solutions to (2), (3).

## SOME USEFUL LEMMAS

In order to prove our main results we need the following inequality.

**Lemma 1 (Ref. 15)** *There exists a uniform positive constant* $C$*, such that*

$$\|\nabla f\|_{L^\infty} \leqslant C(1 + \|f\|_{L^2} + \|\nabla \times f\|_{\mathrm{BMO}}\sqrt{\ln(e + \|f\|_{H^3})}). \quad (6)$$

*holds for all vectors* $f \in H^3(\mathbb{R}^n)$, $(n = 2, 3)$ *with* $\nabla \cdot f = 0$.

The following inequality is the well-known Gagliardo-Nirenberg inequality.

**Lemma 2** *Let* $j, m$ *be any integers satisfying* $0 \leqslant j < m$, *and let* $1 \leqslant q$, $r \leqslant \infty$, *and* $p \in \mathbb{R}$, $j/m \leqslant \theta \leqslant 1$ *such that*

$$\frac{1}{p} - \frac{j}{n} = \theta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\theta)\frac{1}{q}.$$

*Then for all* $f \in L^q(\mathbb{R}^n) \bigcap W^{m,r}(\mathbb{R}^n)$, *there is a positive constant* $C$ *depending only on* $n$, $m$, $j$, $q$, $r$, $\theta$ *such that the following inequality holds:*

$$\|\nabla^j f\|_{L^p} \leqslant C\|f\|_{L^q}^{1-\theta}\|\nabla^m f\|_{L^r}^{\theta} \quad (7)$$

*with the following exception: if* $1 < r < \infty$ *and* $m - j - n/r$ *is a nonnegative integer then* (7) *holds only for* $\theta$ *satisfying* $j/m \leqslant \theta < 1$.

**Lemma 3 (Ref. 16)** *The following inequality holds:*

$$\|\nabla^m(f\cdot\nabla g) - f\cdot\nabla\nabla^m g\|_{L^2} \leqslant C(\|\nabla f\|_{L^\infty}\|\nabla^m g\|_{L^2} + \|\nabla g\|_{L^\infty}\|\nabla^m f\|_{L^2}). \quad (8)$$

In order to prove Theorem 1 we need the following interpolation inequalities in three space dimensions. In fact, we can obtain them by Sobolev embedding and scaling techniques.

**Lemma 4** *In three space dimensions, the inequalities*

$$\begin{cases} \|\nabla f\|_{L^2} \leqslant C\|f\|_{L^2}^{\frac{2}{3}}\|\nabla^3 f\|_{L^2}^{\frac{1}{3}}. \\ \|f\|_{L^\infty} \leqslant C\|f\|_{L^2}^{\frac{1}{4}}\|\nabla^2 f\|_{L^2}^{\frac{3}{4}}. \\ \|f\|_{L^4} \leqslant C\|f\|_{L^2}^{\frac{3}{4}}\|\nabla^3 f\|_{L^2}^{\frac{1}{4}} \\ \|\nabla f\|_{L^3} \leqslant C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla^3 f\|_{L^2}^{\frac{1}{2}} \\ \|\nabla f\|_{L^\infty} \leqslant C\|\nabla f\|_{L^2}^{\frac{3}{4}}\|\nabla^3 f\|_{L^2}^{\frac{1}{4}} \end{cases} \quad (9)$$

*hold.*

## PROOF OF THEOREM 1

Multiplying (2a) by $u$ and using integration by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} \Delta\phi \cdot \nabla\phi \cdot u \,\mathrm{d}x. \quad (10)$$

Applying $\nabla$ to (2b), then taking the inner product with $\nabla\phi$ and using integration by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi(t)\|_{L^2}^2 + \|\nabla^2\phi(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla(u\cdot\nabla\phi)\cdot\nabla\phi \,\mathrm{d}x + \int_{\mathbb{R}^3} \nabla(|\nabla\phi|^2\phi)\nabla\phi \,\mathrm{d}x. \quad (11)$$

Summing (10) and (11), we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|\nabla\phi(t)\|_{L^2}^2) + \|\nabla^2\phi(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^3} \Delta\phi\cdot\nabla\phi\cdot u \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \nabla(u\cdot\nabla\phi)\cdot\nabla\phi \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} \nabla(|\nabla\phi|^2\phi)\nabla\phi \,\mathrm{d}x. \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (12)$$

In what follows, we estimate $I_i$, $(i = 1, 2, 3)$. Using Hölder's inequality and Young's inequality, we have

$$I_1 \leqslant \|\nabla\phi\|_{L^\infty}\|u\|_{L^2}\|\nabla^2\phi\|_{L^2} \leqslant \frac{1}{6}\|\nabla^2\phi\|_{L^2}^2 + C\|\nabla\phi\|_{L^\infty}^2\|u\|_{L^2}^2. \quad (13)$$

Using integration by parts, Hölder's inequality and Young's inequality, we obtain

$$I_2 \leqslant C\|\nabla\phi\|_{L^\infty}\|u\|_{L^2}\|\nabla^2\phi\|_{L^2} \leqslant \frac{1}{6}\|\nabla^2\phi\|_{L^2}^2 + C\|\nabla\phi\|_{L^\infty}^2\|u\|_{L^2}^2. \quad (14)$$

From integration by parts, Hölder's inequality and Young's inequality we obtain

$$I_3 \leqslant -\int_{\mathbb{R}^3} |\nabla\phi|^2\phi\nabla^2\phi \,\mathrm{d}x \leqslant \frac{1}{6}\|\nabla^2\phi\|_{L^2}^2 + C\|\nabla\phi\|_{L^\infty}^2\|\nabla\phi\|_{L^2}^2. \quad (15)$$

Combining (12), (13), (14), (15) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|\nabla\phi(t)\|_{L^2}^2) + \|\nabla^2\phi(t)\|_{L^2}^2 \leqslant C\|\nabla\phi\|_{L^\infty}^2(\|u\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2).$$

From Gronwall's inequality we get

$$\|u(t)\|_{L^2}^2 + \|\nabla\phi(t)\|_{L^2}^2 + \int_0^t \|\nabla^2\phi(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leqslant C(\|u_0\|_{L^2}^2 + \|\nabla\phi_0\|_{L^2}^2). \quad (16)$$

Applying $\nabla$ to (2a), multiplying the resulting equation by $\nabla u$, and integrating with respect to $x$ over $\mathbb{R}^3$ using integration by parts we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla(u\cdot\nabla u)\nabla u \,\mathrm{d}x - \int_{\mathbb{R}^3} \nabla(\Delta\phi\cdot\nabla\phi)\nabla u \,\mathrm{d}x. \quad (17)$$

As with the proof of (17), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\phi(t)\|_{L^2}^2 + \|\nabla^3\phi(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla\phi)\nabla^2\phi \,\mathrm{d}x + \int_{\mathbb{R}^3} \nabla^2(|\nabla\phi|^2\phi)\nabla^2\phi \,\mathrm{d}x. \quad (18)$$

From (17), (18) and $\nabla\cdot u = 0$, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2\phi(t)\|_{L^2}^2) + \|\nabla^3\phi(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^3} [\nabla(u\cdot\nabla u) - u\cdot\nabla\nabla u]\nabla u \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \nabla(\Delta\phi\cdot\nabla\phi)\nabla u \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} [\nabla^2(u\cdot\nabla\phi) - u\cdot\nabla\nabla^2\phi]\nabla^2\phi \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} \nabla^2(|\nabla\phi|^2\phi)\nabla^2\phi \,\mathrm{d}x \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned} \quad (19)$$

It follows from Lemma 3 that

$$J_1 \leqslant C\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2. \quad (20)$$

From Hölder's inequality and Young's inequality we obtain

$$\begin{aligned} J_2 &\leqslant \|\nabla\phi\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla^3\phi\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^2\phi\|_{L^2}^2 \\ &\leqslant \frac{1}{6}\|\nabla^3\phi\|_{L^2}^2 + C(\|\nabla u\|_{L^\infty} + \|\nabla\phi\|_{L^\infty}^2) \times (\|\nabla u\|_{L^2}^2 + \|\nabla^2\phi\|_{L^2}^2). \end{aligned} \quad (21)$$

Using integrating by parts, Hölder's inequality, and Young's inequality we obtain

$$\begin{aligned} J_3 &\leqslant 3\int_{\mathbb{R}^3} |\nabla u\nabla^2\phi\nabla^2\phi| \,\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla u\nabla\phi\nabla^3\phi| \,\mathrm{d}x \\ &\leqslant 3\|\nabla u\|_{L^\infty}\|\nabla^2\phi\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\nabla^3\phi\|_{L^2} \\ &\leqslant \frac{1}{6}\|\nabla^3\phi\|_{L^2}^2 + C(\|\nabla u\|_{L^\infty} + \|\nabla\phi\|_{L^\infty}^2) \times (\|\nabla u\|_{L^2}^2 + \|\nabla^2\phi\|_{L^2}^2). \end{aligned} \quad (22)$$

We apply integration by parts, Hölder's inequality and Young's inequality. This yields

$$\begin{aligned} J_4 &\leqslant 6\int_{\mathbb{R}^3} |\nabla\phi\nabla^2\phi\,\phi\nabla^3\phi| \,\mathrm{d}x + 3\int_{\mathbb{R}^3} |\nabla\phi\nabla\phi\nabla^2\phi\nabla^2\phi| \,\mathrm{d}x \\ &\leqslant 6\|\nabla\phi\|_{L^\infty}\|\nabla^2\phi\|_{L^2}\|\nabla^3\phi\|_{L^2} + \|\nabla\phi\|_{L^\infty}^2\|\nabla^2\phi\|_{L^2}^2 \\ &\leqslant \frac{1}{6}\|\nabla^3\phi\|_{L^2}^2 + C\|\nabla\phi\|_{L^\infty}^2\|\nabla^2\phi\|_{L^2}^2. \end{aligned} \quad (23)$$

Combining (19), (20), (21), (22), (23) and using Gronwall's inequality, we get

$$\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2\phi(t)\|_{L^2}^2 + \int_{t_1}^t \|\nabla^3\phi(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leqslant (\|\nabla u(t_1)\|_{L^2}^2 + \|\nabla^2\phi(t_1)\|_{L^2}^2) \times \exp\left\{C\int_{t_1}^t \|\nabla u(\tau)\|_{L^\infty}\,\mathrm{d}\tau\right\}. \quad (24)$$

From (4) we know that for any small constant $\varepsilon > 0$, there exists $T_* < T$ such that

$$\int_{T_*}^T \frac{\|\nabla\times u(t)\|_{\mathrm{BMO}}}{\sqrt{\ln(e+\|\nabla\times u(t)\|_{\mathrm{BMO}})}}\,\mathrm{d}t \leqslant \varepsilon. \quad (25)$$

Let

$$\Theta(t) = \sup_{T_*\leqslant\tau\leqslant t}(\|\nabla^3 u(\tau)\|_{L^2}^2 + \|\nabla^4\phi(\tau)\|_{L^2}^2) \quad (26)$$

for $T_* \leqslant t < T$. It follows from (16), (24), (25), (26) and Lemma 1 that

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2\phi(t)\|_{L^2}^2 + \int_{T_*}^t \|\nabla^3\phi(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\
&\leqslant C_1\exp\left\{C_0\int_{T_*}^t\|\nabla\times u\|_{\mathrm{BMO}}\sqrt{\ln(e+\|u\|_{H^3})}\,\mathrm{d}\tau\right\} \\
&\leqslant C_1\exp\{C_0\varepsilon\ln(e+\Theta(t))\} \\
&\leqslant C_1(e+\Theta(t))^{C_0\varepsilon},\quad T_*\leqslant t<T, \qquad (27)
\end{aligned}$$

where $C_1$ depends on $\|\nabla u(T_*)\|_{L^2}^2 + \|\nabla^2\phi(T_*)\|_{L^2}^2$, while $C_0$ is an absolute positive constant.

Applying $\nabla^m$ to (2a), then taking the $L^2$ inner product of the resulting equation with $\nabla^m u$ and using integration by parts, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3}\nabla^m(u\cdot\nabla u)\nabla^m u\,\mathrm{d}x - \int_{\mathbb{R}^3}\nabla^m(\Delta\phi\cdot\nabla\phi)\nabla^m u\,\mathrm{d}x. \quad (28)$$

Applying the method used to get (28), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{m+1}\phi(t)\|_{L^2}^2 + \|\nabla^{m+2}\phi(t)\|_{L^2}^2 = -\int_{\mathbb{R}^3}\nabla^{m+1}(u\cdot\nabla\phi)\nabla^{m+1}\phi\,\mathrm{d}x + \int_{\mathbb{R}^3}\nabla^{m+1}(|\nabla\phi|^2\phi)\nabla^{m+1}\phi\,\mathrm{d}x. \quad (29)$$

It follows from (28), (29), $\nabla\cdot u = 0$, and integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^m u(t)\|_{L^2}^2 + \|\nabla^{m+1}\phi(t)\|_{L^2}^2) + \|\nabla^{m+2}\phi(t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3}[\nabla^m(u\cdot\nabla u) - u\cdot\nabla\nabla^m u]\nabla^m u\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3}\nabla^m(\Delta\phi\cdot\nabla\phi)\nabla^m u\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3}[\nabla^{m+1}(u\cdot\nabla\phi) - u\cdot\nabla\nabla^{m+1}\phi]\nabla^{m+1}\phi\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\nabla^{m+1}(|\nabla\phi|^2\phi)\nabla^{m+1}\phi\,\mathrm{d}x. \qquad (30)
\end{aligned}$$

In what follows, for simplicity, we will set $m = 3$. From Hölder's inequality and Lemma 3, we derive

$$\left|\int_{\mathbb{R}^3}[\nabla^3(u\cdot\nabla u) - u\cdot\nabla\nabla^3 u]\nabla^3 u\,\mathrm{d}x\right| \leqslant C\|\nabla u(t)\|_{L^\infty}\|\nabla^3 u(t)\|_{L^2}^2. \quad (31)$$

Using integration by parts and Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\nabla^3(\Delta\phi\cdot\nabla\phi)\nabla^3 u\,\mathrm{d}x\right| \\
&\leqslant \|\nabla\phi(t)\|_{L^\infty}\|\nabla^5\phi\|_{L^2}\|\nabla^3 u(t)\|_{L^2} \\
&\quad + 4\|\nabla^2\phi(t)\|_{L^4}\|\nabla^2 u(t)\|_{L^4}\|\nabla^5\phi(t)\|_{L^2} \\
&\quad + 10\|\nabla u(t)\|_{L^\infty}\|\nabla^3\phi(t)\|_{L^2}\|\nabla^5\phi(t)\|_{L^2} \\
&\quad + 10\|\nabla u\|_{L^\infty}\|\nabla^4\phi\|_{L^2}^2. \qquad (32)
\end{aligned}$$

Using Young's inequality, we obtain

$$\|\nabla\phi(t)\|_{L^\infty}\|\nabla^5\phi\|_{L^2}\|\nabla^3 u(t)\|_{L^2} \leqslant \frac{1}{20}\|\nabla^5\phi\|_{L^2}^2 + C\|\nabla\phi(t)\|_{L^\infty}^2\|\nabla^3 u(t)\|_{L^2}^2. \quad (33)$$

From Lemma 2, Lemma 4, Young's inequality, and (27), we get

$$\begin{aligned}
&4\|\nabla^2\phi(t)\|_{L^4}\|\nabla^2 u(t)\|_{L^4}\|\nabla^5\phi(t)\|_{L^2} \\
&\leqslant C\|\nabla^2\phi\|_{L^2}^{\frac{3}{4}}\|\nabla^5\phi(t)\|_{L^2}^{\frac{5}{4}}\|\nabla u\|_{L^\infty}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}} \\
&\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}^{\frac{4}{3}}\|\nabla^2\phi(t)\|_{L^2}^2\|\nabla^3 u(t)\|_{L^2}^{\frac{4}{3}} \\
&\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty} \times \|\nabla u(t)\|_{L^2}^{\frac{1}{12}}\|\nabla^3 u(t)\|_{L^2}^{\frac{19}{12}}\|\nabla^2\phi(t)\|_{L^2}^2 \\
&\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e+\Theta(t))^{\frac{25}{24}C_0\varepsilon}\Theta^{\frac{19}{24}}(t) \qquad (34)
\end{aligned}$$

and

$$
\begin{aligned}
&10\|\nabla u(t)\|_{L^\infty}\|\nabla^3\phi(t)\|_{L^2}\|\nabla^5\phi(t)\|_{L^2}\\
&\quad\leqslant C\|\nabla u(t)\|_{L^\infty}\|\nabla^2\phi(t)\|_{L^2}^{\frac{2}{3}}\|\nabla^5\phi(t)\|_{L^2}^{\frac{4}{3}}\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}^3\|\nabla^2\phi(t)\|_{L^2}^2\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}\\
&\qquad\quad\times\|\nabla u(t)\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u(t)\|_{L^2}^{\frac{3}{2}}\|\nabla^2\phi(t)\|_{L^2}^2\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2\\
&\qquad\quad + C\|\nabla u(t)\|_{L^\infty}(e+\Theta(t))^{\frac{5}{4}C_0\varepsilon}\Theta^{\frac{3}{4}}(t). \quad (35)
\end{aligned}
$$

Combining (32), (33), (34), (35) yields

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\nabla^3(\Delta\phi\cdot\nabla\phi)\nabla^3 u\,\mathrm{d}x\right|\\
&\quad\leqslant \frac{3}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla\phi(t)\|_{L^\infty}^2\|\nabla^3 u(t)\|_{L^2}^2\\
&\qquad + C\|\nabla u(t)\|_{L^\infty}(e+\Theta(t))^{\frac{25}{24}C_0\varepsilon}\Theta^{\frac{19}{24}}(t)\\
&\qquad + C\|\nabla u(t)\|_{L^\infty}(e+\Theta(t))^{\frac{5}{4}C_0\varepsilon}\Theta^{\frac{3}{4}}(t)\\
&\qquad + C\|\nabla u\|_{L^\infty}\|\nabla^4\phi\|_{L^2}^2. \quad (36)
\end{aligned}
$$

Using integration by parts and Hölder's inequality, we get

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}[\nabla^4(u\cdot\nabla\phi)-u\cdot\nabla\nabla^4\phi]\nabla^4\phi\,\mathrm{d}x\right|\\
&\quad\leqslant 15\|\nabla u(t)\|_{L^\infty}\|\nabla^4\phi\|_{L^2}^2\\
&\qquad + 11\|\nabla u(t)\|_{L^\infty}\|\nabla^3\phi(t)\|_{L^2}\|\nabla^5\phi(t)\|_{L^2}\\
&\qquad + \|\nabla\phi(t)\|_{L^\infty}\|\nabla^3 u(t)\|_{L^2}\|\nabla^5\phi(t)\|_{L^2}\\
&\qquad + 5\|\nabla^2 u\|_{L^4}\|\nabla^2\phi\|_{L^4}\|\nabla^5\phi\|_{L^2}. \quad (37)
\end{aligned}
$$

As with the estimate of (36), we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}[\nabla^4(u\cdot\nabla\phi)-u\cdot\nabla\nabla^4\phi]\nabla^4\phi\,\mathrm{d}x\right|\\
&\quad\leqslant \frac{3}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla\phi(t)\|_{L^\infty}^2\|\nabla^3 u(t)\|_{L^2}^2\\
&\qquad + C\|\nabla u(t)\|_{L^\infty}(e+\Theta(t))^{\frac{25}{24}C_0\varepsilon}\Theta^{\frac{19}{24}}(t)\\
&\qquad + C\|\nabla u(t)\|_{L^\infty}(e+\Theta(t))^{\frac{5}{4}C_0\varepsilon}\Theta^{\frac{3}{4}}(t)\\
&\qquad + \|\nabla u\|_{L^\infty}\|\nabla^4\phi\|_{L^2}^2. \quad (38)
\end{aligned}
$$

Making use of integration by parts and Hölder's inequality, we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\nabla^4(|\nabla\phi|^2\phi)\nabla^4\phi\,\mathrm{d}x\right|\\
&\quad = \left|\int_{\mathbb{R}^3}\nabla^3(|\nabla\phi|^2\phi)\nabla^5\phi\,\mathrm{d}x\right|\\
&\quad\leqslant 7\|\nabla\phi(t)\|_{L^\infty}\|\nabla\phi\|_{L^6}\|\nabla^3\phi\|_{L^3}\|\nabla^5\phi\|_{L^2}\\
&\qquad + 12\|\nabla\phi(t)\|_{L^\infty}\|\nabla^2\phi(t)\|_{L^4}^2\|\nabla^5\phi(t)\|_{L^2}\\
&\qquad + 6\|\nabla^2\phi(t)\|_{L^4}\|\nabla^3\phi(t)\|_{L^4}\|\nabla^5\phi(t)\|_{L^2}\\
&\qquad + 2\|\nabla\phi\|_{L^\infty}\|\nabla^4\phi\|_{L^2}\|\nabla^5\phi\|_{L^2}. \quad (39)
\end{aligned}
$$

It follows from Lemma 2, Lemma 4, Young's inequality, and (27) that

$$
\begin{aligned}
&7\|\nabla\phi(t)\|_{L^\infty}\|\nabla\phi\|_{L^6}\|\nabla^3\phi\|_{L^3}\|\nabla^5\phi\|_{L^2}\\
&\leqslant C\|\nabla\phi(t)\|_{L^\infty}\|\nabla^2\phi(t)\|_{L^2}\|\nabla^2\phi(t)\|_{L^2}^{\frac{1}{2}}\|\nabla^5\phi(t)\|_{L^2}^{\frac{3}{2}}\\
&\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla\phi(t)\|_{L^\infty}^4\|\nabla^2\phi(t)\|_{L^2}^6\\
&\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2\\
&\qquad + C\|\nabla\phi(t)\|_{L^\infty}^2\|\nabla^2\phi(t)\|_{L^2}^{\frac{15}{2}}\|\nabla^4\phi(t)\|_{L^2}^{\frac{1}{2}}\\
&\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2\\
&\qquad + C\|\nabla\phi(t)\|_{L^\infty}^2(e+\Theta(t))^{\frac{15}{4}C_0\varepsilon}\Theta^{\frac{1}{4}}(t) \quad (40)
\end{aligned}
$$

$$
\begin{aligned}
&12\|\nabla\phi(t)\|_{L^\infty}\|\nabla^2\phi(t)\|_{L^4}^2\|\nabla^5\phi(t)\|_{L^2}\\
&\quad\leqslant C\|\nabla\phi(t)\|_{L^\infty}\|\nabla^2\phi(t)\|_{L^2}^{\frac{3}{2}}\|\nabla^5\phi(t)\|_{L^2}^{\frac{3}{2}}\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla\phi(t)\|_{L^\infty}^4\|\nabla^2\phi(t)\|_{L^2}^6\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2\\
&\qquad + C\|\nabla\phi(t)\|_{L^\infty}^2\|\nabla^2\phi(t)\|_{L^2}^{\frac{15}{2}}\|\nabla^4\phi(t)\|_{L^2}^{\frac{1}{2}}\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2\\
&\qquad + C\|\nabla\phi(t)\|_{L^\infty}^2(e+\Theta(t))^{\frac{15}{4}C_0\varepsilon}\Theta^{\frac{1}{4}}(t). \quad (41)
\end{aligned}
$$

$$
\begin{aligned}
&6\|\nabla^2\phi(t)\|_{L^4}\|\nabla^3\phi(t)\|_{L^4}\|\nabla^5\phi(t)\|_{L^2}\\
&\quad\leqslant C\|\nabla\phi(t)\|_{L^\infty}^{\frac{9}{10}}\|\nabla^5\phi(t)\|_{L^2}^{\frac{1}{10}}\|\nabla\phi(t)\|_{L^\infty}^{\frac{1}{2}}\\
&\qquad\times\|\nabla^5\phi(t)\|_{L^2}^{\frac{1}{2}}\|\nabla^5\phi(t)\|_{L^2}\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla\phi(t)\|_{L^\infty}^7\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2\\
&\qquad + C\|\nabla\phi(t)\|_{L^\infty}^2\|\nabla^2\phi(t)\|_{L^2}^{\frac{15}{4}}\|\nabla^4\phi(t)\|_{L^2}^{\frac{5}{4}}\\
&\quad\leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2\\
&\qquad + C\|\nabla\phi(t)\|_{L^\infty}^2(e+\Theta(t))^{\frac{15}{8}C_0\varepsilon}\Theta^{\frac{5}{8}}(t) \quad (42)
\end{aligned}
$$

and

$$2\|\nabla\phi\|_{L^\infty}\|\nabla^4\phi\|_{L^2}\|\nabla^5\phi\|_{L^2} \leqslant \frac{1}{20}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla\phi(t)\|_{L^\infty}^2\|\nabla^4\phi(t)\|_{L^2}^2 \tag{43}$$

It follows from (39), (40), (41), (42), (43) that

$$\left|\int_{\mathbb{R}^3}\nabla^3(|\nabla\phi|^2\phi)\nabla^5\phi\,\mathrm{d}x\right| \leqslant \frac{1}{5}\|\nabla^5\phi(t)\|_{L^2}^2 + C\|\nabla\phi(t)\|_{L^\infty}^2(e+\Theta(t))^{\frac{15}{4}C_0\varepsilon}\Theta^{\frac{1}{4}}(t) + C\|\nabla\phi(t)\|_{L^\infty}^2(e+\Theta(t))^{\frac{15}{8}C_0\varepsilon}\Theta^{\frac{5}{8}}(t) + C\|\nabla\phi(t)\|_{L^\infty}^2\|\nabla^4\phi\|_{L^2}^2. \tag{44}$$

For $T_* \leqslant t < T$, collecting (31), (36), (38) and (44) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^3u(t)\|_{L^2}^2 + \|\nabla^4\phi(t)\|_{L^2}^2) + \|\nabla^5\phi(t)\|_{L^2}^2 \leqslant C(\|\nabla u(t)\|_{L^\infty} + \|\nabla\phi\|_{L^\infty}^2)(e+\Theta(t)), \tag{45}$$

provided that $\varepsilon \leqslant 1/5C_0$. Integrating (45) with respect to time from $T_*$ to $\tau$ and using Lemma 1, we have

$$e + \|\nabla^3u(\tau)\|_{L^2}^2 + \|\nabla^4\phi(\tau)\|_{L^2}^2 \leqslant e + \|\nabla^3u(T_*)\|_{L^2}^2 + \|\nabla^4\phi(T_*)\|_{L^2}^2 + C_2\int_{T_*}^{\tau}[\|\nabla\phi\|_{L^\infty}^2 + 1 + \|u\|_{L^2} + \frac{\|\nabla\times u(s)\|_{\mathrm{BMO}}}{\sqrt{\ln(e+\|\nabla\times u(s)\|_{\mathrm{BMO}})}}\ln(e+\Theta(s))] \times (e+\Theta(s))\,\mathrm{d}s. \tag{46}$$

From (46) we get

$$e+\Theta(t) \leqslant e + \|\nabla^3u(T_*)\|_{L^2}^2 + \|\nabla^4\phi(T_*)\|_{L^2}^2 + C_2\int_{T_*}^{t}[\|\nabla\phi\|_{L^\infty}^2 + 1 + \|u\|_{L^2} + \frac{\|\nabla\times u(s)\|_{\mathrm{BMO}}}{\sqrt{\ln(e+\|\nabla\times u(s)\|_{\mathrm{BMO}})}} \times \ln(e+\Theta(\tau))](e+\Theta(\tau))\,\mathrm{d}\tau. \tag{47}$$

For all $T_* \leqslant t < T$, with help of Gronwall's inequality and (47), we have

$$e + \|\nabla^3u(t)\|_{L^2}^2 + \|\nabla^4\phi(t)\|_{L^2}^2 \leqslant C, \tag{48}$$

where $C$ depends on $\|\nabla u(T_*)\|_{L^2}^2 + \|\nabla^2\phi(T_*)\|_{L^2}^2$.

Noting that (16) and the right-hand sides of (16) and (48) are independent of $t$ for $T_* \leqslant t < T$, we know that $u(T,\cdot) \in H^3(\mathbb{R}^3,\mathbb{R}^3)$ and $\phi(T,\cdot) \in H^4(\mathbb{R}^3,S^2)$. Thus Theorem 1 is proved.

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