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Piecewise continuous mild solutions of a system governed by impulsive differential equations in locally convex spaces

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ABSTRACT: Existence and uniqueness problems of piecewise continuous mild solutions for a system governed by impulsive differential equations in locally convex spaces are solved. The global existence problem is proved for the uniformly Lipschitz case while the local existence problem is proved for the locally Lipschitz case. A priori estimate is given and used as an important tool for proving the global existence of a mild solution. The continuous dependence on impulsive conditions of the system is also proved. Our main results are obtained by using the fixed point theorem of a seminorm contraction. Some examples are given.

KEYWORDS: impulsive conditions, γ-contraction, a priori estimate, continuous dependence

INTRODUCTION

Impulsive differential equations have been proved to be a useful subject for applications in various fields of sciences and other disciplines that involve a process caused by a short-time perturbation neglecting the duration of perturbation.

Discontinuous solutions often occur in the process described by impulsive differential equations. To our knowledge, there are very few researchers studying piecewise continuous mild solutions of impulsive systems in locally convex spaces. This kind of solutions occurs in dynamical systems with impacts and frictions.

We consider the existence problems of piecewise continuous mild solutions for a system governed by the following impulsive differential equations in a locally convex space X:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f_{x;B}(t),\ t \in [0,T] - D \qquad (1)$$

$$\Delta x(t_i) = J_i(x(t_i)),\ t_i \in D,\ x(0) = x_0 \in X,$$

where $T > 0$, $f_{x;B}(t) := f(t, x(t), Bx(t))$, $D = \{t_1, t_2, \dots, t_n\}$ is the set of resetting times with $0 < t_1 < t_2 < \cdots < t_n < T$, and A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ on a sequentially complete locally convex space X, f and B are suitable operators and $J_i : X \to X$ is an operator determining the size of the jumps, $\Delta x(t_i) = x(t_i^+) - x(t_i) = x(t_i^+) - x(t_i^-)$ denotes the jump of x at the point t_i. Impulsive differential equations were considered for the first time by Milman and Myshkis [1,2].

A period of active research culminated with the monograph by Halaney and Wexler [3,4]. Many authors studied properties of mild solutions for systems or controlled systems governed by impulsive differential equations in Banach spaces [5–7].

It is known that Peano's theorem may not be valid in locally convex spaces, even in Banach spaces, i.e., the differential equation $\mathrm{d}x/\mathrm{d}t = f(t, x(t))$, $t \geqslant 0$ with $x(0) = x_0$ may fail to have solution [8–10], e.g., Dieudonne [8] showed that there is no solution for the IVP, $\mathrm{d}x/\mathrm{d}t = f(x(t))$, $x(0) = 0$, where f is continuous in X with

$$f(x(t)) = f(x_1, x_2, \dots, x_n) = (|x_n|^{\frac{1}{2}} + \tfrac{1}{n}),$$

$x = (x_1, \dots, x_n) \in X = c_0$. Some authors studied the existence of solutions for differential equations in locally convex spaces [11–13].

PRELIMINARIES

Let (X, P), or X in short, be a sequentially complete locally convex Hausdorff space, or an $\mathcal{S}$-space in short, topologized by a family of continuous seminorms $P = \{|\cdot|_\gamma : \gamma \in \Gamma\}$. It is well-known that every Hausdorff locally convex space can be topologized by a family of seminorms that separates points, or a sufficient family of seminorms, i.e., $|x|_\gamma = 0$ for all $\gamma \in \Gamma$ if and only if $x = 0$. The space of all continuous linear operators from X into itself will be denoted by $L(X)$. For each $A \in L(X)$ and $\beta, \gamma \in \Gamma$, we define

$$\|A\|_{\beta,\gamma} := \sup\{|Ax|_\gamma : |x|_\beta \leqslant 1\}.$$

It is clear that $A \in L(X)$ if and only if for every $\gamma \in \Gamma$ there exists $\beta \in \Gamma$ such that $\|A\|_{\beta,\gamma} < \infty$. Let $\{S(t)\}_{t\geqslant 0}$ denote a C_0-semigroup of linear operators on X, that is for each $t \geqslant 0$, $S(t) \in L(X)$ with,

(i) $S(0) = I$,
(ii) $S(t)S(h) = S(t + h)$ for all $t, h \geqslant 0$ and
(iii) $\lim_{h\downarrow 0} S(h)x = x,$ for all $x \in X$.

The generator A of the C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ is a linear operator whose domain is $D(A)$,

$$D(A) = \{x \in X : \lim_{h\downarrow 0} \frac{S(h)x - x}{h} \text{ exists in } X\}$$

and

$$Ax = \lim_{h\downarrow 0} \frac{S(h)x - x}{h} \text{ for all } x \in D(A).$$

Definition 1 Let $L_0(X) = \{S \in L(X) : \forall\gamma \in \Gamma,\ \exists M(\gamma) > 0,\ \forall x \in X,\ |Sx|_\gamma \leqslant M(\gamma)|x|_\gamma\}$. For each $\gamma \in \Gamma$, we define $\|\cdot\|_\gamma : L_0(X) \to (-\infty, \infty)$ as follows:

$$\|S\|_\gamma = \sup\left\{\frac{|Sx|_\gamma}{|x|_\gamma} : x \in X,\ |x|_\gamma \neq 0\right\}.$$

The following useful facts are known[14].

Proposition 1 *The following assertions hold:*

(i) $\{\|\cdot\|_\gamma : \gamma \in \Gamma\}$ *is a sufficient family of seminorms on* $L_0(X)$.

(ii) $L_0(X)$ *is an m-convex algebra, i.e., for every* $\gamma \in \Gamma$ *and* $S, T \in L_0(X)$, $\|ST\|_\gamma \leqslant \|S\|_\gamma\|T\|_\gamma$.

(iii) If X *is sequentially complete, then* $L_0(X)$ *is also sequentially complete.*

The followings are some examples of locally convex Hausdorff spaces and C_0-semigroups of linear operators on X.

Example 1 Let $X = C((-\infty, \infty))$ be the space of real continuous functions equipped with the family of seminorms $P = \{|\cdot|_\gamma : \gamma \in \Gamma\}$ defined by

$$|x|_\gamma = \sup\{|x(\xi)| : |\xi| < \gamma\},$$

where $\gamma \in \Gamma$, $\Gamma = (0, \infty)$ and $x \in X$. It is obvious that P separates points in X. Then (X, P) is a locally convex Hausdorff space. Also let $S : X \to X$ be defined by

$$S(x)(\xi) = \begin{cases} \int_0^\xi x(\eta)\mathrm{d}\eta,\ \xi > 0, \\ 0,\ \xi \leqslant 0. \end{cases}$$

It is clear that S is a linear operator and $|S(x)|_\gamma \leqslant \gamma|x|_\gamma$ for all $x \in X$ and $\gamma \in \Gamma$. Furthermore, $S \in L_0(X)$ and $\|S\|_\gamma \leqslant \gamma$ for all $\gamma \in \Gamma$.

Example 2 Let $X = C((-\infty, \infty))$ be equipped with the same topology as above and let $S : X \to X$ be defined by

$$S(x)(\xi) = \begin{cases} \mathrm{e}^{-\xi} \int_0^\xi x(\eta)\mathrm{d}\eta,\ \xi > 0, \\ 0,\ \xi \leqslant 0. \end{cases}$$

Let $\gamma \in \Gamma$ and $\xi \in [-\gamma, \gamma]$. Then

$$|S(x)(\xi)| \leqslant \mathrm{e}^{-\xi} \int_0^\xi x(\eta)\mathrm{d}\eta \leqslant M(\gamma)|x|_\gamma,$$

where

$$M(\gamma) = \begin{cases} \gamma\mathrm{e}^{-\gamma},\ \gamma \leqslant 1 \\ \mathrm{e}^{-1},\ \gamma \geqslant 1. \end{cases}$$

Therefore $|S(x)|_\gamma \leqslant M(\gamma)$ for all $\gamma \in \Gamma$ and for all $x \in X$. Furthermore, $|Sx|_\gamma \leqslant |x|_\gamma$, for all $\gamma \in \Gamma$.

Example 3 Let $X = C((-\infty, \infty))$ equipped with the same topology as in Example 2. Let $S : [0, \infty) \to L(X)$ be defined by $(S(t)x)(\xi) = \mathrm{e}^{-t}x(\xi)$, for all $\xi \in (-\infty, \infty)$ and $x \in X$. Then S is a C_0-semigroup on X whose infinitesimal generator is the operator $A : X \to X$ defined by $Ax = -x$. Indeed, for each $\gamma > 0$, we have for each $t \in [0, \infty)$

$$|S(t)x - x|_\gamma \leqslant (1 - \mathrm{e}^{-t})|x|_\gamma,$$

and

$$\left|\frac{S(t)x - x}{t} - x\right| \leqslant \frac{|\mathrm{e}^{-t} + t - 1|}{t}|x|_\gamma.$$

Definition 2 A semigroup $S : [0, \infty) \to L_0(X)$ is called locally bounded, if for each $\gamma \in \Gamma$ and $T > 0$, there exists $M(\gamma, T) \geqslant 0$ such that $\|S(t)\|_\gamma \leqslant M(\gamma, T)$, for all $t \in [0, T]$.

A semigroup $S : [0, \infty) \to L_0(X)$ is called bounded, if for each $\gamma \in \Gamma$, there exists $M(\gamma) > 0$ such that $\|S(t)\|_\gamma \leqslant M(\gamma)$, for all $t > 0$.

Proposition 2 and Theorem 1 below are used to obtain our main results[14].

Proposition 2 *Let $S : [0,\infty) \to L_0(X)$ be a C_0-semigroup. Then the following statements are equivalent:*

1. *S is locally bounded.*
2. *For each $\gamma \in \Gamma$, there exist $M(\gamma) \geqslant 1$ and $\omega(\gamma) > 0$ such that $\|S(t)\|_\gamma \leqslant M(\gamma)\mathrm{e}^{\omega(\gamma)t}$, for all $t \geqslant 0$.*

Theorem 1 *Let (X, P) be an $\mathcal{S}$-space. Suppose that $S : [0,\infty) \to L(X)$ is a C_0-semigroup whose generator is $A : D(A) \to X$. Then the following statements are true:*

(i) For $t \geqslant 0$ and $x \in X$,

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} S(s)(x)\mathrm{d}s = S(t)(x).$$

(ii) For $t \geqslant 0$ and $x \in X$, $\int_0^t S(\tau)(x)\mathrm{d}\tau \in D(A)$ and

$$A\left(\int_0^t S(\tau)(x)\mathrm{d}\tau\right) = S(t)(x) - x.$$

(iii) For $x \in D(A)$ and $t \geqslant 0$, $S(t)(x) \in D(A)$ and

$$AS(t)(x) = S(t)A(x).$$

(iv) If in addition, $S(t) \in L_0(X)$, for all $t \geqslant 0$ and S is locally bounded, then

$$\frac{\mathrm{d}}{\mathrm{d}t}S(t)(x) = S(t)A(x) = AS(t)(x)$$

and

$$\int_s^t S(\tau)A(x)\mathrm{d}\tau = S(t)(x) - S(s)(x),$$

for all $x \in D(A)$, and $s, t \geqslant 0$.

Definition 3 Let Y be an $\mathcal{S}$-space topologized by a family of seminorms $\{\|\cdot\|_\gamma : \gamma \in \Gamma\}$. Let $V \subset Y$ and $\gamma \in \Gamma$. A mapping $G : V \to V$ is called a γ-contraction if there is a constant l_γ with $0 \leqslant l_\gamma < 1$ such that for all $x, y \in V$

$$\|Gx - Gy\|_\gamma \leqslant l_\gamma \|x - y\|_\gamma.$$

We employ the fixed point theorem of a γ-contraction[15].

Theorem 2 *Suppose V is a sequentially complete subset of an $\mathcal{S}$-space Y topologized by the family of seminorms $\{\|\cdot\|_\gamma : \gamma \in \Gamma\}$, and the mapping $G : V \to V$ is a γ-contraction for every $\gamma \in \Gamma$. Then G has a unique fixed point $\overline{x}$ in V, i.e., $\overline{x} = G\overline{x}$.*

In proving a priori estimate, we recall the standard Gronwall's lemma[16].

Lemma 1 (Gronwall's lemma) *If*

$$x(t) \leqslant a(t) + \int_{t_0}^t b(s)x(s)\mathrm{d}s,\ t \in [t_0, T),$$

where x, a, b are continuous on $[t_0, T)$, $0 \leqslant t_0 < T \leqslant \infty$, and $b(t) \geqslant 0$ on $[t_0, T)$, then $x(t)$ satisfies

$$x(t) \leqslant a(t) + \int_{t_0}^t a(s)b(s)\exp\left[\int_s^t b(u)\mathrm{d}u\right]\mathrm{d}s,$$

for any $t \in [t_0, T)$. If in addition, $a(t)$ is nondecreasing on $[t_0, T)$, then

$$x(t) \leqslant a(t)\exp\left[\int_{t_0}^t b(s)\mathrm{d}s\right],\ t \in [t_0, T).$$

MAIN RESULTS

We state and prove our main results for the existence and the uniqueness of the PC-mild solution for the system (1). Finally, we give some examples to illustrate our abstract results. We state some hypotheses for our main results.

(H1) X is an $\mathcal{S}$-space topologized by the family of seminorms $\{|\cdot|_\gamma : \gamma \in \Gamma\}$ and $PC([0,T];X)$ is the space of all piecewise continuous functions from $[0,T]$ into X topologized by the family of seminorms $\{\|\cdot\|_{\gamma,PC} : \gamma \in \Gamma\}$ where

$$\|x\|_{\gamma,PC} := \max\{\sup_{t\in[0,T]} |x(t^+)|_\gamma,\ \sup_{t\in[0,T]} |x(t^-)|_\gamma\},$$

$x \in PC([0,T];X)$, $\gamma \in \Gamma$, together with the discontinuity points $\{t_i\}$, where $0 = t_0 < t_1 < t_2, \ldots, t_n < T$, and prescribed impulsive conditions $x(t_i) = x(t_i^-), i = 1, 2, \ldots, n$.

(H2) A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ on X such that $S(t) \in L_0(X)$, and $S(\cdot) : [0,\infty) \to L_0(X)$ is locally bounded.

(H3) $B : PC([0,T];X) \to PC([0,T];X)$ is an operator such that for any $\gamma \in \Gamma$ there exists $k_{\gamma,B} \in L^1_{loc}([0,T];[0,\infty))$ such that

$$|Bx(t) - By(t)|_\gamma \leqslant k_{\gamma,B}(t)|x(t) - y(t)|_\gamma,$$

for all $x, y \in PC([0,T];X)$ and all $t \in [0,T]$.

(H4) $B : PC([0,T];X) \to PC([0,T];X)$ is an operator such that for any $\gamma \in \Gamma$, and $\rho > 0$, there exists

$$k_{\gamma,B,\rho}(\cdot) \in L^1_{loc}([0,T];[0,\infty))$$

such that

$$|Bx(t) - By(t)|_\gamma \leqslant k_{\gamma,B,\rho}(t)|x(t) - y(t)|_\gamma,$$

for all $x, y \in PC([0,T];X)$ such that

$$\|x\|_{\gamma,PC} \leqslant \rho, \|y\|_{\gamma,PC} \leqslant \rho, \text{ and all } t \in [0,T].$$

(H5) $f : [0,T] \times X \times X \to X$ is continuous a.e. in t and $f(t,\cdot,\cdot)$ is measurable in $X \times X$ and there exists a function $K_f(\cdot) \in L^1_{loc}([0,T];[0,\infty))$, such that

$$|f(t,u_1,v_1) - f(t,u_2,v_2)|_\gamma \leqslant K_f(t)(|u_1-u_2|_\gamma + |v_1-v_2|_\gamma),$$

for all $t \in [0,T]$, all u_1, u_2, v_1, v_2 in X and all $\gamma \in \Gamma$.

(H6) $f : [0,T] \times X \times X \to X$ is continuous a.e. in $t \in [0,T]$ and locally Lipschitz in $(u,v) \in X \times X$, i.e., for each $\rho > 0$, there exists a constant $K_{f,\rho}(\cdot) \in L^1_{loc}([0,T];[0,\infty))$ such that

$$|f(t,u_1,v_1) - f(t,u_2,v_2)|_\gamma \leqslant K_{f,\rho}(t)(|u_1-u_2|_\gamma + |v_1-v_2|_\gamma),$$

for all u_1, u_2, v_1, v_2 in X such that

$$|u_1|_\gamma \leqslant \rho, |u_2|_\gamma \leqslant \rho, |v_1|_\gamma \leqslant \rho, |v_2|_\gamma \leqslant \rho,$$

for all $t \in [0,T]$.

(H7) $f : [0,T] \times X \times X \to X$ is continuous in $t \in [0,T]$ and satisfies the linear growth condition, i.e., for each $\gamma \in \Gamma$, there exists a constant $K_1(\gamma) > 0$ such that

$$|f(t,u,v)|_\gamma \leqslant K_1(\gamma)(1 + |u|_\gamma + |v|_\gamma),$$

for all $t \in [0,T]$, and all $u, v \in X$.

(H8) For each $i = 1,\dots,n$ the mapping $J_i : X \to X$, satisfies the following property: for each $\gamma \in \Gamma$, there exists $\eta_i(\gamma) \geqslant 0$ such that

$$|J_i(x) - J_i(y)|_\gamma \leqslant \eta_i(\gamma)|x-y|_\gamma$$

for all $x, y \in X$.

Definition 4 A function $x \in PC([0,T];X)$ satisfying the integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f_{x;B}(s)\mathrm{d}s + \sum_{0<t_i<t} S(t-t_i)J_i(x(t_i)),$$

is called a PC-mild solution of the system (1).

Indeed, suppose x is a solution of the system (1). On the interval $[t_i, t_{i+1}]$, $i = 0,1,\dots,n$ where $t_0 = 0$ and $t_{n+1} = T$, we let

$$w(s) = S(t-s)x(s).$$

Then we obtain

$$w'(s) = S(t-s)f_{x;B}(s),$$

and

$$w(t) - w(t_i^+) = \int_{t_i}^t w'(s)\mathrm{d}s = \int_{t_i}^t S(t-s)f_{x;B}(s)\mathrm{d}s.$$

Then

$$x(t) = w(t) = S(t-t_i)x(t_i) + \int_{t_i}^t S(t-s)f_{x;B}(s)\mathrm{d}s + S(t-t_i)J_i(x(t_i)),$$

where $t \in [t_i, t_{i+1}]$, $i = 0,1,\dots,n$. Hence for $t \in [0,t_1]$, we have

$$x(t) = S(t)x(0) + \int_0^t S(t-s)f_{x;B}(s)\mathrm{d}s.$$

For $t \in [0,t_2]$, we have

$$\begin{aligned} x(t) &= S(t-t_1)x(t_1) + \int_{t_1}^t S(t-s)f_{x;B}(s)\mathrm{d}s \\ &= S(t-t_1)\Big(S(t_1)x(0) + \int_0^{t_1} S(t_1-s)f_{x;B}(s)\mathrm{d}s\Big) \\ &\quad + \int_{t_1}^t S(t-s)f_{x;B}(s)\mathrm{d}s + S(t-t_1)J_1(x(t_1)) \\ &= S(t)x_0 + \int_0^t S(t-s)f_{x;B}(s)\mathrm{d}s \\ &\quad + S(t-t_1)J_1(x(t_1)). \end{aligned}$$

Similarly, for $t \in [0,T]$, we have

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f_{x;B}(s)\mathrm{d}s + \sum_{0<t_i<t} S(t-t_i)J(x(t_i))$$

where $0 < t_1 < t_2 < \dots < t_n < T$.

Our main results consist of solving existence and uniqueness problems of PC-mild solutions for the system (1), which are divided into 2 cases. The first case is when f is uniformly Lipschitz. The latter case is when f is locally Lipschitz, and we also prove a priori estimate as it is an important tool for proving the global existence of a mild solution. Finally, we show that the continuous dependence on impulsive conditions is satisfied by the system. Some examples are used to demonstrate our main results.

Case 1. Global existence with uniformly Lipschitz conditions

Theorem 3 *Assume the hypotheses (H1), (H2), (H3), (H5) and (H8) hold. Then the system* (1) *has a unique PC-mild solution on* $[0,T]$ *provided that*

$$0 \leqslant M(\gamma)\left(\int_0^T K_f(s)(1+k_{\gamma,B}(s))\mathrm{d}s + \sum_{i=1}^n \eta_i(\gamma)\right) = l_\gamma < 1.$$

Proof: Let $x_0 \in X$. Set

$$PC_{x_0}([0,T];X) = \{x \in PC([0,T];X) : x(0) = x_0\}.$$

Since X is sequentially complete, then $PC([0,T];X)$ and $PC_{x_0}([0,T];X)$ are sequentially complete. Define an operator G on $PC_{x_0}([0,T];X)$ by

$$(Gx)(t) = S(t)x_0 + \int_0^t S(t-s)f_{x;B}(s)\mathrm{d}s + \sum_{0<t_i<t} S(t-t_i)J_i(x(t_i)),$$

provided that $x \in PC_{x_0}([0,T];X)$. It is clear that $G : PC_{x_0}([0,T];X) \to PC_{x_0}([0,T];X)$. Indeed, for $0 \leqslant t' < t \leqslant t_1$, we have

$$\begin{aligned}
&|Gx(t) - Gx(t')|_\gamma \leqslant |S(t)x_0 - S(t')x_0|_\gamma \\
&\quad + \left|\int_0^t S(t-s)f_{x;B}(s)\mathrm{d}s - \int_0^{t'} S(t'-s)f_{x;B}(s)\mathrm{d}s\right|_\gamma \\
&\leqslant |S(t)x_0 - S(t')x_0|_\gamma + \int_{t'}^t |S(t-s)f_{x;B}(s)|_\gamma \mathrm{d}s \\
&\quad + \int_0^{t'} |S(t-s)f_{x;B}(s) - S(t'-s)f_{x;B}(s)|_\gamma \mathrm{d}s \\
&\leqslant |S(t)x_0 - S(t')x_0|_\gamma + \int_{t'}^t |S(t-s)f_{x;B}(s)|_\gamma \mathrm{d}s \\
&\quad + \int_0^{t'} |S(t-t')\{(S(t'-s) - I)f_{x;B}(s)\}|_\gamma \mathrm{d}s.
\end{aligned}$$

Since $\{S(t)\}_{t\geqslant 0}$ is continuous at any $t \geqslant 0$, by the above inequality, the mapping Gx is left continuous at each $t \in [0,t_1]$. For $0 \leqslant t < t' \leqslant t_1$, we have

$$\begin{aligned}
&|Gx(t') - Gx(t)|_\gamma \leqslant |S(t')x_0 - S(t)x_0|_\gamma \\
&\quad + \left|\int_0^{t'} S(t'-s)f_{x;B}(s)\mathrm{d}s - \int_0^{t} S(t-s)f_{x;B}(s)\mathrm{d}s\right|_\gamma \\
&\leqslant |S(t')x_0 - S(t)x_0|_\gamma + \int_{t}^{t'} |S(t'-s)f_{x;B}(s)|_\gamma \mathrm{d}s \\
&\quad + \int_0^{t} |S(t-s)\{(S(t'-t) - I)f_{x;B}(s)\}|_\gamma \mathrm{d}s.
\end{aligned}$$

By the continuity of $\{S(t)\}_{t\geqslant 0}$ and f, Gx is right continuous at $t \in [0,t_1]$. Therefore, Gx is continuous on $[0,t_1]$, and we can conclude that $Gx \in C([0,t_1];X)$ with $Gx(0) = x_0$. On the interval $[t_1,t_2]$, for any $t_1 \leqslant t' < t \leqslant t_2$, we have

$$\begin{aligned}
&|Gx(t) - Gx(t')|_\gamma \leqslant |S(t)x_0 - S(t')x_0|_\gamma \\
&\quad + \left|\int_0^t S(t-s)f_{x;B}(s)\mathrm{d}s - \int_0^{t'} S(t'-s)f_{x;B}(s)\mathrm{d}s\right|_\gamma \\
&\qquad + |S(t-t_1)J_1(x(t_1)) - S(t'-t_1)J_1(x(t_1))|_\gamma \\
&\quad \leqslant |S(t)x_0 - S(t')x_0|_\gamma + \int_{t'}^t |S(t-s)f_{x;B}(s)|_\gamma \mathrm{d}s \\
&\quad + \int_0^{t'} |(S(t-t') - I)\{S(t'-s)f_{x;B}(s)\}|_\gamma \mathrm{d}s \\
&\quad + |S(t-t_1)J_1(x(t_1)) - S(t'-t_1)J_1(x(t_1))|_\gamma.
\end{aligned}$$

By the continuity of the semigroup $\{S(t)\}_{t\geqslant 0}$ and f, the mapping Gx is left continuous at t. Similarly, one can show that Gx is right continuous at t. So Gx is continuous at $t \in [t_1,t_2]$. By the same manner, we also obtain that $Gx \in C([t_i,t_{i+1}];X)$, $i = 2,\dots,n-1$ and $Gx \in C([t_n,T];X)$. Therefore $Gx \in PC_{x_0}([0,T];X)$.

Next, we must show that G is a γ-contraction on $PC_{x_0}([0,T];X)$ for each $\gamma \in \Gamma$. Let $\gamma \in \Gamma$ be arbitrary. For any $x, y \in PC_{x_0}([0,T];X)$ and any $t \in [0,T]$, we have

$$\begin{aligned}
&|Gx(t) - Gy(t)|_\gamma \\
&\leqslant \int_0^t |S(t-s)\left(f_{x;B}(s) - f_{y;B}(s)\right)|_\gamma \mathrm{d}s \\
&\quad + \sum_{0<t_i<t} |S(t-t_i)\{J_i(x(t_i)) - J_i(y(t_i))\}|_\gamma \\
&\leqslant \int_0^t \|S(t-s)\|_\gamma \left|f_{x;B}(s) - f_{y;B}(s)\right|_\gamma \mathrm{d}s \\
&\quad + \sum_{0<t_i<t} \|S(t-t_i)\|_\gamma |J_i(x(t_i)) - J_i(y(t_i))|_\gamma \\
&\leqslant M(\gamma)\int_0^t K_f(s)|x(s) - y(s)|_\gamma \mathrm{d}s \\
&\quad + \int_0^t K_f(s)|Bx(s) - By(s)|_\gamma \mathrm{d}s \\
&\quad + M(\gamma)\sum_{i=1}^n \eta_i(\gamma)|x(t_i) - y(t_i)|_\gamma.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\|Gx - Gy\|_{\gamma,PC} \\
&\quad \leqslant M(\gamma)\left(\int_0^T K_f(s)(1+k_{\gamma,B}(s))\mathrm{d}s + \sum_{i=1}^n \eta_i(\gamma)\right)
\end{aligned}$$

$= l_\gamma < 1.$

Under the assumption above, by Theorem 2, G has a unique fixed point in $PC_{x_0}([0,T];X)$, which is a unique PC-mild solution of the system (1). Hence the proof is complete. □

Case 2. Local existence, a priori estimate and continuous dependence on impulsive conditions

Next, we consider proving the existence and the uniqueness of a PC-mild solution for the system (1) by assuming the local Lipschitz continuity of the function f in the system (1). A priori estimate is a powerful tool for proving the global existence of a mild solution. Then by showing that a priori estimate of any solution of the system (1) exists, we can prove the global existence of a PC-mild solution for the system (1) on the interval $[0,T]$.

Lemma 2 (A priori estimate) *Assume the hypotheses (H1), (H2), (H3), (H7) and (H8) hold. Then for any $\gamma \in \Gamma$, there exists a constant $\rho(\gamma,T) > 0$ such that if x is any PC-mild solution of the system (1) on the interval $[0,T]$ then*

$$|x(t)|_\gamma \leqslant \rho(\gamma,T),$$

for all $t \in [0,T]$.

Proof: Suppose $x(\cdot)$ is a mild solution of the system (1) on $[0,T]$. For any $\gamma \in \Gamma$, we have

$$\begin{aligned}
|x(t)|_\gamma &\leqslant |S(t)x_0|_\gamma + \int_0^t |S(t-s)f_{x;B}(s)|_\gamma \mathrm{d}s \\
&\quad + \sum_{0<t_i<t} |S(t-t_i)J_i(x(t_i))|_\gamma \\
&\leqslant M(\gamma)\left(|x_0|_\gamma + \int_0^t |f_{x;B}(s)|_\gamma \mathrm{d}s + \sum_{i=1}^n |J_i(x(t_i))|_\gamma\right) \\
&\leqslant M(\gamma)\left(\int_0^t K_1(\gamma)(1+|x(s)|_\gamma + |Bx(s)|_\gamma)\mathrm{d}s\right) \\
&\quad + M(\gamma)\left(|x_0|_\gamma + \sum_{i=1}^n \eta_i(\gamma)|x(t_i)|_\gamma\right) \\
&\leqslant M(\gamma)\left(|x_0|_\gamma + \int_0^t K_1(\gamma)\mathrm{d}s + \sum_{i=1}^n \eta_i |x(t_i)|_\gamma\right) \\
&\quad + M(\gamma)K_1(\gamma)\int_0^t |x(s)|_\gamma + k_{\gamma,B}(s)|x(s)|_\gamma \mathrm{d}s \\
&\leqslant a(t) + \int_0^t \beta(s)|x(s)|_\gamma \mathrm{d}s,
\end{aligned}$$

where

$$a(t) = M(\gamma)\left(|x_0|_\gamma + K_1(\gamma)t + \sum_{i=1}^n \eta_i(\gamma)|x(t_i)|_\gamma\right)$$

is a nondecreasing function, independent of the solution x, and

$$\beta(s) = M(\gamma)K_1(\gamma)(1 + k_{\gamma,B}(s)) \geqslant 0$$

is integrable for all $s \in [0,T]$. By using Gronwall's lemma (Lemma 1), we obtain

$$\begin{aligned}
|x(t)|_\gamma &\leqslant a(t)\exp\left(\int_0^t \beta(s)\mathrm{d}s\right) \\
&\leqslant a(T)\exp\left(\int_0^T \beta(s)\mathrm{d}s\right) = \rho(\gamma,T),
\end{aligned}$$

where $\rho(\gamma,T) \geqslant 0$ is independent of the solution x. So a priori estimate is proved for any PC-mild solution of the system (1). □

Theorem 4 (Local existence of a PC-mild solution) *Assume that (H1), (H2), (H4), (H6) and (H8) hold. Then the system (1) has a unique local PC-mild solution in $PC_{x_0}([0,T];X)$ on $[0,\delta]$, for some $0 < \delta \leqslant t_1$, such that*

$$0 \leqslant M(\gamma)\int_0^\delta K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s < 1$$

and

$$\delta M(\gamma)\Big(\int_0^\delta K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s + |f_{x;B}(0)|_\gamma\Big) \leqslant 1 - \epsilon$$

for a fixed constant $\epsilon \in (0,1)$, where t_1 is the first resetting time.

Proof: Let $x_0 \in X$. Let $\delta \in (0,t_1]$ such that

$$0 \leqslant M(\gamma)\int_0^\delta K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s < 1$$

and

$$\delta M(\gamma)\Big(\int_0^\delta K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s + |f_{x;B}(0)|_\gamma\Big) \leqslant 1 - \epsilon,$$

for a fixed constant $\epsilon \in (0,1)$, where t_1 is the first resetting time. Set

$$\begin{aligned}
\Omega(x_0) = \{&x \in PC([0,T];X) : \\
&x(0) = x_0,\ |x(t) - x_0|_\gamma \leqslant 1, t \in [0,\delta]\}.
\end{aligned}$$

Then $\Omega(x_0)$ is a nonempty sequentially complete, closed and convex subset of $PC([0,T];X)$. Define a mapping G on $\Omega(x_0)$ by

$$(Gx)(t) = S(t)x_0 + \int_0^t S(t-s)f_{x;B}(s)\mathrm{d}s.$$

We show that (i) $G : \Omega(x_0) \to \Omega(x_0)$ and (ii) G is a γ-contraction on $\Omega(x_0)$. To prove (i), let $x \in \Omega(x_0)$. Then

$$|x(t)|_\gamma \leqslant 1 + |x_0|_\gamma =: \rho, \text{ for all } t \in [0, \delta],$$

for any $\gamma \in \Gamma$. Since $\{S(t)\}_{t\geqslant 0}$ is continuous at $t = 0$, we have a fixed constant $\epsilon \in (0, 1)$ and

$$\begin{aligned}|Gx(t) - x_0|_\gamma &\leqslant |S(t)x_0 - x_0|_\gamma \\ &\quad + \int_0^t |S(t-s) f_{x;B}(s)|_\gamma \mathrm{d}s \\ &\leqslant M(\gamma) \int_0^t |f_{x;B}(s) - f_{x;B}(0)|_\gamma \mathrm{d}s \\ &\quad + \epsilon + M(\gamma) \int_0^t |f_{x;B}(0)|_\gamma \mathrm{d}s \\ &\leqslant M(\gamma) \int_0^t K_{f,\rho}(s)|x(s) - x(0)|_\gamma \mathrm{d}s \\ &\quad + M(\gamma) \int_0^t K_{f,\rho}(s)|Bx(s) - Bx(0)|_\gamma \mathrm{d}s \\ &\quad + \epsilon + M(\gamma)\delta|f_{x;B}(0)|_\gamma \\ &\leqslant \epsilon + \delta M(\gamma)\Big(\int_0^\delta K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s\Big) \\ &\quad + |f_{x;B}(0)|_\gamma\} \leqslant 1.\end{aligned}$$

Thus $Gx \in \Omega(x_0)$. To prove (ii), suppose x and $\overline{x} \in \Omega(x_0)$. We have $x(0) = \overline{x}(0) = x_0$, and for $\gamma \in \Gamma$,

$$\begin{aligned}&|Gx(t) - G\overline{x}(t)|_\gamma \\ &\leqslant \int_0^t |S(t-s)\{f_{x;B}(s) - f_{\overline{x};B}(s)\}|_\gamma \mathrm{d}s \\ &\leqslant M(\gamma) \int_0^t |f_{x;B}(s) - f_{\overline{x};B}(s)|_\gamma \mathrm{d}s \\ &\leqslant M(\gamma) \int_0^t K_{f,\rho}(s)|x(s) - \overline{x}(s)|_\gamma \mathrm{d}s \\ &\quad + \int_0^t K_{f,\rho}(s)|Bx(s) - B\overline{x}(s)|_\gamma \mathrm{d}s \\ &\leqslant M(\gamma)\int_0^t K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))|x(s) - \overline{x}(s)|_\gamma \mathrm{d}s \\ &\leqslant M(\gamma)\int_0^\delta K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s\|x - \overline{x}\|_{\gamma,\Omega(x_0)}.\end{aligned}$$

Therefore,

$$\|Gx - G\overline{x}\|_{\gamma,\Omega(x_0)} \leqslant l_\gamma \|x - \overline{x}\|_{\gamma,\Omega(x_0)},$$

where

$$0 \leqslant l_\gamma = M(\gamma) \int_0^\delta K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s < 1.$$

By using Theorem 2, we have a unique fixed point x in $\Omega(x_0)$, so this fixed point is a unique local PC-mild solution of the system (1) on the interval $[0, \delta]$. □

Theorem 5 (Global existence of a mild solution) *Assume that (H1), (H2), (H4), (H6), (H7) and (H8) hold. Then the system* (1) *has a unique global PC-mild solution on* $[0, T]$*, provided that*

$$M(\gamma)\Big(\int_{t_i}^{t_{i+1}} K_{f,\rho^*}(1 + k_{\gamma,B,\rho^*}(s))\mathrm{d}s + K_1(\gamma)\delta\Big) < 1 - \epsilon,$$

for some $\delta \in (0, t_{i+1} - t_i)$, $i = 1, \dots, n$ *and* $\epsilon \in (0, 1)$ *is a fixed constant.*

Proof: By the assumptions and Theorem 4, the system (1) has a unique local PC-mild solution, say x_1 on the interval $[0, \delta]$, for some $\delta \in (0, t_1]$, where t_1 is the first resetting time. By Lemma 2, a priori bound exists, so there is a constant $\rho^* > 0$ such that

$$|x_1(t)|_\gamma \leqslant \rho^*,$$

for all $t \in [0, \delta]$. Set

$$\begin{aligned}\Omega(x_1) = \{&y \in PC([\delta, T]; X) : \\ &y(\delta) = x_1(\delta), |y(t) - x_1(\delta)|_\gamma \leqslant 1, t \in [\delta, 2\delta]\}.\end{aligned}$$

Then $\Omega(x_1)$ is a nonempty, sequentially complete, closed and convex subset of $PC([\delta, T]; X)$. Define a mapping G on $\Omega(x_1)$ by

$$Gy(t) = \int_\delta^t S(t - \delta - s) f_{y;B}(s)\mathrm{d}s + S(t - \delta)x_1(\delta),$$

$t \in [\delta, 2\delta]$, provided that $y \in \Omega(x_1)$. By the same argument as in Theorem 4, we have the same constant $\delta > 0$ as found in Theorem 4 that δ depends only on ρ^* such that

$$\begin{cases}\frac{\mathrm{d}y}{\mathrm{d}t} = Ay(t) + f_{y;B}(t), \ t \in [\delta, t_1] \\ y(\delta) = x_1(\delta),\end{cases}$$

has a unique mild solution x_2 on $[\delta, 2\delta]$. Let

$$z(t) = \begin{cases}x_1(t), \ t \in [0, \delta], \\ x_2(t), \ t \in [\delta, 2\delta].\end{cases}$$

Then z is the unique mild solution of the system (1) on the interval $[0, 2\delta]$. By the same procedure, since δ depends only on ρ^*, z can be extended to the interval $[2\delta, 3\delta]$. We then obtain the intervals of existence of mild solutions with equal length δ, i.e.,

$[\delta, 2\delta], \dots, [n\delta, (n+1)\delta]$ so that $t_1 \in [n\delta, (n+1)\delta]$, for some n. Hence the system (1) has a unique global mild solution on $[0, t_1]$.

Now, we consider the existence problem of the system (1) on the interval $[t_1, t_2]$, where t_1, t_2 are the resetting times of the system. We use $x(t_1^+) = x(t_1) + J_1(x(t_1))$ as a prescribed condition. Then on $[t_1, t_2]$, we let $\epsilon \in (0, 1)$ be a fixed constant and set

$$\begin{aligned}\Omega(x_1(t_1^+)) = \{y \in PC([t_1, T]; X) :\\ y(t_1) = x_1(t_1^+),\\ |y(t) - x_1(t_1^+)|_\gamma \leqslant 1, t \in [t_1, t_1 + \delta]\},\end{aligned}$$

where $\delta \in (0, t_2 - t_1)$ such that

$$\begin{aligned}1 - \epsilon > M(\gamma) \int_{t_1}^{t_1+\delta} K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s\\ + M(\gamma)K_1(\gamma)\delta.\end{aligned}$$

Then $\Omega(x_1(t_1^+))$ is a nonempty sequentially complete, closed and convex subset of $PC([t_1, T]; X)$. Define a mapping G on $\Omega(x_1(t_1^+))$ by

$$\begin{aligned}Gy(t) = S(t - t_1)x_1(t_1) + S(t - t_1)J_1(x(t_1))\\ + \int_{t_1}^{t} S(t - t_1 - s)f_{x;B}(s)\mathrm{d}s.\end{aligned}$$

provided that $y \in \Omega(x_1(t_1^+))$. We show that $Gy \in \Omega(x_1(t_1^+))$. For any $\gamma \in \Gamma$, we have $Gy(t_1) = x_1(t_1) + J_1(x(t_1)) = x(t_1^+)$, and for any $t \in [t_1, t_1 + \delta]$, we have

$$\begin{aligned}|Gy(t) - x_1(t_1^+)|_\gamma &\leqslant |S(t - t_1)x_1(t_1) - x_1(t_1)|_\gamma\\ &\quad + \int_{t_1}^{t} |S(t - t_1 - s)f_{y;B}(s)|_\gamma \mathrm{d}s\\ &\leqslant \epsilon + M(\gamma)\int_{t_1}^{t} |f_{y;B}(s)|_\gamma \mathrm{d}s\\ &\leqslant \epsilon + M(\gamma)\int_{t_1}^{t} K_1(\gamma)\{1 + |y(s)|_\gamma + |By(s)|_\gamma\}\mathrm{d}s\\ &\leqslant \epsilon + M(\gamma)K_1(\gamma)\delta\\ &\quad + M(\gamma)\int_{t_1}^{t_1+\delta} K_{f,\rho}(s)(1 + k_{\gamma,B,\rho}(s))\mathrm{d}s \leqslant 1.\end{aligned}$$

Then $Gy \in \Omega(x_1(t_1^+))$. Hence

$$G : \Omega(x_1(t_1^+)) \to \Omega(x_1(t_1^+)).$$

Similarly, we can show that G is a γ-contraction on $\Omega(x_1(t_1^+))$. So G has a fixed point x_2 in $\Omega(x_1(t_1^+))$. We can extend this x_2 to a unique mild solution of the system (1) on the interval $[t_1, t_2]$. By using this procedure, we have a unique mild solution x_n of the system (1) on the interval $[t_{n-1}, t_n]$, $n = 1, 2, \dots, n+1$, where $t_0 = 0$, $t_{n+1} = T$. Define

$$x(t) = \begin{cases} x_1(t),\ t \in [0, t_1],\\ x_2(t),\ t \in [t_1, t_2],\\ \vdots\\ x_{n+1}(t),\ t \in [t_n, t_{n+1}].\end{cases}$$

Then it is easy to show that x is the unique PC-mild solution of the system (1) on the interval $[0, T]$. This completes the proof. □

Now we investigate the continuous dependence on impulsive conditions of the system (1).

CONTINUOUS DEPENDENCE OF SOLUTIONS

Theorem 6 *Assume that x and $\overline{x}$ are PC-mild solutions of the system (1) corresponding to the prescribed impulsive conditions*

$$x(t_i),\ \overline{x}(t_i),\ i = 0, 1, \dots, n,$$

respectively, where $t_0 = 0$, and $0 < t_1 < t_2 < \cdots < t_n < T$. Then for each $\gamma \in \Gamma$,

$$\begin{aligned}&|x(t) - \overline{x}(t)|_\gamma \leqslant\\ &M(\gamma)\exp\left(\int_0^t b(s)\mathrm{d}s\right)\sum_{i=0}^{n}\eta_i(\gamma)|x(t_i) - \overline{x}(t_i)|_\gamma,\end{aligned}$$

for all $t \in [0, T]$.

Proof:

$$\begin{aligned}|x(t) - \overline{x}(t)|_\gamma &\leqslant |S(t)x(0) - S(t)\overline{x}(0)|_\gamma\\ &\quad + \int_0^t \left|S(t - s)\left(f_{x;B}(s) - f_{\overline{x};B}(s)\right)\right|_\gamma \mathrm{d}s\\ &\quad + \sum_{0<t_i<t} |S(t - t_i)(J_i(x(t_i)) - J_i(\overline{x}(t_i)))|_\gamma\\ &\leqslant M(\gamma)|x(0) - \overline{x}(0)|_\gamma\\ &\quad + M(\gamma)\int_0^t |f_{x;B}(s) - f_{\overline{x};B}(s)|_\gamma \mathrm{d}s\\ &\quad + M(\gamma)\sum_{i=1}^{n}\eta_i(\gamma)|x(t_i) - \overline{x}(t_i)|_\gamma\\ &\leqslant M(\gamma)\sum_{i=1}^{n}\eta_i(\gamma)|x(t_i) - \overline{x}(t_i)|_\gamma\\ &\quad + M(\gamma)\int_0^t K_f(s)|x(s) - \overline{x}(s)|_\gamma \mathrm{d}s\\ &\quad + M(\gamma)\int_0^t K_f(s)|Bx(s) - B\overline{x}(s)|_\gamma \mathrm{d}s\end{aligned}$$

$$\leqslant M(\gamma)\sum_{i=1}^{n}\eta_i(\gamma)|x(t_i)-\overline{x}(t_i)|_\gamma + M(\gamma)\int_0^t K_f(s)(1+k_{\gamma,B}(s))|x(s)-\overline{x}(s)|_\gamma \mathrm{d}s$$

$$\leqslant M(\gamma)\sum_{i=1}^{n}\eta_i(\gamma)|x(t_i)-\overline{x}(t_i)|_\gamma + \int_0^t b(s)|x(s)-\overline{x}(s)|_\gamma \mathrm{d}s.$$

By using Lemma 1, we have

$$|x(t)-\overline{x}(t)|_\gamma \leqslant M(\gamma)\exp\left(\int_0^t b(s)\mathrm{d}s\right)\sum_{i=0}^{n}\eta_i(\gamma)|x(t_i)-\overline{x}(t_i)|_\gamma.$$

This shows the continuous dependence on impulsive conditions of the mild solution. □

We demonstrate our main results via some concrete examples.

Example 4 Let X be the space of rapidly decreasing functions on $\mathbb{R}$ with the family of seminorms P, where $P=\{\overline{p}_{k,n}: k,n=0,1,\dots\}$ defined by

$$\overline{p}_{k,n}(f)=\sup_{s\geqslant 0} p_{k,n}(\mathrm{e}^{-s}H(s)f)$$

where $f\in X$, $(H(s)f)(\xi):=f(s+\xi)$, $s\geqslant 0, \xi\in\mathbb{R}$, and

$$p_{k,n}(g):=\max_{0\leqslant j\leqslant k}\sup_{\xi\in\mathbb{R}}|\xi^j g^{(n)}(\xi)|.$$

So

$$\overline{p}_{k,n}(f):= \sup_{s\geqslant 0}\left[\max_{0\leqslant j\leqslant k}\left\{\sup_{\xi\in\mathbb{R}}\left|\xi^j\frac{\partial^n}{\partial\xi^n}\left(\mathrm{e}^{-s}f(s+\xi)\right)\right|\right\}\right],$$

provided $f\in X$. The family P induces a locally convex topology on X. Actually, the space (X,P) is a Fréchet space. Consider the following dynamical system with impulsive conditions

$$\begin{aligned}\tfrac{\partial}{\partial t}y(t,\xi)&=\tfrac{\partial^2}{\partial\xi^2}y(t,\xi)+\mathrm{e}^{-t}(1+y(t,\xi))\\&\quad+\textstyle\int_0^t h(t,s)y(s,\xi)\mathrm{d}s,\ t\in[0,1]-\{\tfrac12\}\\ \Delta y(\tfrac12,\xi)&=\eta(k,n)y(\tfrac12,\xi),\ \xi\in(-\infty,\infty)\\ y(0,\xi)&=y_0(\xi),\ y_0\in X,\end{aligned} \tag{2}$$

where $h:[0,1]^2\to\mathbb{R}$ is continuous, $\eta:\mathbb{N}_0^2\to(-1,1)$, $y_0\in X$, $\mathbb{N}_0=\{0,1,2,\dots\}$ such that

$$\int_0^1(\mathrm{e}^{-s}+h(1,s))\mathrm{d}s+|\eta(k,n)|<1.$$

The operator $A=\partial^2/\partial\xi^2$ generates a P-contraction C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ defined by $S(0)=I$, I the identity operator, and for $t>0$,

$$(S(t)y)(\xi)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\mathrm{e}^{-(\xi-\eta)^2/4t}y(\eta)\mathrm{d}\eta$$

for all $\xi\in\mathbb{R}$. Define $x:[0,1]\to X$, by $x(t)(\xi)=y(t,\xi)$, for all $\xi\in\mathbb{R}$,

$$Bx(t)(\xi)=\int_0^t h(t,s)y(s,\xi)\mathrm{d}s,$$
$$J_1(x(\cdot))(\xi)=\eta(k,n)y(\cdot,\xi),\ \xi\in\mathbb{R},\ \text{and}$$

$$f_{x;B}(t)(\xi)=\mathrm{e}^{-t}(1+y(t,\xi)) + \int_0^t h(t,s)y(s,\xi)\mathrm{d}s.$$

It is clear that f satisfies Lipschitz condition in $x\in X$ and continuous in $t\in[0,1]$. Then the problem (2) can be written as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=Ax+f_{x;B}(t), t\in[0,1]-\tfrac12,\\ \Delta x(\tfrac12)&=J_1(x(\tfrac12)), t_1=\tfrac12; x(0)=x_0\in X.\end{aligned}$$

It is clear that the system satisfies the hypotheses of Theorem 3. Hence the system has a unique mild solution on the interval $[0,1]$.

Example 5 Let H be the space of real-valued C^∞ functions on $\mathbb{R}^m$ whose partial derivatives of all orders belong to $L^2(\mathbb{R}^m)$. It is known[17,18] that H is a pre-Hilbert space with inner product

$$\langle f,g\rangle_m=\sum_{|\alpha|\leqslant m}\int_{\mathbb{R}^m}D^\alpha f(\eta)D^\alpha g(\eta)\mathrm{d}\eta$$

for all $f,g\in H$. For each m, $\langle f,f\rangle_m=\|f\|_m^2$, $\forall f\in H$, defines a norm on H. For each multi-index $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)$, a seminorm p_α is defined on H by

$$p_\alpha(f)=\|D^\alpha f\|_0=\left(\int_{\mathbb{R}^m}(D^\alpha f(\eta))^2\mathrm{d}\eta\right)^{\frac12}$$

for all $f\in H$.

The totality P of these seminorms p_α corresponding to all multi-indices α induces a metrizable locally convex topology on H. For each seminorm p_α, α is a multi-index. Consider the differential equation

$$\frac{\partial}{\partial t}y(t,\xi)=Ay(t,\xi)+\mathrm{e}^{-t}y^2(t,\xi) + \int_0^1 h(t,s)y(s,\xi)\mathrm{d}s,\ t\in[0,1]-\{\tfrac12\},$$

$$\Delta y(\tfrac{1}{2},\xi) = \eta(\alpha)y(\tfrac{1}{2},\xi),\ \xi \in \mathbb{R}^n,$$
$$y(0,\xi) = y_0(\xi),\ y_0 \in H,\ \xi \in \mathbb{R}^n,$$

where $h : [0,1]^2 \to \mathbb{R}$ is continuous, $\eta : \mathbb{N}_0^n \to (-1,1)$, $\mathbb{N}_0 = \{0,1,2,3,\dots\}$, such that

$$\int_0^1 (\mathrm{e}^{-s} + h(1,s))\mathrm{d}s + |\eta(\alpha)| < 1,$$

$y_0 \in X$, $A = \sum_{i=1}^m \partial^2/\partial\xi_i^2$ is the Laplacian operator. It is known that A generates the contraction C_0-semigroup $\{G(t) : t \geqslant 0\}$ on H defined by $G(0) = I$ and for $t > 0$,

$$(G(t)y)(\xi) = \frac{1}{(\sqrt{4\pi t})^{\frac{m}{2}}} \int_{\mathbb{R}^m} \mathrm{e}^{-|\xi-\eta|^2/4t} y(\eta)\mathrm{d}\eta.$$

Again a map $x : [0,1] \to H$ is defined by

$$x(t)(\xi) = y(t,\xi),\ \text{for all } \xi \in \mathbb{R}^m,\ \text{and}$$

$$Bx(t)(\xi) = \int_0^t h(t,s)y(s,\xi)\mathrm{d}s.$$

$J_1(x(\cdot))(\xi) = \eta(\alpha)y(\cdot,\xi)$ is defined on $[0,1]$ for all $\xi \in (-\infty,\infty)$, and

$$f_{x;B}(t)(\xi) = \mathrm{e}^{-t}y^2(t,\xi) + \int_0^t h(t,s)y(s,\xi)\mathrm{d}s.$$

which is locally Lipschitz in x, and continuous in $t \in [0,1]$, reduces the above problem in the abstract form and can be studied similarly.

CONCLUSIONS

We have shown that under suitable conditions, the impulsive system (1) has a unique PC-mild solution with values in locally convex spaces. The first case we considered is when f is uniformly Lipschitz while in the second case is when f is locally Lipschitz. A priori estimate is proved and used for proving the global existence of a PC-mild solution. The fixed point theorem of a seminorm contraction is used to guarantee the existence and the uniqueness of the solution. The continuous dependence on impulsive conditions is also proved for wellposedness of the system. Some examples are given to illustrate our results.

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REFERENCES

1. Milman VD, Myshkis AA (1960) On the stability of motion in the presence of impulses. *Siberian Math J* **1**, 233–7 (in Russian).
2. Milman VD, Myshkis AA (1963) Random impulses in linear dynamical systems. In: *Approximate Methods for Solving Differential Equations*, Publishing House of the Academy of Sciences of Ukrainian SSR, Kiev, pp 64–81 (in Russian).
3. Halanay A, Wexler D (1968) *Teoria Calitativa a Systeme cu Impulduri*. Editura Republicii Socialiste Romania, Bucharest.
4. Halanay A, Wexler D (1971) *The Qualitative Theory of Systems with Impulses* [Russian translation]. Mir, Moscow.
5. Chen F, Chen A, Wang X (2009) On the solutions for impulsive fractional functional differential equations. *Differ Equ Dyn Syst* **17**, 379–91.
6. Peng Y, Xiang X, Wei W (2007) Necessary conditions of optimality for second-order nonlinear impulsive differential equations. *Advances in Differential Equations*, Hindawi publishing corporation.
7. Wei W, Xiang X, Peng Y (2006) Nonlinear impulsive integro-differential equations of mixed type and optimal controls. *Optimization* **55**, 141–56.
8. Dieudonné J (1950) Deux exemples singuliers d'équations différentielles. *Acta Sci Math (Szeged)* **12**, 38–40.
9. Shkarin SA (1993) Peano's theorem fails for infinite-dimensional Fréchet spaces. *Funct Anal Appl*, **27**, 149–51.
10. Yorke JA (1970) A continuous differential equation in Hilbert space without existence. *Funkcial Ekvac* **13**, 19–21.
11. Agase SB (1987) Existence and stability of mild solutions of semilinear differential equations in locally convex spaces. *Yokohama Math J* **35**, 33–45.
12. Lemle LD, Wu L (2010) Uniqueness of C_0-semigroups on a general locally convex vector space and an application. *Semigroup Forum*, Springer Science+Business Media, LLC.
13. Teixeira EV (2005) Strong solutions for differential equations in abstract spaces. *J Differ Equat* **214**, 65–91.
14. Voicu F, Voicu M (2006) Semigroups of linear operators on locally convex spaces. *Math Track* **2**, 28–46.
15. Cain GL Jr, Nashed MZ (1971) Fixed points and stability for a sum of two operators in locally convex spaces. *Pac J Math* **39**, 581–92.
16. Corduneanu C (1971) *Principles of Differential and Integral Equations*. Allyn and Bacon, Boston.
17. Choe YH (1985) C_0-semigroups on a locally convex space. *J Math Anal Appl* **106**, 293–320.
18. Miyadera I (1959) Semigroups of operators in Fréchet space and applications to partial differential equations. *Tohoku Math J* **11**, 162–83.