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# Relative non nil-$n$ graphs of finite groups

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**ABSTRACT**: Suppose $G$ is not a nilpotent group of class at most $n$ (a non nil-$n$ group). Consider a subgroup $H$ of $G$. In this paper, we introduce the relative non nil-$n$ graph $\Gamma^{(n)}{}_{H,G}$ of a finite group $G$. It is a graph with vertex set $G \setminus C^{(n)}{}_G(H)$ and two distinct vertices $x$ and $y$ are adjacent if at least one of them belongs to $H$ and $[x, y] \notin Z_{n-1}(G)$, where the subgroup $C^{(n)}{}_G(H)$ contains $g \in G$ such that $[g, h] \in Z_{n-1}(G)$ for all $h \in H$. We present some general information about the graph. Moreover, we define the probability which shows how close a group is to being a nil-$n$ group. It is proved that two $n$-isoclinic groups which are not nil-$n$ groups have isomorphic graphs under special conditions.

**KEYWORDS**: nilpotent groups, $n$th nilpotency degree

## INTRODUCTION

Graphs provide some tools for studying algebraic structures. There are some ways to make a graph associated with a group or semigroup. We may refer to the works of Bertram et al[1], Grunewald et al[2], Moghadamfar et al[3] and Williams[4] or recent papers on non-commuting graphs, Engel graphs and non-cyclic graphs given in Ref. 5, Ref. 6 and Ref. 7, respectively.

In the next section, we introduce the relative non nil-$n$ graph. We discuss the diameter, dominating set, and domination number of this graph. Defining such a graph is the key to establishing the new probability, namely, the $n$th nilpotency degree, which shows how much a group is near to becoming a nil-$n$ group. Moreover, we prove that there is no relative non nil-$n$ star graph, non nil-$n$ complete graph or relative non nil-$n$ complete bipartite graph. We discuss the non-regularity of the relative non nil-$n$ graph in some special cases too. Furthermore, it is proved that two $n$-isoclinic groups which are not nil-$n$ groups have isomorphic graphs under special conditions.

## THE RELATIVE NON NIL-$n$ GRAPH

Initially we define the relative non nil-$n$ graph for any group $G$ which is not a nil-$n$ group. Some graph theoretical properties such as diameter, dominating set, and domination number will be presented. We also discuss the planarity of this graph.

**Definition 1** We associate a graph $\Gamma^{(n)}{}_G$ with the group $G$ which is not a nil-$n$ group. The vertex set of this graph is $G \setminus Z_n(G)$ and two distinct vertices $x$ and $y$ are adjacent if $[x, y] \notin Z_{n-1}(G)$, where $Z_n(G)$ is the $n$th term of the upper central series of $G$.

It is clear that when $n = 1$ then $\Gamma^{(n)}{}_G$ is the non-commuting graph of $\Gamma_G$. Therefore in the rest of the paper suppose $n > 1$. Obviously, if $Z_n(G) = \{1\}$ then $\Gamma^{(n)}{}_G = \Gamma_G$. By similar methods to those used to prove Propositions 2 and 3 in Ref. 3 one can conclude that there is no non nil-$n$ group $G$ with a normal subgroup $N \neq 1$ such that $\Gamma^{(n)}{}_G \cong \Gamma^{(n)}{}_{G/N}$. Moreover, there is no non nil-$n$ group $G$ with $H < G$ such that $\Gamma^{(n)}{}_G \cong \Gamma^{(n)}{}_H$.

Now, we generalize the non nil-$n$ graph to the relative non nil-$n$ graph for any subgroup $H$ of $G$.

**Definition 2** The relative non nil-$n$ graph, $\Gamma^{(n)}{}_{H,G}$, is associated with the non nil-$n$ group $G$ and $H \leqslant G$. The vertex set is $G \setminus C^{(n)}{}_G(H)$, where

$$C^{(n)}{}_G(H) = \{x \in G : [x, y] \in Z_{n-1}(G), \forall y \in H\}.$$

Moreover, two vertices $x$ and $y$ are adjacent if at least one of them belongs to $H$ and $[x, y] \notin Z_{n-1}(G)$. If $H = G$ then we write $\Gamma^{(n)}{}_{G,G} = \Gamma^{(n)}{}_G$.

Obviously, the above graphs are simple. If $G$ is a nil-$n$ group then both graphs are null and for abelian subgroup $H$ of $G$ the relative non nil-$n$ graph is an empty graph. In this paper we always assume $G$ is a group which is not nil-$n$, unless stated. We define

$$C^{(n)}{}_H(y) = \{x \in H : [x, y] \in Z_{n-1}(G)\}$$

for $H \leqslant G$, which is a generalization of the ordinary centralizer of an element $y$ in a group. For every subgroup $H$ of $G$ one can verify the equality $C^{(n)}{}_G(H) = \cap_{x\in H} C^{(n)}{}_G(x)$. The degree of each vertex can be easily computed. If the vertex $h \in H$ then $\deg(h) = |G| - |C^{(n)}{}_G(h)|$ and for vertex $g \in G \setminus H$ we have $\deg(g) = |H| - |C^{(n)}{}_H(g)|$. Isolated vertices do not exist, and if two vertices in $H$ are in the same conjugacy classes of $G$ then they have the same vertex degrees. Moreover, for every vertex $x$ of $\Gamma^{(n)}{}_G$ we have $[G : C^{(n)}{}_G(x)] \geqslant 2$ which implies that the graph is Hamiltonian.

**Theorem 1** *If $H$ is a subgroup of $G$ such that $Z_n(H) = 1$, then $\operatorname{diam}(\Gamma^{(n)}{}_{H,G}) = 2$ and the girth of the graph is 3.*

*Proof*: Let $g_1$ and $g_2$ be two vertices which are not adjacent. There are vertices $h_1, h_2 \in H$ by hypothesis which join $g_1$ and $g_2$, respectively. If we have $[g_2, h_1] \notin Z_{n-1}(G)$ or $[g_1, h_2] \notin Z_{n-1}(G)$ then $d(g_1, g_2) = 2$. Suppose this does not occur. Hence, $h_1h_2$ is a vertex and a neighbour of $g_1$ and $g_2$. By similar an argument, if $g$ and $h$ are adjacent then there is a triangle of the form $\{g, h, h_1\}$ or $\{g, h, h_2\}$ or $\{g, h, h_1h_2\}$, where $h, h_1, h_2 \in H$ and $[g, h_1] \notin Z_{n-1}(G)$, $[h, h_2] \notin Z_{n-1}(G)$. □

Note that $Z_n(H) = 1$ is a necessary condition which makes us sure that $h_1, h_2 \in H$ are vertices. As a consequence of the above theorem we can say that the graph $\Gamma^{(n)}{}_{H,G}$ is connected whenever $Z_n(H) = 1$. Moreover, by the same argument as in Theorem 1 one can deduce for any non nil-$n$ group $G$, $\operatorname{diam}(\Gamma^{(n)}{}_G) = 2$ and $\operatorname{girth}(\Gamma^{(n)}{}_G) = 3$ which is a generalization of Proposition 2.1 in Ref. 5.

Now let us deal with the dominating set of the graph. The following results are generalizations of Remark 2.5, Proposition 2.12 part (1), Remark 2.13 and Proposition 2.14 in Ref. 5 so we omit the proof.

If $\{h\}$ is a dominating set for $\Gamma^{(n)}{}_{H,G}$, then $C^{(n)}{}_G(H) = 1$, $h^2 = 1$ and $C^{(n)}{}_G(h) = \langle h \rangle$, where $h \in H$. Moreover, if $S \subseteq V(\Gamma^{(n)}{}_{H,G}) \cap H$ then $S$ is a dominating set of the graph $\Gamma^{(n)}{}_{H,G}$ if and only if

$$C^{(n)}{}_G(S) \subseteq C^{(n)}{}_G(H) \cup S.$$

It is not hard to verify that if $H = \langle Y \rangle$ then $Y$ is a dominating set for $\Gamma^{(n)}{}_{H,G}$. Now, if $H$ is a finite simple non-abelian subgroup of $G$ then the domination number of $\Gamma^{(n)}{}_{H,G}$ is less than 3. For every maximal independent set $S$ for $\Gamma^{(n)}{}_G$, $S \cup Z_n(G)$ is a maximal nil-$n$ subgroup of $G$.

The following two results will give some ways of obtaining a dominating set for $\Gamma_{H,G}$.

**Theorem 2** *Let $G$ be a non nil-$n$ group, $H$ a subgroup of $G$, and $X = \{h_1, \dots, h_k\}$ a generating set for $H$. If $X \cap C^{(n)}{}_G(H) = \{h_{m+1}, \dots, h_k\}$ then*

$$S = \{h_1, \dots, h_m\} \cup \{h_1h_{m+1}, h_1h_{m+2} \cdots, h_1h_k\}$$

*is a dominating set for $\Gamma^{(n)}{}_{H,G}$.*

*Proof*: Let $t$ be a vertex which does not belong to $S$. Consider two cases. If $t \in H$ then there exists an element $h = h_1^{\beta_1} h_2^{\beta_2} \cdots h_m^{\beta_m}$, $\beta_i \in \mathbb{Z}$ with $[t, h] \notin Z_{n-1}(G)$. Hence, $[t, h_j] \notin Z_{n-1}(G)$ for some $h_j \in S$, $1 \leqslant j \leqslant m$ and it implies that $t$ joins $h_j$ as required. If $t \in G \setminus H$ then there is an element $h = h_1^{\gamma_1} h_2^{\gamma_2} \cdots h_k^{\gamma_k} \in H$, $\gamma_i \in \mathbb{Z}$ such that $[t, h] \notin Z_{n-1}(G)$. Now if $[t, h_j] \notin Z_{n-1}(G)$ for some $1 \leqslant j \leqslant m$ then $t$ meets $h_j$ where $h_j \in S$. Otherwise if $[t, h_j] \in Z_{n-1}(G)$ for all $1 \leqslant j \leqslant m$ and as $t \notin C^{(n)}{}_G(H)$ there will exist $h_s$ for some $m+1 \leqslant s \leqslant k$ such that $[t, h_s] \notin Z_{n-1}(G)$. Hence $t$ is adjacent to $h_1h_s$ and the proof is completed. □

**Proposition 1** *Assume $H$ is a subgroup of $G$ which is not nil-$n$. Then $S = HC^{(n)}{}_G(H) - C^{(n)}{}_G(H)$ is a dominating set for $\Gamma^{(n)}{}_{H,G}$.*

*Proof*: Let $x \notin S$ be a vertex, so $[x, h] \notin Z_{n-1}(G)$ for some $h \in H$. If $h \notin C^{(n)}{}_G(H)$ then $h$ should be in $S$ and so $x$ joins $h$. If $h \in C^{(n)}{}_G(H)$, then there is $t \in H \setminus C^{(n)}{}_G(H)$, since $H$ is not a nil-$n$ group. If $[x, t] \notin Z_{n-1}(G)$ then $t$ and $x$ are adjacent where $t \in S$. Otherwise $[x, t] \in Z_{n-1}(G)$ and implies that $th \notin C^{(n)}{}_G(H)$, $th \in S$, and $x$ meets $th$. □

If $S$ is a dominating set for $\Gamma^{(n)}{}_H$ then it is a dominating set for $\Gamma^{(n)}{}_{H,G}$ whenever $C^{(n)}{}_G(H) = C^{(n)}{}_G(S)$. It is not hard to conclude that $\Gamma^{(n)}{}_H$ is a subgraph of $\Gamma^{(n)}{}_{H,G}$ while $\Gamma^{(n)}{}_{H,G}$ itself is subgraph of $\Gamma^{(n)}{}_G$ which is a subgraph of the non-commuting graph $\Gamma_G$. Abdollahi et al[5] proved that $\Gamma_G$ is planar if and only if $G \cong S_3$ or $D_8$ or $Q_8$. Clearly, if $G \cong D_8$ or $Q_8$ then $\Gamma^{(n)}{}_G$ is a null graph for $n > 1$. Moreover, the above argument implies that $\Gamma^{(n)}{}_{S_3}$ is planar for $n > 1$.

**Theorem 3** *Suppose $n > 1$ and $G$ is not a nil-$n$ group. Then $\Gamma^{(n)}{}_G$ is planar if and only if $G \cong S_3$.*

*Proof*: Suppose $\Gamma^{(n)}{}_G$ is planar. The clique number of the graph $\omega(\Gamma^{(n)}{}_G)$ is less than 5. Therefore the number of vertices that satisfy $[x, y] \notin Z_{n-1}(G)$ is at most 4. Now, the main result of Ref. 8 implies $G/Z(G)$ is finite. We can also define the epimorphism $G/Z(G) \twoheadrightarrow G/Z_n(G)$ which implies $G/Z_n(G)$ is finite. The assertion follows by similar methods to the proof of Proposition 2.3 in Ref. 5. □

We are interested in the properties which can be inherited via isomorphic non nil-$n$ graphs. For instance, if $G$ is a finite group which is not nil-$n$ and

$\Gamma^{(n)}{}_G \cong \Gamma^{(n)}{}_H$ for some group $H$ then one can deduce $H$ is a finite group which is not nil-$n$. Moreover, if $\Gamma^{(n)}{}_G \cong \Gamma^{(n)}{}_H$ then $|Z_n(G)|$ divides $\gcd(|H| - |Z_n(H)|, |H| - |C^{(n)}{}_H(x)|, |C^{(n)}{}_H(x)| - |Z_n(H)|)$ for a vertex $x \in V(\Gamma^{(n)}{}_H)$.

**Theorem 4** *Let $P$ be a finite simple group which is not a nil-$n$ group. If $\Gamma^{(n)}{}_G \cong \Gamma^{(n)}{}_P$ for some group $G$, then $|P| = |G|$.*

*Proof*: The proof follows from the classification of finite simple groups and the fact that $C^{(n)}{}_P(x) = C_P(x)$ for all types of simple groups (see Theorem 1 in Ref. 9 for more details). □

## RELATIVE $n$th NILPOTENCY DEGREE AND $\Gamma^{(n)}{}_{H,G}$

For any positive integer $n$, we introduce the probability that the commutator of two arbitrary elements $h \in H$ and $g \in G$ belongs to $Z_{n-1}(G)$. Let us define the probability and give some lower and upper bounds for it.

**Definition 3** The relative $n$th nilpotency degree of the subgroup $H$ in the group $G$, which is denoted by $P^{(n)}{}_{\text{nil}}(H, G)$, is the ratio

$$P^{(n)}{}_{\text{nil}}(H,G) = \frac{|\{(h,g) \in H \times G : [h,g] \in Z_{n-1}(G)\}|}{|H||G|},$$

or equivalently

$$P^{(n)}{}_{\text{nil}}(H,G) = \frac{1}{|H||G|} \sum_{g \in G} |C^{(n)}{}_H(g)| = \frac{1}{|H||G|} \sum_{h \in H} |C^{(n)}{}_G(h)|.$$

It is clear that if $n = 1$ then $P^{(1)}{}_{\text{nil}}(H,G) = d(H,G)$ which is the relative commutativity degree of a subgroup $H$ in the group $G$ (see Ref. 10) and when $H = G$ then $P^{(1)}{}_{\text{nil}}(G) = d(G)$ is the commutativity degree of the group $G$ (see Refs. 11, 12). Moreover, $P^{(n)}{}_{\text{nil}}(G) = 1$ if and only if $G$ is a nil-$n$ group and also $P^{(n)}{}_{\text{nil}}(G) \leqslant P^{(n+1)}{}_{\text{nil}}(G)$.

If $H \leqslant Z_n(G)$ then $P^{(n)}{}_{\text{nil}}(H,G) = 1$ and the definition implies $P^{(n)}{}_{\text{nil}}(H,G) \leqslant P^{(n)}{}_{\text{nil}}(H)$. The following theorem is an improvement of Theorem 3.5 in Ref. 10.

**Theorem 5** *Let $H$ be a subgroup of $G$ and $p$ be a smallest prime which divides the order of $G$. Then*

(i)

$$\frac{|Z_n(G)|}{|G|} + \frac{p(|G| - |Z_n(G)|)}{|G|^2} \leqslant P^{(n)}{}_{\text{nil}}(G) \leqslant \frac{|Z_n(G)| + |G|}{2|G|}$$

(ii)

$$\frac{|H \cap Z_n(G)|}{|H|} + \frac{p(|H| - |Z_n(G) \cap H|)}{|G||H|} \leqslant P^{(n)}{}_{\text{nil}}(H,G) \leqslant \frac{|Z_n(G) \cap H| + |H|}{2|H|}.$$

*Proof*:

(i) Definition 3 implies

$$|G|^2 P^{(n)}{}_{\text{nil}}(G) = \sum_{x \in G} |C^{(n)}{}_G(x)| = \sum_{x \in Z_n(G)} |C^{(n)}{}_G(x)| + \sum_{x \notin Z_n(G)} |C^{(n)}{}_G(x)| = |G||Z_n(G)| + \sum_{x \in G \setminus Z_n(G)} |C^{(n)}{}_G(x)|.$$

Clearly we have $p \leqslant |C^{(n)}{}_G(x)| \leqslant |G|/2$ for a non-central element $x$. Therefore, we can easily conclude that

$$p(|G| - |Z_n(G)|) \leqslant \sum_{x \in G \setminus Z_n(G)} |C^{(n)}{}_G(x)| \leqslant \frac{(|G| - |Z_n(G)|)|G|}{2}$$

and the assertion follows just by substitution.

(ii) Again, we use Definition 3:

$$|G||H| P^{(n)}{}_{\text{nil}}(H,G) = \sum_{x \in H} |C^{(n)}{}_G(x)| = \sum_{x \in H \cap Z_n(G)} |C^{(n)}{}_G(x)| + \sum_{x \in H \setminus H \cap Z_n(G)} |C^{(n)}{}_G(x)| = |G||H \cap Z_n(G)| + \sum_{x \in H \setminus H \cap Z_n(G)} |C^{(n)}{}_G(x)|.$$

The result follows in a similar way to the proof of the first part.

□

If $H \nsubseteq Z_n(G)$ then $P^{(n)}{}_{\text{nil}}(H,G) \leqslant \frac{3}{4}$ for non nil-$n$ group $G$, by the above theorem.

**Theorem 6** *Let $H$ be a subgroup of $G$ and $N$ a normal subgroup of $G$ which is contained in $H$. We have*

$$P^{(n)}{}_{\text{nil}}(H,G) \leqslant P^{(n)}{}_{\text{nil}}\left(\frac{H}{N}, \frac{G}{N}\right) P^{(n)}{}_{\text{nil}}(N),$$

*and the equality holds if* $N \cap [H, {}_nG] = 1$ *where* $[H, {}_nG]$ *is the commutator subgroup of* $H$ *and* $n$ *copies of* $G$.

*Proof*: First we show that for every $x \in G$, ${C^{(n)}}_H(x)N/N \leqslant {C^{(n)}}_{H/N}(xN)$ and the equality holds if $N \cap [H, {}_nG] = 1$ since if $h \in {C^{(n)}}_H(x)$ then $[hN, xN] \in Z_{n-1}(G)N/N \leqslant Z_{n-1}(G/N)$. Moreover, if $hN \in {C^{(n)}}_{H/N}(xN)$ then $[hN, xN] \in Z_{n-1}(G/N)$. This means $[hN, xN, g_1N, \dots, g_{n-1}N] = N$ for all $g_iN \in G/N$. Thus $[h, x, g_1, \dots, g_{n-1}] \in N \cap [H, {}_nG] = 1$. The rest of the proof is very similar to the proof of Theorem 3.9 in Ref. 10. □

We state the following formula which is the number of edges of the non nil-$n$ graph.

$$|E({\Gamma^{(n)}}_G)| = \frac{|G|^2}{2}(1 - {P^{(n)}}_{\mathrm{nil}}(G)). \qquad (1)$$

Moreover, any lower or upper bounds for ${P^{(n)}}_{\mathrm{nil}}(G)$ will give lower or upper bounds for $|E({\Gamma^{(n)}}_G)|$ and vice versa. It is clear that for every graph, the number of edges is at most $t(t-1)/2$, where $t$ is the number of vertices. Thus by using (1) we can obtain

$$\begin{aligned}{P^{(n)}}_{\mathrm{nil}}(G) \geqslant \frac{2|Z_n(G)|}{|G|} + \frac{1}{|G|} - \frac{|Z_n(G)|^2}{|G|^2} \\ - \frac{|Z_n(G)|}{|G|^2}.\end{aligned}$$

Furthermore, if $G_1$ and $G_2$ are groups with $|Z_n(G_1)| = |Z_n(G_2)|$ such that ${\Gamma^{(n)}}_{G_1} \cong {\Gamma^{(n)}}_{G_2}$, then ${P^{(n)}}_{\mathrm{nil}}(G_1) = {P^{(n)}}_{\mathrm{nil}}(G_2)$.

Now, we recall that a star graph is a tree on $n$ vertices in which one vertex has degree $n-1$ and the others have degree 1.

**Theorem 7** *There is no group* $G$ *and subgroup* $H$ *with a relative non nil-*$n$ *star graph.*

*Proof*: Suppose on the contrary that ${\Gamma^{(n)}}_{H,G}$ is relative non nil-$n$ star graph. Assume $h \in H$ is the unique vertex of degree $(|V({\Gamma^{(n)}}_{H,G})| - 1)$. Then we conclude that ${C^{(n)}}_G(H) = 1$. On the other hand, for a vertex $g \in G \setminus H$ we have $|{C^{(n)}}_H(g)| = |H| - 1$ so $[H : {C^{(n)}}_H(g)] = |H|/(|H|-1)$ which is impossible. Secondly, assume $g \in G \setminus H$ is the unique vertex of degree $(|V({\Gamma^{(n)}}_{H,G})| - 1)$ and all the other vertices. For instance $h \in H$ has degree 1. Then we have $|G| = |{C^{(n)}}_G(h)| + 1$. Therefore, $|{C^{(n)}}_G(H)| = 1$ and similarly, $[G : {C^{(n)}}_G(h)] = |G|/(|G|-1)$ which is impossible. Hence such a graph does not exist. □

Non nil-$n$ star graphs do not exist and so there is no group with $n$th nilpotency degree ${P^{(n)}}_{\mathrm{nil}}(G) = 1 - 2/|G| + 4/|G|^2$ whenever $Z_n(G) = 1$.

**Theorem 8** *There is no relative non nil-*$n$ *complete graph* ${\Gamma^{(n)}}_{H,G}$.

*Proof*: Suppose ${\Gamma^{(n)}}_{H,G}$ is a relative non nil-$n$ complete graph. So for a vertex $x \in H$, we have $|G| - |{C^{(n)}}_G(x)| = |G| - |{C^{(n)}}_G(H)| - 1$. Therefore $|{C^{(n)}}_G(H)| = 1$, $|{C^{(n)}}_G(x)| = 2$, and the order of all non-trivial elements of $H$ is 2. This implies $H$ is an elementary abelian 2-group which is a contradiction. □

It can be proved there is no non nil-$n$ complete graph in a similar way to the proof of the above theorem. A complete bipartite graph is a bipartite graph such that every pair of graph vertices in the two sets are adjacent.

**Theorem 9** *There is no non nil-*$n$ *complete bipartite graph.*

*Proof*: Suppose ${\Gamma^{(n)}}_G$ is a non nil-$n$ complete bipartite graph. All vertices are partitioned into two disjoint sets $V_1$ and $V_2$ such that $|V_1| + |V_2| = |G| - |Z_n(G)|$. We have

$$\deg(x) = |G| - |{C^{(n)}}_G(x)| \leqslant \frac{(|G| - |Z_n(G)|)}{2}$$

and $|Z_n(G)|q = |{C^{(n)}}_G(x)|$ for some $q \in \mathbb{Z}$ and $x \in V({\Gamma^{(n)}}_G)$. Hence $|G| \leqslant |Z_n(G)|(2q-1)$ and so $[G : {C^{(n)}}_G(x)] \leqslant (2 - (1/q)) < 2$ which is a contradiction. □

We claim that there is no relative non nil-$n$ complete bipartite graph. Otherwise, the only possibility is to have two disjoint sets $V_1$ and $V_2$ such that one contains vertices of $H$. Since all vertices of $H$ are not adjacent, if $h \in V_1$ then $[h, x] \in Z_{n-1}(G)$ for all $x \in H \setminus {C^{(n)}}_G(H)$ which implies that $h \in {C^{(n)}}_G(H)$ which is a contradiction.

**Proposition 2** *Let* $H$ *be a subgroup of the non nil-*$n$ *group* $G$. *Then the following hold:*

(i) ${\Gamma^{(n)}}_{H,G}$ *has no vertex of degree 2.*

(ii) *There is no vertex in* $H$ *of degree 4 and if* $g \in G \setminus H$ *is a vertex of degree 4 then* $H \cong S_3$ *or* $D_8$ *or* $Q_8$.

(iii) ${\Gamma^{(n)}}_{H,G}$ *has no vertex of degree* $p$ *in* $H$, *also if* $g \in G \setminus H$ *is a vertex of degree* $p$ *then* $H \cong D_{2p}$ *or* $|H| = p + 1$, *where* $p$ *is an odd prime.*

*Proof*:

(i) Obvious.

(ii) Assume $h \in H$ is a vertex of degree 4.

$$\deg(h) = |{C^{(n)}}_G(h)|([G : {C^{(n)}}_G(h)] - 1) = 4.$$

It implies $|{C^{(n)}}_G(h)| = 2$ or 4 and $G \cong S_3$ or $D_8$ or $Q_8$. But they have no non-abelian subgroup and this means that $\Gamma^{(n)}{}_{H,G}$ has isolated vertices in $G \setminus H$ and no vertices in $H$ which is a contradiction. If $g \in G \setminus H$ is a vertex of degree 4, then the same argument about its degree implies the assertion.

(iii) Suppose $h \in H$ is a vertex and

$$\deg(h) = |C^{(n)}{}_G(h)|([G : C^{(n)}{}_G(h)] - 1) = p.$$

Therefore $|G| = 2p$ and since $G$ is non-abelian and $p$ is an odd number $G \cong D_{2p}$ and again $D_{2p}$ has no non-abelian subgroup which is a contradiction. Similarly, one can conclude that $|C^{(n)}{}_H(g)| = p$ or $|C^{(n)}{}_H(g)| = 1$ for $g \in G \setminus H$ of degree $p$. Hence $H \cong D_{2p}$ or $|H| = p + 1$. □

Obviously, by Proposition 2, $\Gamma^{(n)}{}_{H,G}$ is not $p$-regular or 4-regular. We finish this section with some interesting results about the non-regularity of the relative nil-$n$ graph. Note that $H$ is a non-trivial proper subgroup of $G$.

**Theorem 10** *There is no relative non nil-$n$ graph which is $m$-regular, where $m$ is a square-free positive odd integer.*

*Proof*: Suppose $\Gamma^{(n)}{}_{H,G}$ is a relative non nil-$n$ graph which is $m$-regular and $P = \{p_1, p_2, \dots, p_k\}$ is the set of distinct odd primes which factorize $m$. If $h \in H \cap V(\Gamma^{(n)}{}_{H,G})$ then

$$m = \deg(h) = |C^{(n)}{}_G(h)|([G : C^{(n)}{}_G(h)] - 1),$$

$|C^{(n)}{}_G(h)| = \prod_{p_i \in S} p_i$, and $([G : C^{(n)}{}_G(h)] - 1) = \prod_{p_j \in S^c} p_j$, where $S$ and $S^c$ are subsets of $P^* = P \cup \{1\}$ such that $|C^{(n)}{}_G(h)| \neq 1$. Thus $|G| = \prod_{p_i \in S} p_i(\prod_{p_j \in S^c} p_j + 1)$. By a similar to above, $|H| = \prod_{p_i \in T} p_i(\prod_{p_j \in T^c} p_j + 1)$, where $T$ and $T^c$ are subsets of $P^*$. Since $H$ is a subgroup of $G$ we have $\prod_{p_i \in T \setminus T \cap S} p_i(\prod_{p_j \in T^c} p_j + 1)$ divides $(\prod_{p_j \in S^c} p_j + 1)$ which is impossible. □

**Theorem 11** *The relative non nil-$n$ graph $\Gamma^{(n)}{}_{H,G}$ is not $2k$-regular, where $k$ is a square free positive odd integer.*

*Proof*: By similar method to the proof of the previous theorem, we obtain several cases for the orders of $H$ and $G$ for which none of them is valid. □

If $\Gamma^{(n)}{}_{H,G}$ is a graph associated with a group $G$ of odd order which is not a nil-$n$ group then the degrees of its vertices are even numbers. By Theorem 11 it is not a $2k$-regular graph, for square free positive odd integer $k$. It is not a $2^r$-regular graph because otherwise we have $2^r = |C^{(n)}{}_G(h)|([G : C^{(n)}{}_G(h)] - 1)$ for $h \in H \cap V(\Gamma^{(n)}{}_{H,G})$. Since $|C^{(n)}{}_G(h)| \neq 1$ it follows that $|C^{(n)}{}_G(h)| = 2^\alpha$, $1 \leqslant \alpha \leqslant r$ and it is a contradiction. We guess $\Gamma^{(n)}{}_{H,G}$ is not regular at all when $G$ is of odd order.

## $\Gamma^{(n)}{}_{H,G}$ AND RELATIVE $n$-ISOCLINIC GROUPS

In this section, we are going to consider the known conjecture posed by Thompson[13] which states that if two graphs associated with the groups $G$ and $H$ are isomorphic then the groups $G$ and $H$ are isomorphic as well. Although we may easily check that the conjecture is not always true but we may find some conditions in which the above conjecture is valid.

**Theorem 12** *Let $H$ be a non nil-$n$ subgroup of group $G$ such that $|C^{(n)}{}_G(H)| \leqslant m$, where $m \geqslant 3$. If $\Gamma^{(n)}{}_{H,G} \cong \Gamma^{(n)}{}_{S_m}$ then $|G| = |S_m|$.*

*Proof*: Isomorphism between graphs implies that $|G| - |C^{(n)}{}_G(H)| = m! - 1$ or equivalently

$$\frac{|G|}{|C^{(n)}{}_G(H)|} = \frac{m! - 1}{|C^{(n)}{}_G(H)|} + 1.$$

Since $m! - 1$ and $k$ are coprime for $1 \leqslant k \leqslant m$ so $|C^{(n)}{}_G(H)| > m$ or equal to 1. Hence, the only possibility is $|C^{(n)}{}_G(H)| = 1$ and so the assertion follows. □

Let us recall the definition of $n$-isoclinism (see Ref. 14 for more details).

**Definition 4** Let $H$ and $G$ be groups. Then the pair $(\alpha, \beta)$ is called $n$-isoclinism from $H$ to $G$ whenever

(i) $\alpha$ is an isomorphism from $H/Z_n(H)$ to $G/Z_n(G)$, where $Z_n(H)$ and $Z_n(G)$ are the n-th term of the upper central series of $H$ and $G$, respectively.

(ii) $\beta$ is an isomorphism from $\gamma_{n+1}(H)$ to $\gamma_{n+1}(G)$, with the law

$$[h_1, \dots, h_n, h_{n+1}] \mapsto [g_1, \dots, g_n, g_{n+1}]$$

in which $g_j \in \alpha(h_j Z_n(H))$ for every $1 \leqslant j \leqslant n + 1$. If there is such a pair $(\alpha, \beta)$ with the above properties then we say that $H$ and $G$ are $n$-isoclinic and denoted by $H \overset{n}{\sim} G$.

**Theorem 13** *Let $G_1 \overset{n}{\sim} G_2$ be $n$-isoclinic groups. If $|Z_n(G_1)| = |Z_n(G_2)|$ then $\Gamma^{(n)}{}_{G_1} \cong \Gamma^{(n)}{}_{G_2}$.*

*Proof*: By hypothesis we have the bijection $\theta$ between $Z_n(G_1)$ and $Z_n(G_2)$. Moreover,

$$\alpha : \frac{G_1}{Z_n(G_1)} \to \frac{G_2}{Z_n(G_2)}$$

is an isomorphism which maps $g_i Z_n(G_1)$ to $g'_i Z_n(G_1)$ for $1 \leqslant i \leqslant k$, where $\{g_1, \dots, g_k\}$ and $\{g'_1, \dots, g'_k\}$ are the sets of transversals of $G_1/Z_n(G_1)$ and $G_2/Z_n(G_2)$, respectively. Now, we introduce $\psi : G_1 \backslash Z_n(G_1) \to G_2 \backslash Z_n(G_2)$ such that $g_i z \mapsto g'_i \theta(z)$ where, $z \in Z_n(G_1)$. Therefore $\psi$ is our favourite bijection between the set of vertices of $\Gamma^{(n)}{}_{G_1}$ and $\Gamma^{(n)}{}_{G_2}$ since if $x$ meets $y$ then $[x, y] \notin Z_{n-1}(G_1)$. The isomorphism $\beta : \gamma_{n+1}(G_1) \to \gamma_{n+1}(G_2)$ implies that $\psi(x)$ and $\psi(y)$ are adjacent and the result follows. □

One can improve the above result for relative $n$-isoclinism (see Ref. 15) and associated relative graphs by a similar method to the proof of Theorem 13.

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