RESEARCH ARTICLE
doi: 10.2306/scienceasia1513-1874.2011.37.152

ScienceAsia 37 (2011): **152–159**

Some families of Diophantine quadruples

Supawadee Prugsapitak[a,*], **Vichian Laohakosol**[b]

[a] Department of Mathematics and Statistics, Faculty of Science, Prince of Songkla University
[b] Department of Mathematics, Faculty of Science,
Kasetsart University and Centre of Excellence in Mathematics, CHE, Si Ayutthya Road, Bangkok 10400, Thailand

*Corresponding author, e-mail: supawadee.p@psu.ac.th

Received 23 Dec 2010
Accepted 4 Apr 2011

ABSTRACT: A Diophantine quadruple is a set of four positive integers such that the product of any two increased by one is a perfect square of an integer. Here we find general families of the Diophantine quadruple problem using solutions of a second order recurrence relation over a ring of integers in a number field and a polynomial ring over a field of characteristic zero.

KEYWORDS: generating function, linear second order recurrence relation

INTRODUCTION

The Diophantine quadruple problem (DQP) asks for four positive integers with the property that the product of any two increased by one is a square. Connections among solutions of the DQP with Fibonacci numbers were first noted by Hoggatt and Bergum[1]. This direction has been much generalized. Shannon[2] showed that the product of any two distinct elements of the set

$$\{W_n, W_{n+2r}, W_{n+4r}, 4W_{n+r}W_{n+2r}W_{n+3r}\} \quad (n, r \in \mathbb{N}) \tag{1}$$

increased by $q^{mt}\left(W_hW_k - aW_{h+k}\right)^t$ is a perfect square where W_j is the generalized Fibonacci sequence. Morgado[3] further generalized the set (1) to

$$\{W_n, W_{n+2r}, W_{n+2r+2s}, 4W_{n+r}W_{n+r+s}W_{n+2r+s}\} \tag{2}$$

$(n, r, s \in \mathbb{N})$ with the conclusion that the product of any two elements of the set (2) increased by $q^{nt}\left(W_{h'}W_{k'} - W_hW_k\right)^t$ is a perfect square, where $t = 1$, $h' = k' = \frac{h+k}{2}$ if there are only two W-factors in the product and $t = 2$, $h + k = h' + k'$ if there are four W-factors in the product. There is also a related result in Ref. 4.

Besides real numbers, Udrea[5] considered a polynomial DQP and showed that if $(U_n)_{n\geqslant 0}$ is the sequence of Chebyshev polynomials of the second kind, then the product of any two distinct elements taken from the set

$$\{U_m,\ U_{m+2r},\ U_{m+4r},\ 4U_{m+r}U_{m+2r}U_{m+3r}\}, \quad (m, r \in \mathbb{N})$$

increased by $U_a^2 \cdot U_b^2$ for suitable $a, b \in \mathbb{N}$, is a perfect square. Udrea's approach is based on identities satisfied by Chebyshev polynomials of the second kind. Morgado[6] gave a similar result for the Chebyshev polynomials of the first kind $(T_n)_{n\geqslant 0}$ which asserts that the product of any two distinct elements of the set

$$\{T_n, T_{n+2r}, T_{n+4r}, 4T_{n+r}T_{n+2r}T_{n+3r}\} \quad (n, r \in \mathbb{N})$$

increased by $[(T_h - T_k)/2]^t$ is a perfect square, where T_h, T_k $(k > h \geqslant 0)$ are suitable terms of the sequence (T_n) and $t = 1$ or 2 if the number of T-factors in the product is 2 or 4, respectively. The approach of Morgado is based on solving a second order recurrence and establishing a crucial identity involving four specific elements which forms a solution to the DQP. The above results were put into a unified perspective in Ref. 7 which is based on identities satisfied by solutions of a linear second order difference equation. These identities were derived purely in the spirit of real and complex numbers and applications to the matrix DQP were given[7].

Since the set of integers is the ring of integers of the field of rational numbers, it is natural to ask whether a solution for the DQP can be found by appropriately finding four elements among the solutions of a linear second order recurrence relation in a ring of integers over any algebraic number fields. Our objective here is to give an affirmative answer. We also find here the DQP over the ring of polynomials over a field of characteristic zero.

Throughout, let $\mathcal{F}$ be an algebraic number field or a field of rational functions over a field of characteristic 0 and let $O_{\mathcal{F}}$ denote its ring of integers, and

$\mathcal{F}[\S]$ its polynomial ring. We consider a sequence $(A_n)_{n\geqslant 0} \subset O_{\mathcal{F}}$ satisfying a second order recurrence of the form

$$A_{n+1} = pA_n - qA_{n-1}\ (n \in \mathbb{N}),\ A_0 = a, A_1 = b, \text{ where } \quad a, b, p, q \in O_{\mathcal{F}}. \tag{3}$$

THE MAIN THEOREM

Our main result is a generalization of Theorem 2 in Ref. 3. The proof here is done via a generating function approach.

Theorem 1 *Let $m, r, s \in \mathbb{N}$. Assume that the sequence $(A_n)_{n\geqslant 0} \subset O_{\mathcal{F}}$ satisfies (3). If $p^2 - 4q \neq 0$, then each product of any two distinct elements of the set*

$$\{A_m, A_{m+2r}, A_{m+2r+2s}, 4A_{m+r}A_{m+r+s}A_{m+2r+s}\} \tag{4}$$

increased by $q^{mt}(A_{h'}A_{k'} - A_hA_k)^t$, for explicitly given t, h, k, h', k', is a perfect square. In addition, $t = 1$, $h' = k' = \frac{h+k}{2}$ if there are only two A-factors in the product and $t = 2$, $h + k = h' + k'$ if there are four A-factors in the product.

Proof: Let

$$g(y) = A_0 + A_1y + A_2y^2 + \cdots + A_ny^n + \cdots$$

be the generating function of the sequence $(A_n)_{n\geqslant 0}$. Then, using (3), we get

$$\begin{aligned}&(1 - py + qy^2)g(y)\\ &= A_0 + y(A_1 - pA_0) + y^2(A_2 - pA_1 + qA_0) +\\ &\quad y^3(A_3 - pA_2 + qA_1) + \ldots\\ &= A_0 + y(A_1 - pA_0) = a + y(b - ap).\end{aligned}$$

Working formally in the ring of the formal power series we let

$$L = \frac{p + \sqrt{p^2 - 4q}}{2}, R = \frac{p - \sqrt{p^2 - 4q}}{2}$$

be the two roots of the characteristic equation of the recurrence relation (3). Both roots exist in the algebraic closure of F since $p^2 - 4q \neq 0$. We have

$$\begin{aligned}g(y) &= \frac{a + y(b - ap)}{1 - py + qy^2}\\ &= \frac{1}{\sqrt{p^2 - 4q}}\left(\frac{(b - ap) + aL}{1 - yL} + \frac{-(b - ap) - aR}{1 - yR}\right)\\ &= \frac{1}{\sqrt{p^2 - 4q}}\Bigg((b - ap)\sum_{n\geqslant 0} L^ny^n + a\sum_{n\geqslant 0} L^{n+1}y^n\\ &\qquad - (b - ap)\sum_{n\geqslant 0} R^ny^n - a\sum_{n\geqslant 0} R^{n+1}y^n\Bigg).\end{aligned}$$

Equating the coefficient of y^n and setting $c_0 = b - ap + aL$, $c_1 = b - ap + aR$, we have

$$A_n = \frac{1}{\sqrt{p^2 - 4q}}(c_0L^n - c_1R^n) \qquad (n \geqslant 0). \tag{5}$$

This formula for A_n is all we need here. An alternative approach avoiding the above formal calculation is to directly check it with (3) and use the initial values to ensure its uniqueness. We next show that the following crucial identity holds.

$$A_mA_{m+r+s} + q^m(A_rA_s - aA_{r+s}) = A_{m+r}A_{m+s}. \tag{6}$$

To prove this identity, we first note that $LR = q$. For $s \geqslant r$, we have

$$\begin{aligned}&A_mA_{m+r+s} - A_{m+r}A_{m+s}\\ &= \frac{1}{p^2 - 4q}(-c_0c_1q^mL^{r+s} - c_0c_1q^mR^{r+s}\\ &\quad + c_0c_1q^{m+r}L^{s-r} + c_0c_1q^{m+r}R^{s-r})\\ &= \frac{q^m}{p^2 - 4q}(-c_0c_1L^{r+s} - c_0c_1R^{r+s} + c_0c_1R^rL^s\\ &\quad + c_0c_1L^rR^s)\\ &= \frac{q^m}{p^2 - 4q}(-c_0c_1L^{r+s} - c_0c_1R^{r+s} +\\ &(c_0c_1R^rL^s + c_0c_1L^rR^s - c_0^2L^{r+s} - c_1^2R^{r+s})\\ &\quad + c_0^2L^{r+s} + c_1^2R^{r+s})\\ &= -\frac{q^m}{p^2 - 4q}(c_0L^r - c_1R^r)(c_0L^s\\ &\quad - c_1R^s) + \frac{q^m}{p^2 - 4q}(-c_0c_1L^{r+s} - c_0c_1R^{r+s}\\ &\quad + c_0^2L^{r+s} + c_1^2R^{r+s})\\ &= -q^mA_rA_s + \frac{q^m}{p^2 - 4q}(c_0 - c_1)(c_0L^{r+s} - c_1R^{r+s})\\ &= -q^mA_rA_s + q^maA_{r+s}.\end{aligned}$$

The proof for the case $s < r$ is similar.

We are now ready to prove our theorem. Letting $s = r$ in (6), we have

$$A_mA_{m+2r} + q^m(A_r^2 - aA_{2r}) = A_{m+r}^2. \tag{7}$$

Replacing r by $r + s$ in (7), one gets

$$A_mA_{m+2r+2s} + q^m(A_{r+s}^2 - aA_{2r+2s}) = A_{m+r+s}^2. \tag{8}$$

Let $r',\ s' \in \mathbb{N}$ be such that $r + s = r' + s'$. In (6), substituting r and s by r' and s', respectively, and then subtracting and adding it to (6), we get

$$\begin{aligned}&q^m(A_rA_s - A_{r'}A_{s'})\\ &= A_{m+r}A_{m+s} - A_{m+r'}A_{m+s'}\end{aligned} \tag{9}$$

$$2A_mA_{m+r+s}+q^m(A_rA_s+A_{r'}A_{s'}-2aA_{r+s}) \\ =A_{m+r}A_{m+s}+A_{m+r'}A_{m+s'}. \quad (10)$$

In (9), replacing r, s, r' and s' by $2r+s,\ 2r+s,\ 2r$ and $2r+2s$, respectively, we have

$$A_{m+2r}A_{m+2r+2s}+q^m(A_{2r+s}^2-A_{2r}A_{2r+2s})= \\ A_{m+2r+s}^2. \quad (11)$$

Squaring (9) and (11) and subtracting, we get

$$4A_{m+r}A_{m+s}A_{m+r'}A_{m+s'}+q^{2m}(A_rA_s-A_{r'}A_{s'})^2 \\ =(2A_mA_{m+r+s}+q^m(A_rA_s+A_{r'}A_{s'}-2aA_{r+s}))^2. \quad (12)$$

Letting $r'=0$ and replacing s and s' by $r+s$ and $2r+s$, respectively, in (12), we get

$$4A_{m+r}A_{m+r+s}A_mA_{m+2r+s}+q^{2m}(A_rA_{r+s}-aA_{2r+s})^2 \\ =(2A_mA_{m+2r+s}+q^m(A_rA_{r+s}-aA_{2r+s}))^2. \quad (13)$$

Replacing s, r' and s' by $2r+s$, $2r$ and $r+s$, respectively, in (12), we get

$$4A_{m+r}A_{m+2r+s}A_{m+2r}A_{m+r+s} \\ +q^{2m}(A_rA_{2r+s}-A_{2r}A_{r+s})^2 \\ =(2A_mA_{m+3r+s}+q^m(A_rA_{2r+s}+A_{2r}A_{r+s} \\ -2aA_{3r+s}))^2. \quad (14)$$

Replacing s, r' and s' by $2r+2s$, $r+s$ and $2r+s$, respectively, in (12), we get

$$4A_{m+r}A_{m+2r+2s}A_{m+r+s}A_{m+2r+s}+ \\ q^{2m}(A_rA_{2r+2s}-A_{r+s}A_{2r+s})^2 \\ =(2A_mA_{m+3r+2s}+ \\ q^m(A_rA_{2r+2s}+A_{r+s}A_{2r+s}-2aA_{3r+2s}))^2. \quad (15)$$

The identities (7), (8), (11), and (13)-(15) prove the theorem. □

The case $s=1;\ a,b,p,q\in\mathbb{Z}$ is due to Horadam[8] and the case $s=r;\ a,b,p,q\in\mathbb{Z}$ is due to Shannon[2].

COROLLARIES

The crucial identity (6) in Theorem 1 can be simplified when A_0 and A_1 are explicitly given.

Proposition 1 *Let $m,r,s\in\mathbb{N}$. If $(A_n)_{n\geqslant 0}$ is a sequence defined by (3) with $A_0=1$ and $A_1=p$, then*

$$A_mA_{m+r+s}+q^{m+1}A_{r-1}A_{s-1}=A_{m+r}A_{m+s}. \quad (16)$$

Proof: If $A_0=1$ and $A_1=p$, then (5) becomes

$$A_n=\frac{1}{\sqrt{p^2-4q}}(L^{n+1}-R^{n+1}).$$

Thus for $s\geqslant r$,

$$\begin{aligned} &A_rA_s-A_{r+s} \\ &=\frac{1}{p^2-4q}(L^{r+s+2}-q^{r+1}L^{s-r}-q^{s+1}R^{s-r} \\ &\quad +R^{r+s+2})-\frac{1}{\sqrt{p^2-4q}}(L^{r+s+1}-R^{r+s+1}) \\ &=\frac{q}{p^2-4q}(-q^rL^{s-r}-q^sR^{s-r}) \\ &\quad +\frac{L^{r+s+1}}{p^2-4q}(L-\sqrt{p^2-4q}) \\ &\quad +\frac{R^{r+s+1}}{p^2-4q}(R+\sqrt{p^2-4q}) \\ &=\frac{q}{p^2-4q}(-q^rL^{s-r}-q^sR^{s-r}) \\ &\quad +\frac{q}{p^2-4q}(L^{r+s}+R^{r+s}) \\ &=\frac{q}{p^2-4q}(L^r-R^r)(L^s-R^s) \\ &=qA_{r-1}A_{s-1}. \end{aligned} \quad (17)$$

The proof for the case $s<r$ is similar. Substituting (17) into (6), the desired assertion follows. □

Using the identity (16), Theorem 1 takes a slightly simplified form.

Corollary 1 *Let $m,\ r\in\mathbb{N}$. Assume $(A_n)_{n\geqslant 0}\subset O_{\mathcal{F}}$ satisfies (3) with $A_0=1$ and $A_1=p$. If $p^2-4q\neq 0$, then each product of any two distinct elements of the set*

$$\{A_m,\ A_{m+2r},\ A_{m+4r},\ 4A_{m+r}A_{m+2r}A_{m+3r}\},$$

increased by $q^cA_d^2\cdot A_e^2$, for explicitly given c, d, and $e\in\mathbb{N}$, is a perfect square.

Proof: Putting $s=r$, the identity (16) becomes

$$A_mA_{m+2r}+q^{m+1}A_{r-1}^2=A_{m+r}^2. \quad (18)$$

Replacing r by $2r$ in (18), we get

$$A_mA_{m+4r}+q^{m+1}A_{2r-1}^2=A_{m+2r}^2. \quad (19)$$

Replacing m by $m+2r$ in (18), we get

$$A_{m+2r}A_{m+4r}+q^{m+2r+1}A_{r-1}^2=A_{m+3r}^2. \quad (20)$$

From (16), we have

$$4A_mA_{m+r}A_{m+s}A_{m+r+s}+q^{2m+2}A_{r-1}^2A_{s-1}^2 \\ =(A_{m+r}A_{m+s}+A_mA_{m+r+s})^2. \quad (21)$$

Putting $s = 2r$ in (21), we get

$$4A_m A_{m+r} A_{m+2r} A_{m+3r} + q^{2m+2} A_{r-1}^2 A_{2r-1}^2 = (A_{m+r} A_{m+2r} + A_m A_{m+3r})^2. \quad (22)$$

Replacing m by $m + r$ in (22), one gets

$$4A_{m+r} A_{m+2r} A_{m+3r} A_{m+4r} + q^{2m+2r+2} A_{r-1}^2 A_{2r-1}^2 = (A_{m+2r} A_{m+3r} + A_{m+r} A_{m+4r})^2. \quad (23)$$

Replacing m by $m + r$ and letting $s = r$ in (21), we have

$$4A_{m+r} A_{m+2r} A_{m+2r} A_{m+3r} + q^{2m+2r+2} A_{r-1}^2 A_{r-1}^2 = (A_{m+2r} A_{m+2r} + A_{m+r} A_{m+3r})^2. \quad (24)$$

The identities (18)-(20) and (22)-(24) prove the corollary. □

With another set of initial values, we deduce the following proposition.

Proposition 2 *If $(A_n)_{n\geqslant 0}$ is a sequence defined by (3) with $A_0 = 1$ and $A_1 = p/2$, then*

$$A_m A_{m+r+s} + \frac{q^m}{2}(q^s A_{r-s} - A_{r+s}) = A_{m+r} A_{m+s} \quad (r \geqslant s \geqslant 0, m \geqslant 0). \quad (25)$$

Proof: Putting $A_0 = 1$ and $A_1 = p/2$ in the proof of Theorem 1, the expression (5) becomes

$$A_n = \frac{1}{2}(L^n + R^n).$$

Thus for $r \geqslant s$,

$$\begin{aligned} & A_r A_s - A_{r+s} \\ =& \frac{1}{4}(L^r + R^r)(L^s + R^s) - \frac{1}{2}\left(L^{r+s} + R^{r+s}\right) \\ =& \frac{1}{4}(q^s L^{r-s} + q^s R^{r-s}) - \frac{1}{4}(L^{r+s} + R^{r+s}) \\ =& \frac{1}{2}q^s\left(\frac{1}{2}L^{r-s} + \frac{1}{2}R^{r-s}\right) - \frac{1}{2}\left(\frac{1}{2}L^{r+s} + \frac{1}{2}R^{r+s}\right) \\ =& \frac{1}{2}(q^s A_{r-s} - A_{r+s}). \end{aligned}$$

Substituting $A_r A_s - A_{r+s}$ into the identity (6), the desired assertion follows. □

Using the identity (25), we obtain the following corollary.

Corollary 2 *Let $m, r \in \mathbb{N}$. Assume that $(A_n)_{n\geqslant 0} \subset O_{\mathcal{F}}$ satisfies (3) with $A_0 = 1$ and $A_1 = p/2$. If $p^2 - 4q \neq 0$, then each product of any two distinct elements of the set*

$$\{A_m,\ A_{m+2r},\ A_{m+4r},\ 4A_{m+r}A_{m+2r}A_{m+3r}\},$$

increased by $\left\{\frac{1}{2}(q^c A_d - q^e A_f)\right\}^t$, for explicitly given c, d, e, f, and $t \in \mathbb{N}$, is a perfect square.

Proof: Letting $s = r$ in (25), we have

$$A_m A_{m+2r} + \frac{q^m}{2}(q^r A_0 - A_{2r}) = A_{m+r}^2. \quad (26)$$

Replacing r by $2r$ in (26), one gets

$$A_m A_{m+4r} + \frac{q^m}{2}(q^{2r} A_0 - A_{4r}) = A_{m+2r}^2. \quad (27)$$

Replacing m by $m + 2r$ in (26), one gets

$$A_{m+2r} A_{m+4r} + \frac{q^{m+2r}}{2}(q^r A_0 - A_{2r}) = A_{m+3r}^2. \quad (28)$$

From (25), we have

$$4A_m A_{m+r} A_{m+s} A_{m+r+s} + \left(\frac{q^m}{2}(q^s A_{r-s} - A_{r+s})\right)^2 = (A_{m+r} A_{m+s} + A_m A_{m+r+s})^2. \quad (29)$$

Observe that the recurrence (3) with two fixed initial values uniquely determines the sequence elements A_n for all integer indices both positive and negative, i.e., for all $n \in \mathbb{Z}$. This is in agreement with defining the sequence elements of negative suffixes as

$$A_{-n} := \frac{1}{2}\left(\frac{1}{L^n} + \frac{1}{R^n}\right)$$

and so

$$A_{-n} = \frac{1}{2}\left(\frac{R^n}{q^n} + \frac{L^n}{q^n}\right) = \frac{1}{q^n} A_n. \quad (30)$$

Letting $s = 2r$ in (29) and using (30), we get

$$4A_m A_{m+r} A_{m+2r} A_{m+3r} + \left(\frac{q^m}{2}(q^r A_r - A_{3r})\right)^2 = (A_{m+r} A_{m+2r} + A_m A_{m+3r})^2. \quad (31)$$

Replacing m by $m + r$ in (31), we get

$$4A_{m+r} A_{m+2r} A_{m+3r} A_{m+4r} + \left(\frac{q^{m+r}}{2}(q^r A_r - A_{3r})\right)^2 = (A_{m+2r} A_{m+3r} + A_{m+r} A_{m+4r})^2. \quad (32)$$

Replacing m by $m+r$ and letting $s=r$ in (29), we have

$$\begin{aligned}&4A_{m+r}A_{m+2r}A_{m+2r}A_{m+3r}\\&\quad+\left(\frac{q^{m+r}}{2}(q^rA_0-A_{2r})\right)^2\\&=(A_{m+2r}A_{m+2r}+A_{m+r}A_{m+3r})^2.\end{aligned}\tag{33}$$

The result follows from identities (26)-(28) and (31)-(33). □

APPLICATIONS

A large number of known solutions are special cases of our results as we now show.

Example 1 (Fibonacci Sequence)
Let $\mathcal{F}=\mathbb{Q}$. Taking

$$A_0=a=A_1=b=p=1, q=-1,$$

the recurrence (3) becomes

$$A_{n+2}=A_{n+1}+A_n\ (n\geqslant 0).$$

The sequence (A_n) so obtained is the classical Fibonacci sequence, (F_n). Corollary 1 shows that a product of any two distinct elements of the set

$$S=\{F_m,F_{m+2r},F_{m+4r},4F_{m+r}F_{m+2r}F_{m+3r}\}\quad(m,r\in\mathbb{N})$$

increased by $F_d^2\cdot F_e^2$, for suitable positive integers d and e, is a perfect square. Indeed, the proof of Corollary 1 gives

$$\begin{aligned}F_mF_{m+2r}+(-1)^{m+1}F_{r-1}^2&=F_{m+r}^2.\\F_mF_{m+4r}+(-1)^{m+1}F_{2r-1}^2&=F_{m+2r}^2.\\F_{m+2r}F_{m+4r}+(-1)^{m+1}F_{r-1}^2&=F_{m+3r}^2.\\4F_mF_{m+r}F_{m+2r}F_{m+3r}+F_{r-1}^2F_{2r-1}^2&\\=\{F_{m+r}F_{m+2r}+F_mF_{m+3r}\}^2&.\\4F_{m+r}F_{m+2r}F_{m+3r}F_{m+4r}+F_{r-1}^2F_{2r-1}^2&\\=\{F_{m+2r}F_{m+3r}+F_{m+r}F_{m+4r}\}^2&.\\4F_{m+r}F_{m+2r}F_{m+2r}F_{m+3r}+F_{r-1}^2F_{r-1}^2&\\=\{F_{m+2r}F_{m+2r}+F_{m+r}F_{m+3r}\}^2&.\end{aligned}$$

The case where m is odd and $r=1$, which shows the set S is a solution of the DQP, is due to Hoggatt and Bergum[1]. The case of even m is due to Morgado[9].

Example 2 (Lucas sequence)
Let $\mathcal{F}=\mathbb{Q}$. Taking

$$A_0=a=2,\ A_1=b=1,\ p=1,\ q=-1$$

the recurrence (3) is the same as the one in the last example and the sequence (A_n) so obtained is the Lucas sequence (L_n). From Theorem 1 and by setting $r=s$, the product of any two distinct elements of the set

$$\{L_m,\ L_{m+2r},\ L_{m+4r},\ 4L_{m+r}L_{m+2r}L_{m+3r}\},$$

increased by $(-1)^t(L_{h'}L_{k'}-L_hL_k)^t$, for explicitly given h', k', h, k, and $t\in\mathbb{N}$, is a perfect square. In particular, if $m=2$ and $r=1$, the product of any two distinct elements of the set $S=\{3,7,8,1232\}$, increased by 25 or decreased by 5, is a perfect square.

Example 3 (Fibonacci polynomials)
Let $\mathcal{F}$=$\mathbb{Q}(x)$. Taking

$$A_0=a(x)=1,\ A_1=b(x)=p(x)=x,q(x)=-1,$$

the resulting sequence (A_n) in (3) is the sequence of Fibonacci polynomials $(F_n(x))_{n\geqslant 0}$ defined by

$$F_{n+2}(x)=xA_{n+1}(x)+F_n(x)\ (n\geqslant 0),\tag{34}$$
$$F_0(x)=1,F_1(x)=x.\tag{35}$$

Corollary 1 implies that each product of any two distinct elements of the set

$$\{F_m(x),\ F_{m+2r}(x),\ F_{m+4r}(x),\ 4F_{m+r}(x)F_{m+2r}(x)F_{m+3r}(x)\}\quad(m,r\in\mathbb{N})$$

increased by $\pm F_d^2(x)\cdot F_e^2(x)$, for suitable positive integers d and e, is a perfect square. In particular, taking $m=r=1$, we deduce that the set

$$\begin{aligned}&\{F_1(x),F_3(x),F_5(x),4F_2(x)F_3(x)F_4(x)\}\\&=\{x,2x+x^3,3x+4x^3+x^5,\\&8x+36x^3+48x^5+24x^7+4x^9\}\end{aligned}$$

solves the DQP with

$$\begin{aligned}&F_1(x)\cdot F_3(x)+1=(x^2+1)^2\\&F_1(x)\cdot F_5(x)+x^2=x^2(x+2)^2\\&F_1(x)\cdot 4F_2(x)F_3(x)F_4(x)+x^2=x^2(x+2)^2\\&F_3(x)\cdot F_5(x)+1=(1+3x^2+x^4)^2\\&F_3(x)\cdot 4F_2(x)F_3(x)F_4(x)+1\\&=(1+8x^2+8x^4+2x^6)^2\\&F_5(x)\cdot 4F_2(x)F_3(x)F_4(x)+x^2\\&=x^2(5+14x^2+10x^4+2x^6)^2.\end{aligned}$$

If we put $x=1$, then the derived set $\{1,3,8,120\}$ gives a Fibonacci quadruple.

Example 4 (Chebyshev polynomials)
Let $\mathcal{F}$=$\mathbb{Q}(x)$. Taking

$$A_0 = a(x) = 1, A_1 = b(x) = p(x) = 2x, q(x) = 1,$$

the resulting sequence (A_n) in (3) is the sequence of Chebyshev polynomials of the second kind, $(U_n(x))_{n\geqslant 0}$. Corollary 1 gives the result proved by Udrea[5]: the product of any two distinct elements of the set

$$\{U_m(x), U_{m+2r}(x), U_{m+4r}(x), \\ U_{m+r}(x)U_{m+2r}(x)U_{m+3r}(x)\} \quad (m, r \in \mathbb{N})$$

increased by $U_d^2(x) \cdot U_e^2(x)$, for suitable $d, e \in \mathbb{N}$, is a perfect square.

Taking $A_0 = a(x) = 1$, $A_1 = b(x) = x$, $p(x) = 2x$, $q(x) = 1$, the resulting sequence (A_n) in (3) is the sequence of Chebyshev polynomials of the first kind, $(T_n(x))_{n\geqslant 0}$. Corollary 2 yields a result of Morgado[6]: the product of any two distinct elements of the set

$$\{T_m(x), T_{m+2r}(x), T_{m+4r}(x), \\ T_{m+r}(x)T_{m+2r}(x)T_{m+3r}(x)\} \quad (m, r \in \mathbb{N})$$

increased by $\left\{\frac{T_h(x)-T_k(x)}{2}\right\}^t$, for suitable integers $k \geqslant h \geqslant 0$ where t is 1 or 2, is a perfect square.

Example 5 (Pell polynomials)
Let $\mathcal{F} = \mathbb{Q}(\S)$. Taking

$$A_0 = a(x) = 0, A_1 = b(x) = 1, p(x) = 2x, q(x) = -1,$$

the resulting sequence (A_n) in (3) is the sequence of Pell polynomials, $(P_n(x))$ which satisfies

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \ (n \geqslant 0).$$

By Corollary 2, the product of any two distinct elements of the set

$$\{P_m(x),\ P_{m+2r}(x),\ P_{m+4r}(x), \\ P_{m+r}(x)P_{m+2r}(x)P_{m+3r}(x)\} \quad (m, r \in \mathbb{N})$$

increased by $(-1)^c \{P_d(x)P_e(x)\}^2$, for suitable positive integers c, d, and e, is a perfect square. In particular, if $m = 2$, $r = 1$, then the set

$$\{P_2(x), P_4(x), P_6(x), 4P_3(x)P_4(x)P_5(x)\} \\ = \{2x,\ 8x^3 + 4x,\ 32x^5 + 32x^3 + 6x, \\ 16x + 288x^3 + 1536x^5 + 3072x^7 + 2048x^9\}$$

solves the DQP because

$$\begin{aligned}
&P_2(x)P_4(x) + 1 = (1 + 4x^2)^2 \\
&P_2(x)P_6(x) + 4x^2 = 16x^2(1 + 2x^2)^2; \\
&P_2(x)(4P_3(x)P_4(x)P_5(x)) + 4x^2 \\
&= 4x^2(3 + 24x^2 + 32x^4)^2 \\
&P_4(x)P_6(x) + 1 = (1 + 12x^2 + 16x^4)^2 \\
&P_4(x)(4P_3(x)P_4(x)P_5(x)) + 1 \\
&= (1 + 32x^2 + 128x^4 + 128x^6)^2 \\
&P_6(x)(4P_3(x)P_4(x)P_5(x)) + 4x^2 \\
&= 4x^2(5 + 56x^2 + 160x^4 + 128x^6)^2.
\end{aligned}$$

If we put $x = \frac{1}{2}$, the resulting set $\{1, 3, 8, 120\}$ gives a Fibonacci quadruple. If we put $x = 1$, then the product of any two distinct elements of the set $\{2, 12, 70, 6960\}$, increased appropriately by 1 or 4, is a perfect square.

Example 6 (Quadratic number fields)
Let $\mathcal{F} = \mathbb{Q}(\sqrt{D})$ be a quadratic number field with $d \in \mathbb{Z}$, D square-free. Taking

$$A_0 = a = A_1 = b = p = 1, q = -\sqrt{D}$$

in (3), we have

$$A_{n+2} = A_{n+1} + \sqrt{D}A_n \ (n \geqslant 0).$$

By Corollary 1, the product of any two distinct elements of the set

$$\{A_m,\ A_{m+2r},\ A_{m+4r}, A_{m+r}A_{m+2r}A_{m+3r}\} \\ (m, r \in \mathbb{N})$$

increased by $(-\sqrt{D})^c A_d^2 A_f^2$, for suitable positive integers c, d, and f, is a perfect square. In particular, if $m = r = 1$, then the set

$$\{A_1,\ A_3,\ A_5,\ 4A_2A_3A_4\}$$

solves the original DQP.

The reader may explore the Diophantine quadruple problem over quadratic number fields in Refs. 10, 11 where the author studies the existence of Diophantine quadruples in $\mathbb{Z}[(1 + \sqrt{d})/2]$ and $\mathbb{Z}[\sqrt{4k + 3}]$.

Example 7 (Cyclotomic field)
Let $x = \zeta_m$ be a primitive mth root of unity and let $\mathcal{F} = \mathbb{Q}(\zeta_m)$. Putting

$$A_0 = a = 1, A_1 = b = p = \zeta_m, q = -1$$

in (3), the derivations follow formally as in Example 3 and we deduce that the product of any two distinct elements of the set

$$\{1, 1+2\zeta_m, 1+4\zeta_m+3\zeta_m^2,\\ 4+24\zeta_m+48\zeta_m^2+36\zeta_m^3+8\zeta_m^4\}$$

increased by 1 or ζ_m^2, is a perfect square of an element in the ring of integers $O_F = \mathbb{Z}[\zeta_m]$. If $m = 2$, then we obtain a well known DQP set $\{1, 3, 8, 120\}$ and a set $\{1, -1, 0, 0\}$ in which a product of two distinct elements increased by 1 is a square.

As a final application of our approach, we derive some analogues of Catalan's identities and generalize some identities for Chebyshev polynomials.

Example 8 (Catalan's identity)
For a sequence $(A_n)_{n\geqslant 0}$ defined by (3) with $A_0 = 1$, $A_1 = p$, replacing m by $m-1$ in Proposition 1 we have

$$A_{m-1}A_{m+r+s-1}+q^m A_{r-1}A_{s-1} = A_{m+r-1}A_{m+s-1}. \tag{36}$$

Taking $p = 2x$, $q = 1$, we get an analogue of the generalized Catalan's identity for Chebyshev polynomials of the second kind, namely,

$$U_{m-1}(x)U_{m+r+s-1}(x)+U_{r-1}(x)U_{s-1}(x) \tag{37}$$
$$= U_{m+r-1}(x)U_{m+s-1}(x). \tag{38}$$

Taking $p = 1$, $q = -1$, we get an analogue of the generalized Catalan's identity for the shifted Fibonacci sequence, namely,

$$A_{m-1}A_{m+r+s-1}+(-1)^m A_{r-1}A_{s-1}\\ = A_{m+r-1}A_{m+s-1}.$$

Since $A_k = F_{k+1}$, we have

$$F_m F_{m+r+s}+(-1)^m F_r F_s = F_{m+r}F_{m+s}, \tag{39}$$

which is a generalization of the Catalan's identity due to Everman, Danese, and Venkannayah [12].

To get more Catalan type identities for Chebyshev polynomials of the second kind and for the Fibonacci sequence, putting $m = n-r$ and $r-s$ in (37) and $m = n-r$ and $r = s$ in (36), we get, respectively,

$$U_{n-r-1}(x)U_{n+r-1}(x)+U_{r-1}^2(x) = U_{n-1}^2(x),\\ F_{n-r}F_{n+r}+(-1)^{n-r}F_r^2 = F_n^2.$$

Similarly, for a sequence $(A_n)_{n\geqslant 0}$ defined by (3) with $A_0 = 1$, $A_1 = p/2$, replacing m by $m-1$ in Proposition 2, we get

$$A_{m-1}A_{m+r+s-1}+\frac{q^{m-1}}{2}(q^s A_{r-s}-A_{r+s})\\ = A_{m+r-1}A_{m+s-1} \qquad (r\geqslant s\geqslant 0, m\geqslant 0).$$

Taking $p = 2x$, $q = 1$, we get an analogue of a generalized Catalan's identity for Chebyshev polynomials of the first kind, namely,

$$T_{m-1}(x)T_{m+r+s-1}(x)+\frac{1}{2}\{T_{r-s}(x)-T_{r+s}(x)\}\\ = T_{m+r-1}(x)T_{m+s-1}(x).$$

Since

$$(x^2-1)U_n^2(x) = \frac{1}{2}\{T_{2n+2}(x)-1\},$$

for positive integers $r > s$ of the same parity, we have

$$T_{m-1}(x)T_{m+r+s-1}(x)+\\ (x^2-1)\left\{U^2_{\frac{r-s-2}{2}}(x)-U^2_{\frac{r+s-2}{2}}(x)\right\}\\ = T_{m+r-1}(x)T_{m+s-1}(x). \tag{40}$$

Replacing m by $m-r+1$ and letting $r = s$ in (40) and noting that $U_{-1}(x) = 0$, we have

$$T_{m-r}(x)T_{m+r}(x)+(1-x^2)U_{r-1}^2(x) = T_m^2(x),$$

which is an identity due to Udrea [7].

FURTHER RESEARCH

In our work, for $\mathcal{F}$ an algebraic number field or a rational function field, we have constructed a finite set $S \subset O_{\mathcal{F}}$ in which products of two distinct elements added by some elements in $O_{\mathcal{F}}$ are square. For convenience, we define the following term: for any finite set S, the set S has the property $D(a_1, a_2, \dots, a_l)$ if for any $s, t \in S$ where $s \neq t$, $st + a$ is a perfect square for some $a \in \{a_1, a_2, \dots, a_l\}$. From Example 2 and Example 5, the set $\{3, 7, 8, 1232\}$ has the property $D(-5, 25)$ and the set $\{2, 12, 70, 6960\}$ has the property $D(1, 4)$. An interesting remark about the property $D(a_1, a_2, \dots, a_l)$ is that if $\{s_1, s_2, \dots, s_n\} \subseteq O_{\mathcal{F}}$ has the property $D(a_1, a_2, \dots, a_l)$, then for any $m \in O_{\mathcal{F}}$ the set

$$\{ms_1, ms_2, \dots, ms_n\}$$

has the property $D(m^2a_1, m^2a_2, \dots, m^2a_l)$.

It has been mentioned in Ref. 13 that there is no set of four natural numbers with the property $D(n)$ if n is an integer of the form $4k+2$, $k \in \mathbb{Z}$. However, this is not true if we consider this problem over rational or other algebraic number fields. For example, the set $\{\frac{1}{4}, 28, \frac{137}{4}, \frac{497}{4}\}$ is $D(2)$ over $\mathbb{Q}$ which is constructed from the set $\{1, 112, 137, 497\}$, a $D(32)$ set. The set $\{\sqrt{2}, 3\sqrt{2}, 8\sqrt{2}, 120\sqrt{2}\}$ is a $D(2)$ set over $\mathbb{Q}(\sqrt{2})$ which is constructed from $\{1, 3, 8, 120\}$, a $D(1)$ quadruple set. However, these two examples are not considered to be new because they can be constructed from the known solutions. On the other hand, if we consider only the integer case, then we know that $\{1, 2, 7, 17\}$ has the property $\{2, -1\}$. So one may ask if it is true that for any integer k there are an integer n and a set of four integers with the property $D(4k + 2, n)$. This question still remains open.

Acknowledgements: This study was supported by the Faculty of Science, Prince of Songkla University, the Commission on Higher Education, the Thailand Research Fund (RTA5180005) and the Centre of Excellence in Mathematics, Thailand.

REFERENCES

1. Hoggatt Jr VE, Bergum GE (1977) A problem of Fermat and the Fibonacci sequence. *Fibonacci Quart* **15**, 323–30.
2. Shannon AG (1988) Fibonacci numbers and Diophantine quadruples: generalizations of results of Morgado and Horadam. *Port Math* **45**, 165–9.
3. Morgado J (1991) Note on a Shannon's theorem concerning the Fibonacci numbers and diophantine quadruples. *Port Math* **48**, 429–39.
4. Dujella A (1995) Diophantine quadruples for squares of Fibonacci and Lucas numbers. *Port Math* **52**, 305–18.
5. Udrea G (1995) A problem of Diophantos-Fermat and Chebyshev polynomials of the second kind. *Port Math* **52**, 301–4.
6. Morgado J (1995) Note on the Chebyshev polynomials and applications to the Fibonacci numbers. *Port Math* **52**, 363–78.
7. Udrea G (1996) A note on the sequence $(W_n)_{n\geqslant 0}$ of A. F. Horadam. *Port Math* **53**, 143–55.
8. Horadam AF (1987) Generalization of a result of Morgado. *Port Math* **44**, 131–6.
9. Morgado J (1983–4) Generalization of a result of Hoggatt and Bergum on Fibonacci numbers. *Port Math* **42**, 441–5.
10. Franušić Z (2008) Diophantine quadruples in $\mathbb{Z}[\sqrt{4k+3}]$. *Ramanujan J* **17**, 77–88.
11. Franušić Z (2009) Diophantine problem in $\mathbb{Z}[(1 + \sqrt{d})/2]$. *Stud Sci Math Hung* **46**, 103–12.
12. Everman D, Danese AE, Venkannayah K (1960) Problem E1396. *Amer Math Mon* **67**, 81–2.
13. Brown E (1985) Sets in which $xy + k$ is always a square. *Math Comput* **45**, 613–20.