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Conjugacy classes and commuting probability in finite metacyclic p-groups

K. Moradipour[a], **N.H. Sarmin**[b,*], **A. Erfanian**[c]

[a] Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

[b] Department of Mathematical Sciences, Faculty of Science and Ibnu Sina Institute for Fundamental Science Studies, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

[c] Department of Pure Mathematics and Centre of Excellence in Analysis on Algebraic Structure, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

*Corresponding author, e-mail: nhs@utm.my

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ABSTRACT: Let G be a finite non-abelian metacyclic p-group where p is any prime. We compute the exact number of conjugacy classes and the commutativity degree of G. In particular, we describe the number of conjugacy classes both in the split and non-split case.

KEYWORDS: split p-group, nilpotency class, commutativity degree

INTRODUCTION

We consider only finite groups. Recently many authors have investigated the number $k(G)$ of conjugacy classes of a group G. There are several papers on the conjugacy classes of finite p-groups[1–3]. Many authors obtained significant results but only on the lower and upper bound of $k(G)$. For instance, Sherman[4] proves that if G is a finite nilpotent group of nilpotency class m, then $k(G) > m|G|^{1/m} - m + 1$. Later Huppert[5] proved that $k(G) > \log n$ for any nilpotent group G of order n. On the other hand, Liebeck and Pyber[6] found an upper bound for $k(G)$ in terms of an arbitrary constant. Lopeze[7] shows that a maximal abelian subgroup A of $|A| = p^\alpha$ of a nilpotent group G of $|G| = p^m$ and $|Z(G)| = p^\beta$ satisfies an equality of the form

$$k(G) = \frac{p^{2\alpha-m} + p^\beta(p+1)(p^{m-\alpha}-1)}{p^{m-\alpha}} + \frac{k(p^2-1)(p-1)}{p^{m-\alpha}}$$

where $k \geqslant 0$. For $k > 0$ this formula provides an upper bound by default but does not determine the exact number of conjugacy classes of G.

A group G is called metacyclic if it contains a normal cyclic subgroup N such that G/N is also cyclic. Concerning these groups, in Ref. 8 it was shown that if G is any finite split metacyclic p-group for an odd prime p, that is, $G = H \ltimes K$ for subgroups H and K, and if $|H| = p^\alpha$ and $|K| = p^{\alpha+\beta}$, then there exist exactly

$$\frac{(\beta-\alpha+1)(p^{\alpha+1}-1)}{(p-1)} + 4\sum_{i=0}^{\alpha-1} p^i(\alpha+i)$$

conjugacy classes of subgroups of G.

The metacyclic p-groups of class 2 have been classified in Ref. 9 where homological methods are used. The case of $p = 2$ needs special attention and was the subject of Ref. 10. Moreover, Beuerle[11] classified the non-abelian metacyclic p-groups of class at least 3 where p is any prime. He showed there are four classes of such groups which have been called of *positive type*. We use these classifications in order to obtain the precise number of conjugacy classes of all non-abelian metacyclic p-groups of class 2 and class at least 3.

Each isomorphism class of metacyclic p-groups can be represented by five parameters p, α, β, ε, and γ. These parameters are used to measure the order, centre and abelianness of the groups, and also their nilpotency class, and whether the groups are split extension or not. We also use the parameters to compute the number of conjugacy classes of the groups. Let us first begin with the some notation. Let

$$G(p,\alpha,\beta,\varepsilon,\gamma) = \langle a,b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^r\rangle$$

where $r = p^{\alpha-\gamma} + 1$. We shorten the notation to $G(p)$ for $G(p,\alpha,\beta,\varepsilon,\gamma)$ and use the notation $[b,a] = bab^{-1}a^{-1} = a^b a^{-1}$ for the commutator of b and a.

Theorem 1 [11] *Let G be a non-abelian metacyclic p-group of nilpotency class 2. Then*

$$G \simeq \langle a,b | a^{p^\alpha} = b^{p^\beta} = 1, [a,b] = a^{p^{\alpha-\gamma}}\rangle,$$

where α, β, $\gamma \in \mathbb{N}$, $\alpha \geqslant 2\gamma$ and $\beta \geqslant \gamma \geqslant 1$.

Theorem 2 [11] *Let p be an odd prime and G a metacyclic p-group of nilpotency class at least 3. Then G is isomorphic to exactly one group in the following list:*

(i) $G \simeq \langle a,b | a^{p^\alpha} = b^{p^\beta} = 1, [b,a] = a^{p^{\alpha-\gamma}}\rangle$, *where α, β, $\gamma \in \mathbb{N}$, $\gamma - 1 < \alpha < 2\gamma$, and $\gamma \leqslant \beta$;*

(ii) $G \simeq \langle a,b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, [b,a] = a^{p^{\alpha-\gamma}}\rangle$, *where α, β, γ, $\varepsilon \in \mathbb{N}$, $\gamma - 1 < \alpha < 2\gamma$, $\gamma \leqslant \beta$, and $\alpha < \beta + \varepsilon$.*

Theorem 3 [11] *Let G be a metacyclic 2-group of nilpotency class at least 3. Then G is isomorphic to exactly one group in the following list:*

(i) $G \simeq \langle a,b | a^{2^\alpha} = b^{2^\beta} = 1, [b,a] = a^{2^{\alpha-\gamma}}\rangle$, *where α, β, $\gamma \in \mathbb{N}$, $1 + \gamma < \alpha < 2\gamma$, and $\beta \geqslant \gamma$;*

(ii) $G \simeq \langle a,b | a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, [b,a] = a^{2^{\alpha-\gamma}}\rangle$, *where α, β, γ, $\varepsilon \in \mathbb{N}$, $1 + \gamma < \alpha < 2\gamma$, $\gamma \leqslant \beta$, and $\alpha < \beta + \varepsilon$.*

In this article we are going to compute the exact number of conjugacy classes of metacyclic p-groups which have been presented in Theorems 1, 2, and 3. We will show in our main result (Theorem 4) that the split and non-split metacyclic p-groups of class greater than 2 and class exactly 2 have precisely

$$p^{\alpha+\beta}\Big(\frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}}\Big)$$

conjugacy classes.

We also investigate the commuting probability in metacyclic p-groups, since conjugacy classes can be used to find the commutativity degree of a group. There are many papers on the lower and upper bound of commutativity degree of some particular groups[12–16]. We recall that Gustafson introduced $Pr(G) = k(G)/|G|$ as the probability that a randomly picked pair of elements of a group G are commuting. Thus finding the commutativity degree of a group is the same of finding the number of conjugacy classes of the group. It was noted in Ref. 16 that $Pr(G) \leqslant \frac{5}{8}$ for any non-abelian group G and equality holds exactly when $[G : Z(G)] = 4$.

In Theorem 5 we will obtain the exact value of $Pr(G)$ for metacyclic p-groups (p is any prime) which have been mentioned in Theorems 1, 2, and 3. Moreover, we show that the commutativity degree of the groups of cases (1)–(5) which are given in Theorem 4 are the same and equal to

$$\frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}}.$$

PRELIMINARIES

This section contains some important results and other preparatory material which will be used in our main theorems. In the following lemma the centre and the order of centre of a metacyclic p-group G are given.

Lemma 1 *If $G = G(p,\alpha,\beta,\varepsilon,\gamma)$, then*

(i) $|G| = p^{\alpha+\beta}$;

(ii) $Z(G) = \langle a^{p^\gamma}, b^{p^\gamma}\rangle$ *and* $|Z(G)| = p^{\alpha+\beta-2\gamma}$.

Proof: (i) $G = \langle a\rangle\langle b\rangle$ and $\langle a\rangle \cap \langle b\rangle = \langle a^{p^{\alpha-\varepsilon}}\rangle$ has order p^ε, then the order of G is $p^{\alpha+\beta}$. Part (ii) is a straightforward consequence of Proposition 4.10 in Ref. 17. □

The following corollary is an immediate consequence of Lemma 1, and also see Ref. 11.

Corollary 1 *Let G be a group of type $G(p,\alpha,\beta,\varepsilon,\gamma)$. If $\beta+\varepsilon \leqslant \alpha$, then G is isomorphic to a split metacyclic p-group and in particular, $G \simeq G(p,\alpha,\beta,0,\gamma)$. Moreover, the class of G is greater than 2 if and only if $\alpha < 2\gamma$.*

Lemma 2 *Let α, β, r, and ε be integers with α, β non-negative and let*

$$G \simeq \langle a,b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^r\rangle$$

be a metacyclic p-group, where $r = p^{\alpha-\gamma} + 1$. If $x, y \in G$ with $x = a^i b^j$ and $y = a^s b^t$, then the following hold in G:

(i) $xy = a^{i+sr^j} b^{j+t}$;

(ii) $x^y = a^{s(1-r^j)+ir^t} b^j$;

(iii) $[x,y] = a^{i(1-r^t)+s(r^j-1)}$.

Proof: This is straightforward. □

Lemma 3 *If $[b,a] = a^{p^{\alpha-\gamma}}$, $r = 1 + p^{\alpha-\gamma}$, and $\ell = p^\delta \ell'$ such that* $\gcd(p,\ell') = 1$*, then $r^\ell - 1 = p^{\alpha-\gamma+\delta}(pk + \ell')$, for some integers δ, ℓ', k, γ, and $\alpha \geqslant 0$.*

Proof: By a direct calculation we get,

$$r^\ell - 1 = \big(1 + p^{\alpha-\gamma}\big)^\ell - 1 = \sum_{i=0}^{\ell} \tbinom{\ell}{i} p^{(\alpha-\gamma)i} = p^{\alpha-\gamma+\delta}(pk + \ell')$$

for some integers δ, ℓ', k, where $\ell = p^\delta \ell'$ and $\gcd(p, \ell') = 1$. □

Lemma 4 *Let* $G_\gamma = G(\alpha,\beta,\gamma) \simeq \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1, [b,a] = a^{p^{\alpha-\gamma}}\rangle$ *and* $G_{\gamma-1} = G_\gamma/\langle z\rangle$, *where* $z = a^{p^{\alpha-1}}$. *Then* $\bar{x}, \bar{y} \in G_{\gamma-1}\backslash Z(G_{\gamma-1})$ *are conjugate if and only if* $x, y \in G_\gamma$ *are conjugate.*

Proof: Clearly, if x, $y \in G_\gamma$ are conjugate, then their images in $G_{\gamma-1}\backslash Z(G_{\gamma-1})$ are conjugate. Now suppose that $\bar{x}$, $\bar{y}$ are conjugate, i.e., $\bar{y} = \bar{x}^{\bar{g}}$ then $y^{-1}x^g \in \langle z\rangle$ and $y^{-1}x^g = z^\ell$ for some ℓ. If $\ell = 0$, the result is trivial. If $\ell \neq 0$ and $x^g = yz^\ell$, we show that y and yz^ℓ are conjugate. Suppose that $y = a^i b^j$, $w = a^s b^t$, and $y^w = yz^\ell$. By Lemma 2, we have $a^{i(1-r^t)+s(r^j-1)} = z^\ell = a^{p^{(\alpha-1)}\ell}$. Therefore

$$i(1 - r^t) + s(r^j - 1) \equiv p^{\alpha-1}\ell \pmod{p^\alpha}.$$

Now if $r^j \not\equiv 1 \pmod{p^\alpha}$ and $t = 0$, then $s(r^j-1) \equiv p^{\alpha-1}\ell \pmod{p^\alpha}$. We can write $r^j - 1 = p^{j'}v$ such that $\gcd(p, v) = 1$ and $j' \leqslant \alpha - 1$. Thus $sp^{j'}v \equiv p^{\alpha-1}\ell \pmod{p^\alpha}$. It follows that $s \equiv \ell v^{-1} \pmod{p}$. Now if we let $r^j \equiv 1 \pmod{p^\alpha}$, then $i(1 - r^t) \equiv p^{\alpha-1}\ell \pmod{p^\alpha}$. Suppose that $i = p^\delta i'$ and $\gcd(p, i') = 1$. Using Lemma 3, we have $i'p^\delta p^{\alpha-\gamma+\sigma}(pk+t') \equiv p^{\alpha-1}\ell \pmod{p^\alpha}$, from which it follows that

$$i'p^{\alpha+\delta-\gamma+\sigma}(pk + t') \equiv p^{\alpha-1}\ell \pmod{p^\alpha},$$

where $t = p^\sigma t'$ such that $\gcd(p, t') = 1$. If $\sigma = \lambda + \delta - 1$, then we have $i'(pk + t') \equiv \ell \pmod{p}$ and hence $t' \equiv -i'^{-1}\ell \pmod{p}$. The proof is then complete. □

Lemma 5 *Let G be a p-group with* $|G/Z(G)| = p^2$, *then* $k(G) = p^{-1}\big(p^2 + p - 1\big)|Z(G)|$.

Proof: It is easy to see that if $|G/Z(G)| = p^2$ and $g \in G\backslash Z(G)$ then $C_G(g) = \big\langle Z(G), g\big\rangle = Z(G)\langle g\rangle$ that is $\big|g^G\big| = \big[G : C_G(g)\big] = p$. Thus each conjugacy classes of G which lies in $G\backslash Z(G)$ has order p and so $G\backslash Z(G)$ has $(|G| - |Z(G)|)/p$ conjugacy classes. Hence G has $(|G| - |Z(G)|)/p + |Z(G)|$ conjugacy classes. □

THE NUMBER OF CONJUGACY CLASSES OF METACYCLIC p-GROUPS

Now we are in a position to prove our main theorem. This theorem gives a formula for the exact number of conjugacy classes of metacyclic p-groups of class 2 and class at least 3 in terms of α, β, γ. By Corollary 1, the group of part (1) in this theorem has class 2 since $\alpha \geqslant 2\gamma$, and the remaining parts have class at least 3 since $\alpha < 2\gamma$.

Theorem 4 ***(Main Theorem)*** *Let G be a non-abelian metacyclic p-group, where p is any prime number. If G is one of the groups in the following list:*

(1) $G \simeq \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1, [a,b] = a^{p^{\alpha-\gamma}}\rangle$, *where* $\alpha, \beta, \gamma \in \mathbb{N}$, $\alpha \geqslant 2\gamma$, *and* $\beta \geqslant \gamma \geqslant 1$;

(2) $G \simeq \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1, [b,a] = a^{p^{\alpha-\gamma}}\rangle$, *where* $\alpha, \beta, \gamma \in \mathbb{N}$, $\gamma - 1 < \alpha < 2\gamma$, *and* $\beta \geqslant \gamma$;

(3) $G \simeq \langle a, b | a^{2^\alpha} = b^{2^\beta} = 1, [b,a] = a^{2^{\alpha-\gamma}}\rangle$, *where* $\alpha, \beta, \gamma \in \mathbb{N}$, $1 + \gamma < \alpha < 2\gamma$, *and* $\gamma \leqslant \beta$;

(4) $G \simeq \langle a, b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, [b,a] = a^{p^{\alpha-\gamma}}\rangle$, *where* $\alpha, \beta, \gamma, \varepsilon \in \mathbb{N}$, $\gamma - 1 < \alpha < 2\gamma$, $\beta \geqslant \gamma$, *and* $\alpha < \beta + \varepsilon$;

(5) $G \simeq \langle a, b | a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, [b,a] = a^{2^{\alpha-\gamma}}\rangle$, *where* $\alpha, \beta, \gamma, \varepsilon \in \mathbb{N}$, $1 + \gamma < \alpha < 2\gamma$, $\gamma \leqslant \beta$, *and* $\alpha < \beta + \varepsilon$, *then*

$$k(G) = p^{\alpha+\beta}\Big(\frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}}\Big).$$

Proof: We compute $k(G)$ in the cases when G is split and non-split separately, and we prove that $k(G)$ for both cases is the same.

Split case. Using Corollary 1 the groups of parts (1)–(3) are split, since $\varepsilon = 0$. Based on Lemma 4 we compute the number of conjugacy classes of G for the split case (2), and then the method of proof can be applied to the other split cases. We denote the split group (2) by showing $G_\gamma(p) = G(p, \alpha, \beta, \varepsilon, \gamma)$. Then by using Lemma 1, $|G_\gamma(p)| = p^{\alpha+\beta}$ and $|Z(G_\gamma(p))| = p^{\alpha+\beta-2\gamma}$. If $z = a^{p^{\alpha-1}}$, then z is a central element of order p and we define the group $G_{\gamma-1} = G_\gamma/\langle z\rangle$. If we let $\bar{a} = a\langle z\rangle$ and $\bar{b} = b\langle z\rangle$, then $|\bar{a}| = p^{\alpha-1}$ and $|\bar{b}| = p^\beta$. Also we have

$$[\bar{b}, \bar{a}] = a^{p^{\alpha-\gamma}}\langle z\rangle = (a\langle z\rangle)^{p^{\alpha-\gamma}} = \bar{a}^{p^{\alpha-\gamma}} = \bar{a}^{p^{(\alpha-1)-(\gamma-1)}}.$$

Hence

$$G_{\gamma-1} = \langle \bar{a}, \bar{b} | \bar{a}^{p^{\alpha-1}} = \bar{b}^{p^\beta} = 1, [\bar{b}, \bar{a}] = \bar{a}^{p^{(\alpha-1)-(\gamma-1)}}\rangle \simeq G_{\gamma-1}(\alpha - 1, \beta, \gamma - 1).$$

Let $\phi : G_\gamma \to G_{\gamma-1}$ be the canonical homomorphism which maps an element $x \in G_\gamma$ to $x\langle z\rangle$. By using Lemma 4, there is a one-to-one correspondence between the conjugacy classes of non-central elements of $G_{\gamma-1}$ with conjugacy classes of G_γ, which are mapped by ϕ to non-central elements of $G_{\gamma-1}$. In fact if we let $Z(G_{\gamma-1}) = K/\langle z\rangle$, then two elements x, y of $G_\gamma\backslash K$ are conjugate in G_γ if and only if $x\langle z\rangle$, $y\langle z\rangle$, as elements of $G_{\gamma-1}\backslash Z(G_{\gamma-1})$ are conjugate in $G_{\gamma-1}$. Since $G_{\gamma-1}\backslash Z(G_{\gamma-1})$ contains $k(G_{\gamma-1})-|Z(G_{\gamma-1})|$ conjugacy classes, we conclude that $G_\gamma\backslash K$ has $k(G_{\gamma-1}) - |Z(G_{\gamma-1})|$ conjugacy classes. We now consider conjugacy classes of G_γ which lie in K. Clearly, $Z(G_\gamma)/\langle z\rangle \subseteq Z(G_{\gamma-1})$. Thus $Z(G_{\gamma-1})$ contains $|Z(G_\gamma)/\langle z\rangle| = |Z(G_\gamma)|/p$ elements which come from $Z(G_\gamma)$. Now if $g \in K\backslash Z(G_\gamma)$ then $g\langle z\rangle$ is a central element in $G_{\gamma-1}$. Hence g^{G_γ} contains p elements and it is exactly the set $g\langle z\rangle$. Thus each conjugacy class of G which lies in $K\backslash Z(G_\gamma)$ maps to a central element of $G_{\gamma-1}$ in $Z(G_{\gamma-1}) - Z(G_\gamma)/\langle z\rangle$ and vice versa. Thus $K\backslash Z(G_\gamma)$ contains exactly

$$\left|Z(G_{\gamma-1}) - \frac{Z(G_\gamma)}{\langle z\rangle}\right| = |Z(G_{\gamma-1})| - \left|\frac{Z(G_\gamma)}{\langle z\rangle}\right| = |Z(G_{\gamma-1})| - \frac{|Z(G_\gamma)|}{p}$$

conjugacy classes. Finally, we know $Z(G_\gamma)$ has $|Z(G_\gamma)|$ conjugacy classes of G_γ. Therefore the conjugacy classes of G_γ is equal to the sum of conjugacy classes of $G_\gamma\backslash K$, $K\backslash Z(G_\gamma)$, and $Z(G_\gamma)$, that is

$$\begin{aligned} k(G_\gamma) &= (k(G_{\gamma-1}) - |Z(G_{\gamma-1})|) + \big(|Z(G_{\gamma-1})| \\ &\quad - \frac{|Z(G_\gamma)|}{p}\big) + |Z(G_\gamma)| \\ &= k(G_{\gamma-1}) + (1-1/p)|Z(G_\gamma)|. \end{aligned}$$

By using Lemma 1 and induction on γ, we have

$$\begin{aligned} k(G_\gamma) &= k(G_{\gamma-1}) + (1-1/p)|Z(G_\gamma)| \\ &= k(G_{\gamma-2}) + (1-1/p)|Z(G_{\gamma-1})| \\ &\quad + (1-1/p)|Z(G_\gamma)| \\ &\quad \vdots \\ &= k(G_1) \\ &\quad + (1-1/p)(|Z(G_2)| + \cdots + |Z(G_\gamma)|). \end{aligned}$$

We complete the proof by computing $k(G_1)$. It is easy to see that $|G_1/Z(G_1)| = p^2$, so according to Lemma 5 we have

$$k(G_1) = p|Z(G_1)| + (1-1/p)|Z(G_1)|.$$

Therefore

$$\begin{aligned} k(G_\gamma) &= p|Z(G_1)| \\ &\quad + \left(1-\frac{1}{p}\right)(|Z(G_1)| + \cdots + |Z(G_\gamma)|) \\ &= pp^{\alpha+\beta-\gamma-1} \\ &\quad + \left(1-\frac{1}{p}\right)\left(p^{\alpha+\beta-\gamma-1} + \cdots + p^{\alpha+\beta-\gamma-\gamma}\right) \\ &= p^{\alpha+\beta-\gamma} \\ &\quad + \left(1-\frac{1}{p}\right)\left(p^{\alpha+\beta-2\gamma}\right)\left(p^{\gamma-1} + \cdots + 1\right) \\ &= p^{\alpha+\beta-\gamma} + \left(1-\frac{1}{p}\right)\left(p^{\alpha+\beta-2\gamma}\right)\left(\frac{p^\gamma-1}{p-1}\right) \\ &= p^{\alpha+\beta-\gamma} + p^{\alpha+\beta-\gamma-1} - p^{\alpha+\beta-2\gamma-1} \\ &= p^{\alpha+\beta}\left(\frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}}\right), \end{aligned}$$

as claimed.

Non-split case. Again by using Corollary 1 the groups of parts (4) and (5) are non-split, since $\alpha < \beta + \varepsilon$. We will use the results of the split case to find a similar formula for the exact number of conjugacy classes of the non-split case. In this case, we need to establish $G_{\gamma-1}$ in terms of α, β, ε, and γ. Also we may verify whether the central factor group $G_1/Z(G_1)$ is isomorphic to the group $\mathbb{Z}_p \times \mathbb{Z}_p$.

Let $G_{\gamma-1} = G_\gamma/\langle z\rangle$, where $z = a^{p^{\alpha-1}}$ is a central element of order p. If $\bar{a} = a\langle z\rangle$ and $\bar{b} = b\langle z\rangle$ then $|\bar{a}| = p^{\alpha-1}$ and $|\bar{b}| = p^{\beta+\varepsilon-1}$. Moreover, $\bar{b}^{p^\beta} = \bar{a}^{p^{\alpha-\varepsilon}} = \bar{a}^{p^{(\alpha-1)-(\varepsilon-1)}}$. We also have

$$[\bar{b}, \bar{a}] = (a\langle z\rangle)^{p^{\alpha-\gamma}} = \bar{a}^{p^{\alpha-\gamma}} = \bar{a}^{p^{(\alpha-1)-(\gamma-1)}}.$$

Thus

$$\begin{aligned} G_{\gamma-1} &= \langle \bar{a}, \bar{b} | \bar{a}^{p^{\alpha-1}} = 1, \bar{b}^{p^\beta} = \bar{a}^{p^{(\alpha-1)-(\varepsilon-1)}}, [\bar{b}, \bar{a}] \\ &= \bar{a}^{p^{(\alpha-1)-(\gamma-1)}}\rangle \\ &\simeq G_{\gamma-1}(p, \alpha-1, \beta, \varepsilon-1, \gamma-1), \end{aligned}$$

and the situation is the same as the split cases. To find out whether the group $G_1/Z(G_1) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, the order of the central factor group $G_\gamma/Z(G_\gamma)$ should be obtained when $\gamma = 1$. By applying Lemma 1 in the following central factor group we have

$$\left|\frac{G_1(p)}{Z(G_1(p))}\right| = \frac{p^{(\alpha-(\gamma-1))+\beta}}{p^{(\alpha-(\gamma-1))+\beta-2(\gamma-(\gamma-1))}} = \frac{p^{\alpha+\beta-\gamma+1}}{p^{\alpha+\beta-\gamma-1}} = p^2.$$

Hence from this equality and by using a similar method as in the proof of the split case, we obtain

$$\begin{aligned} k\left(G_\gamma(p,\alpha,\beta,\varepsilon,\gamma)\right) &= p|Z(G_1)| \\ &\quad + \left(1-\frac{1}{p}\right)\sum_{i=1}^{\gamma}|Z(G_i)| \\ &= pp^{\alpha+\beta-\gamma-1} \\ &\quad + \left(1-\frac{1}{p}\right)\sum_{i=1}^{\gamma}p^{\alpha+\beta-\gamma-i} \\ &= p^{\alpha+\beta-\gamma}+p^{\alpha+\beta-\gamma-1} \\ &\quad - p^{\alpha+\beta-2\gamma-1}, \end{aligned}$$

which completes our proof. □

As mentioned in the introduction, conjugacy classes can be used to find the commutativity degree of a group, so we arrive at the following result.

Theorem 5 *Let G be a non-abelian metacyclic p-group mentioned in Theorem 4. Then*

$$Pr(G) = \frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}}.$$

Proof: Theorem 4 gives the exact number of conjugacy classes in metacyclic p-groups, where p is any prime number. We then use Gustafson's formula $Pr(G) = k(G)/|G|$ which yields

$$Pr(G) = \frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}}.$$

□

Remark 1 This theorem shows directly that $Pr(G) = \frac{5}{8}$, when $p = 2$ and $\gamma = 1$. On the other hand, it is easy to see that $[G : Z(G)] = 2^{2\gamma} = 4$ when $\gamma = 1$. Thus by using Ref. 16 we see again that $Pr(G) = \frac{5}{8}$.

We conclude this section with a direct consequence of Theorem 4 and Theorem 5, given in the following corollary.

Corollary 2 *Let G be the quasi-dihedral group $QD_{2^{\alpha+1}}$. Then $Pr(G) = \frac{5}{8}$.*

Proof: By taking $\beta = \gamma = 1$ in Theorem 4-(3), we have

$$G(2,\alpha,1,0,1) \simeq \langle a,b | a^{2^\alpha} = b^2 = 1, [b,a] = a^{2^{\alpha-1}}\rangle.$$

Thus the exact number of conjugacy classes of quasi-dihedral group G is

$$k(G) = k(QD_{2^{\alpha+1}}) = 2^\alpha\left(\frac{3}{2} - \frac{1}{4}\right) = 2^\alpha + 2^{\alpha-2},$$

and thus $Pr(QD_{2^{\alpha+1}}) = \frac{5}{8}$. □

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