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Halpern iteration of Cesàro means for asymptotically nonexpansive mappings

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ABSTRACT: Using a new proof technique which is independent of the approximation fixed point of T ($\lim_{n\to\infty}\|x_n - Tx_n\| = 0$) and the convergence of the Browder type iteration path ($z_t = tu + (1-t)Tz_t$), the strong convergence of the Halpern iteration $\{x_n\}$ of Cesàro means for asymptotically nonexpansive self-mappings T, defined by $x_{n+1} = \alpha_n u + (1-\alpha_n)(n+1)^{-1}\sum_{j=0}^{n} T^j x_n$ for $n \geqslant 0$, is proved in a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm whenever $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying the conditions $\lim_{n\to\infty} b_n/\alpha_n = 0$ and $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

KEYWORDS: strong convergence, uniformly Gâteaux differentiable norm, uniformly convex

INTRODUCTION

Throughout this paper, a Banach space E will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its dual space by E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x^*\rangle$. The normalized duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\},$$

for all $x \in E$. Let $F(T)$ denote the set of all fixed point for a mapping T, that is $F(T) = \{x \in E : Tx = x\}$, and let $\mathbb{N}$ denote the set of all positive integers.

Let K be a non-empty closed convex subset of a Banach space E. A mapping $T : K \to K$ is said to be *asymptotically non-expansive* if for each $n \geqslant 1$, there exists a non-negative real number k_n satisfying $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leqslant k_n\|x - y\|,\ \forall x, y \in K.$$

When $k_n \equiv 1$, T is called *non-expansive*.

The concept of asymptotically non-expansive mapping which is a natural generalization of the important class of non-expansive mappings was introduced by Goebel et al[1] where the first existent theorem of fixed points was obtained: if K is a nonempty closed convex and bounded subset of a uniformly convex Banach space, then every asymptotically non-expansive self-mapping of K has a fixed point. Kirk et al[2] improved the above result: if a reflexive Banach space E has the property that each of its closed bounded convex sets has the fixed point property for non-expansive mappings (we call this the FPP), then it will also have the fixed point property for any asymptotically non-expansive mapping which has a non-expansive iterate.

Baillon[3] proved the first nonlinear ergodic theorem: suppose that K is a nonempty closed convex subset of Hilbert space E and $T : K \to K$ is a non-expansive mapping such that $F(T) \neq \emptyset$. Then $\forall x \in K$, the Cesàro means

$$T_n x = \frac{1}{n+1}\sum_{i=0}^{n} T^i x \tag{1}$$

weakly converge to a fixed point of T.

Bruck[4,5] studied the property of Cesàro means for non-expansive mapping in a uniformly convex Banach space. Hirano and Takahashi[6] extended Baillon's theorem to asymptotically non-expansive mappings. Several authors have studied methods for the iterative approximation of Cesàro means of (asymptotically) non-expansive mappings. For example, it was studied in Ref. 7 in a Hilbert space, in Refs. 8,9 in a uniformly convex Banach spaces with a uniformly Gâteaux differentiable norm, and in Ref. 10 for a Lipschitz pseudo-contractive mapping.

Halpern[11] ($u = 0$) was the first who introduced the following iteration scheme for a non-expansive mapping T which was referred to as *Halpern iteration*: for $u, x_0 \in K$, $\alpha_n \in [0,1]$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geqslant 0. \tag{2}$$

Subsequently, considerable research efforts, within the past 40 years or so, have been devoted to studying strong convergence of this scheme for approximating fixed points of T with various types of additional conditions. Its strong convergence was obtained by Lions[12] in the condition $\alpha_n = \frac{1}{n^a}(a \in (0,1))$; by Wittmann[13] under the conditions (C1) $\lim_{n\to\infty} \alpha_n = 0$, (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$; by Reich[14–16] in a Hilbert space; by Shioji-Takahashi[17] in a uniformly convex Banach spaces with a uniformly Gâteaux differentiable norm; by Song[18,19] for a non-expansive mapping sequence $\{T_n\}$; by Song-Xu[20] for a non-expansive mapping semigroup. Also see Song-Chen[21–23].

In a uniformly convex and uniformly smooth Banach space, Xu[24] obtained the strong convergence of the Halpern iteration $\{x_n\}$ of Cesàro means for a non-expansive mapping T:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n. \tag{3}$$

Subsequently, many mathematics workers studied the strong convergence of this scheme. For example, it has been investigated carefully by Matsushita and Kuroiwa[25] for non-expansive nonself-mappings in a Hilbert space, by Song-Chen[26] for a non-expansive mapping in a uniformly convex Banach space with a weakly continuous duality mapping, and by Song[27] for an asymptotically non-expansive self-mapping T in a uniformly convex Banach space with with a weakly continuous duality mapping J_φ.

On carefully reading the above results about Halpern iteration, a common ground is found. That is, their proofs all depend upon the approximation fixed point of T ($\lim_{n\to\infty} \|x_n - Tx_n\| = 0$) and the convergence of the Browder type iteration path ($z_t = tu + (1-t)Tz_t$)[28].

In this paper, we will employ a new proof technique which is independent of the approximation fixed point of T and the convergence of the Browder type iteration path to prove the strong convergence of $\{x_n\}$ defined by (3) for an asymptotically non-expansive self-mapping T defined on a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm whenever $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying the conditions: (i) $\lim_{nto\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$; (iii) $\lim_{n\to\infty} b_n/\alpha_n = 0$, where $b_n = \frac{1}{n+1}\sum_{j=0}^{n}(k_j - 1)$.

PRELIMINARIES AND BASIC RESULTS

Let $S(E) := \{x \in E; \|x\| = 1\}$ denote the unit sphere of a Banach space E. E is said to have: (i) a*uniformly Gâteaux differentiable norm*, if for each y in $S(E)$, the limit $\lim_{t\to 0}(\|x+ty\| - \|x\|)/t$ is uniformly attained for $x \in S(E)$; (ii) a *uniformly Fréchet differentiable norm* (we also say that E is *uniformly smooth*) if the above limit is attained uniformly for $(x,y) \in S(E) \times S(E)$. The modulus of convexity of E is defined by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2};\ \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x-y\| \geqslant \varepsilon\}$$

for each $\varepsilon \in (0,2]$. A Banach space E is said to be *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. If E is uniformly convex, then

$$\left\|\frac{x+y}{2}\right\| \leqslant r\,[1 - \delta_E(\varepsilon/r)] \tag{4}$$

for every $x, y \in E$ with $\|x\| \leqslant r$, $\|y\| \leqslant r$, and $\|x-y\| \geqslant \varepsilon > 0$. For more details on the geometry of Banach spaces see Refs. 29, 30.

Lemma 1 *(Theorem 3 of Ref. 8) Let C be a closed, convex subset of a uniformly convex Banach space. Let T be an asymptotically non-expansive mapping from C into itself such that $F(T)$ is non-empty. Then for each $r > 0$, there holds*

$$\limsup_{n\to\infty}\limsup_{m\to\infty}\sup_{x\in C\cap B_r} \left\|\frac{1}{m+1}\sum_{j=0}^{m} T^j x - T^n(\frac{1}{m+1}\sum_{j=0}^{m} T^j x)\right\| = 0, \tag{5}$$

where $B_r = \{x \in E; \|x\| \leqslant r\}$.

Lemma 2 was proved and used by several authors. For details of proofs, see Refs. 24, 31, 32. Furthermore, a variant of Lemma 2 has already been used by Reich in Theorem 1 of Ref. 33.

Lemma 2 *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the property*

$$a_{n+1} \leqslant (1-t_n)a_n + t_n c_n, \quad \forall\, n \geqslant 0,$$

where $\{t_n\}$ and $\{c_n\}$ satisfy the restrictions $\sum_{n=0}^{\infty} t_n = \infty$ and $\limsup_{n\to\infty} c_n \leqslant 0$. Then $\{a_n\}$ converges to zero as $n \to \infty$.

MAIN RESULTS

With the help of the geometric properties of a uniformly convex Banach space, we can obtain the following lemma which extends Lemma 4 of Ref. 7 and Lemma 1 of Ref. 25 from a Hilbert space to a uniformly convex Banach space, and simplifies the proof of Proposition 2.4 of Ref. 27 and is different to the proof of Lemma 3.1 of Ref. 34.

Lemma 3 *Let K be a non-empty closed convex subset of a uniformly convex Banach space E. Suppose that $T : K \to K$ is an asymptotically non-expansive mapping with $k_n \in [1, +\infty)$. Suppose that for the bounded sequence $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ satisfying the condition*

$$\lim_{k\to\infty} \left\| x_{n_k+1} - \frac{1}{n_k+1} \sum_{j=0}^{n_k} T^j x_{n_k} \right\| = 0. \tag{6}$$

Let $h(z) = \limsup_{k\to\infty} \|x_{n_k+1} - z\|$, $\forall z \in K$. Then there exists a unique $x \in K$ such that

$$h(x) = \inf_{z\in K} h(z) \quad \text{and} \quad x = Tx.$$

Proof: (i) First we show the existence and uniqueness of x (also see Ref. 35). Indeed, $h(z)$ is clearly continuous and convex and $\lim_{\|z\|\to\infty} h(z) = +\infty$. There exists x such that $h(x) = \inf_{z\in K} h(z)$ by the uniformly convexity of E (Theorem 1.3.11 of Ref. 29). Suppose there exists $y \in K$ also satisfying

$$h(x) = h(y) = \inf_{z\in K} h(z).$$

If $h(x) = \limsup_{k\to\infty} \|x_{n_k+1} - x\| = 0$, then

$$\|x - y\| \leqslant \limsup_{k\to\infty} \|x - x_{n_k+1}\| + \limsup_{k\to\infty} \|x_{n_k+1} - y\| = 0,$$

and so $x = y$.

When $r = h(x) > 0$ suppose $x \neq y$. There exists $\varepsilon \in (0, 2]$ such that $\|x - y\| \geqslant \varepsilon > 0$. We may choose a positive number a such that

$$(r + a)\left[1 - \delta_E\left(\frac{\varepsilon}{2r}\right)\right] < r,$$

i.e.,

$$0 < a < \frac{r\delta_E(\frac{\varepsilon}{2r})}{1 - \delta_E(\frac{\varepsilon}{2r})},$$

where $\delta_E(\cdot)$ is the modulus of convexity of the norm. Take

$$c = \min\left\{r, \frac{r\delta_E(\frac{\varepsilon}{2r})}{1 - \delta_E(\frac{\varepsilon}{2r})}\right\}$$

and $a \in (0, c)$. Then we have

$$(r + a)\left[1 - \delta_E\left(\frac{\varepsilon}{r + a}\right)\right] < (r + a)\left[1 - \delta_E\left(\frac{\varepsilon}{2r}\right)\right] < r. \tag{7}$$

By the definition of the function h, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$\sup_{k\geqslant N_1} \|x_{n_k+1} - x\| \leqslant r + a$$

and

$$\sup_{k\geqslant N_2} \|x_{n_k+1} - y\| \leqslant r + a.$$

Take $N = \max\{N_1, N_2\}$. Then we have

$$\sup_{k\geqslant N} \|x_{n_k+1} - x\| \leqslant r + a$$

and

$$\sup_{k\geqslant N} \|x_{n_k+1} - y\| \leqslant r + a.$$

Hence, it follows from the uniform convexity of E that for all $k \geqslant N$,

$$\left\| x_{n_k+1} - \frac{x + y}{2} \right\| = \left\| \frac{(x_{n_k+1} - x) + (x_{n_k+1} - y)}{2} \right\| \leqslant (r + a)(1 - \delta_E(\frac{\varepsilon}{r + a})) < r.$$

This implies that

$$h\left(\frac{x + y}{2}\right) = \limsup_{k\to\infty} \left\| x_{n_k+1} - \frac{x + y}{2} \right\| \leqslant (r + a)\left[1 - \delta_E\left(\frac{\varepsilon}{r + a}\right)\right] < r = h(x),$$

which is a contradiction to $h(x) = \inf_{z\in K} h(z)$. Hence $x = y$.

Next we show that $x = Tx$. Let $T_n = \frac{1}{n+1} \sum_{j=0}^{n} T^j$. Since

$$\begin{aligned} \|x_{n_k+1} - T^l x\| &\leqslant \|x_{n_k+1} - T_{n_k} x_{n_k}\| \\ &\quad + \|T_{n_k} x_{n_k} - T^l(T_{n_k} x_{n_k})\| \\ &\quad + \|T^l(T_{n_k} x_{n_k}) - T^l x_{n_k+1}\| + \|T^l x_{n_k+1} - T^l x\| \\ &\leqslant (1 + k_l)\|x_{n_k+1} - T_{n_k} x_{n_k}\| \\ &\quad + \|T_{n_k} x_{n_k} - T^l(T_{n_k} x_{n_k})\| + k_l\|x_{n_k+1} - x\| \\ &\leqslant (1 + k_l)\|x_{n_k+1} - T_{n_k} x_{n_k}\| \\ &\quad + \sup_{x\in K\cap B_r} \|T_{n_k} x - T^l(T_{n_k} x)\| + k_l\|x_{n_k+1} - x\|, \end{aligned}$$

using (6) and Lemma 1 along with the fact that $\lim_{l\to\infty} k_l = 1$, we have

$$\limsup_{l\to\infty}\limsup_{k\to\infty}\|x_{n_k+1}-T^l x\| \leqslant \limsup_{k\to\infty}\|x_{n_k+1}-x\|.$$

Hence

$$0 \leqslant \limsup_{l\to\infty} h(T^l x) \leqslant h(x). \tag{8}$$

We claim that $\lim_{l\to\infty} T^l x = x$. If $h(x) = 0$, then by (8) and the continuity of the function h, we have

$$\lim_{l\to\infty} h(T^l x) = h(\lim_{l\to\infty} T^l x) = h(x),$$

and hence it is done by the uniqueness of x.

We may assume that $r = h(x) > 0$ below. Suppose $\lim_{l\to\infty} T^l x \neq x$. There exists $\varepsilon > 0$, $\forall N_1$, $\exists l_1 > N_1$ such that $\|T^{l_1}x - x\| > \varepsilon$. Without loss of generality, let $\varepsilon \in (0, 2]$. Then we can choose a positive number a satisfying (7). It follows from (8) that for a, there is $N_0 \in \mathbb{N}$ such that

$$\sup_{l'\geqslant N_0} h(T^{l'}x) \leqslant h(x) + \frac{a}{2} = r + \frac{a}{2}.$$

Furthermore, for N_0, there exists $l > N_0$ such that $\|T^l x - x\| \geqslant \varepsilon$. Thus by the definition of $\limsup$, there exists $N \in \mathbb{N}$ such that

$$\sup_{k\geqslant N}\|x_{n_k+1} - T^l x\| \leqslant h(T^l x) + \frac{a}{2} \leqslant r + a$$

and

$$\sup_{k\geqslant N}\|x_{n_k+1} - x\| \leqslant r + a.$$

Hence, it follows from the uniform convexity of E that

$$\left\|x_{n_k+1} - \frac{T^l x + x}{2}\right\| \leqslant (r+a)\left[1 - \delta_E\left(\frac{\varepsilon}{r+a}\right)\right] < r$$

for all $k \geqslant N$. This means that

$$\begin{aligned} h\left(\frac{T^l x + x}{2}\right) &= \limsup_{k\to\infty}\left\|x_{n_k+1} - \frac{x + T^l x}{2}\right\| \\ &\leqslant (r+a)\left[1 - \delta_E\left(\frac{\varepsilon}{r+a}\right)\right] \\ &< r = h(x) \end{aligned}$$

is a contradiction, and hence

$$\lim_{l\to\infty} T^l x = x.$$

As a consequence,

$$\begin{aligned} \|x - Tx\| &\leqslant \|x - T^{l+1}x\| + \|T^{l+1}x - Tx\| \\ &\leqslant \|x - T^{l+1}x\| + k_1\|T^l x - x\|. \end{aligned}$$

Then $\|x - Tx\| = 0$, and so $x = Tx$. This completes the proof. □

Theorem 1 *Let K be a nonempty closed convex subset of a uniformly convex Banach space E with a uniformly Gâteaux differentiable norm. Suppose that $T : K \to K$ is an asymptotically non-expansive mapping with k_n. Let $\{x_n\}$ be defined by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n. \tag{9}$$

It is assumed that $\alpha_n \in (0,1)$ satisfies (i) $\lim_{n\to\infty}\alpha_n = 0$, (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$, (iii) $\lim_{n\to\infty}\dfrac{b_n}{\alpha_n} = 0$, where $b_n = \frac{1}{n+1}\sum_{j=0}^{n}(k_j - 1)$. Then as $n\to\infty$, $\{x_n\}$ converges strongly to some fixed point x^* of T.

Proof: Take $p \in F(T)$. Since $\lim_{n\to\infty} b_n/\alpha_n = 0$, there exists $N \in \mathbb{N}$, for all $n \geqslant N$, $\frac{b_n}{\alpha_n} \leqslant \frac{1}{2}$. Choose a constant $M > 0$ sufficiently large such that

$$\|x_N - p\| \leqslant M \text{ and } \|u - p\| \leqslant \frac{M}{2}.$$

We proceed by induction to show that $\|x_n - p\| \leqslant M$, $\forall n \geqslant 1$. Assume that $\|x_n - p\| \leqslant M$ for some $n > 1$. We show that $\|x_{n+1} - p\| \leqslant M$. From the iteration process (9), we estimate as follows:

$$\begin{aligned} \|x_{n+1}-p\| &\leqslant (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|T^j x_n - p\| + \alpha_n\|u-p\| \\ &\leqslant \alpha_n\|u-p\| + (1-\alpha_n)b_n\|x_n - p\| \\ &\quad + (1-\alpha_n)\|x_n - p\| \\ &\leqslant \frac{M}{2}\alpha_n + \frac{\alpha_n}{2}M + (1-\alpha_n)M = M. \end{aligned}$$

This proves the boundedness of the sequence $\{x_n\}$. Let $T_n = \frac{1}{n+1}\sum_{j=0}^{n} T^j$. Then we also obtain the boundedness of $\{T_n x_n\}$ since $\|T_n x_n - p\| \leqslant (1 + b_n)\|x_n - p\|$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1} - T_n x_n\| = \lim_{n\to\infty}\alpha_n\|u - T_n x_n\| = 0. \tag{10}$$

Let $h(z) = \limsup_{n\to\infty}\|x_{n+1} - z\|$, $\forall z \in K$. Then it follows from Lemma 3 that there exists a unique $x^* \in K$ such that

$$h(x^*) = \inf_{z\in K} h(z) \text{ and } x^* = Tx^*.$$

We claim that

$$\limsup_{n\to\infty}\langle u - x^*, J(x_{n+1} - x^*)\rangle \leqslant 0. \tag{11}$$

In fact, we can take a subsequence $\{x_{n_k+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty}\langle u-x^*, J(x_{n+1}-x^*)\rangle = \lim_{k\to\infty}\langle u-x^*, J(x_{n_k+1}-x^*)\rangle = c. \quad (12)$$

Let $f(z) = \limsup_{k\to\infty}\|x_{n_k+1}-z\|$, $\forall z\in K$. Then using Lemma 3, there exists a unique $x\in K$ such that

$$f(x) = \inf_{z\in K} f(z) \text{ and } x = Tx.$$

Now we show $x^* = x$. In fact, for $p\in F(T)$, we have

$$\begin{aligned}\|x_{n+1}-p\| &\leqslant \|x_{n+1}-T_nx_n\| + \|T_nx_n-p\| \\ &\leqslant \|x_{n+1}-T_nx_n\| + (1+b_n)\|x_n-p\|.\end{aligned}$$

Following (10), for any $\{n_k\}\subset\{n\}$, we have

$$\limsup_{k\to\infty}\|x_{n_k+1}-p\| \leqslant \limsup_{i\to\infty}\|x_{n_k}-p\|. \quad (13)$$

We may choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$h(p) = \limsup_{n\to\infty}\|x_{n+1}-p\| = \lim_{j\to\infty}\|x_{n_j+1}-p\|.$$

When $n_j > n_k$, following (13), we have

$$\begin{aligned}h(p) &= \limsup_{j\to\infty}\|x_{n_j+1}-p\| \leqslant \limsup_{j\to\infty}\|x_{n_j}-p\| \\ &\leqslant \cdots \leqslant \limsup_{k\to\infty}\|x_{n_k+2}-p\| \\ &\leqslant \limsup_{k\to\infty}\|x_{n_k+1}-p\| = f(p).\end{aligned}$$

Clearly,

$$\begin{aligned}f(p) &= \limsup_{k\to\infty}\|x_{n_k+1}-p\| \\ &\leqslant \limsup_{n\to\infty}\|x_{n+1}-p\| = h(p).\end{aligned}$$

So

$$f(p) = h(p) \text{ for all } p\in F(T).$$

Since $x, x^*\in F(T)$, we obtain that $f(x) = h(x)$ and $f(x^*) = h(x^*)$, and hence $x^* = x$ and $f(x^*) = \inf_{z\in K} f(z)$ by the uniqueness.

For any given $t\in(0,1)$, take

$$z_t = x^* + t(u-x^*) = (1-t)x^* + tu.$$

Then $\lim_{t\to 0} z_t = x^*$ and $z_t\in K$ by the convexity of K, and hence $f(x^*)\leqslant f(z_t)$. Since $x_{n_k+1}-z_t = (x_{n_k+1}-x^*) - t(u-x^*)$,

$$\begin{aligned}\|x_{n_k+1}-z_t\|^2 &= \langle x_{n_k+1}-x^*, J(x_{n_k+1}-z_t)\rangle \\ &\quad - t\langle u-x^*, J(x_{n_k+1}-z_t)\rangle \\ &\leqslant \frac{\|x_{n_k+1}-x^*\|^2 + \|x_{n_k+1}-z_t\|^2}{2} \\ &\quad - t\langle u-x^*, J(x_{n_k+1}-z_t)\rangle.\end{aligned}$$

Then,

$$\|x_{n_k+1}-z_t\|^2 \leqslant \|x_{n_k+1}-x^*\|^2 - 2t\langle u-x^*, J(x_{n_k+1}-z_t)\rangle.$$

Thus we have

$$\limsup_{k\to\infty}\|x_{n_k+1}-z_t\|^2 \leqslant \limsup_{k\to\infty}\|x_{n_k+1}-x^*\|^2 - 2t\liminf_{k\to\infty}\langle u-x^*, J(x_{n_k+1}-z_t)\rangle.$$

That is,

$$\liminf_{k\to\infty}\langle u-x^*, J(x_{n_k+1}-z_t)\rangle \leqslant \frac{f^2(x^*)-f^2(z_t)}{2t} \leqslant 0. \quad (14)$$

On the other hand, since J is uniformly continuous on bounded set from norm topology to weak star topology and $\lim_{t\to 0} z_t = x^*$, then for any $\varepsilon > 0$, $\exists\delta > 0, \forall t\in(0,\delta)$, for all k, we have

$$\langle u-x^*, J(x_{n_k+1}-x^*)\rangle < \langle u-x^*, J(x_{n_k+1}-z_t)\rangle + \varepsilon.$$

By (14), we have that

$$\liminf_{k\to\infty}\langle u-x^*, J(x_{n_k+1}-x^*)\rangle \leqslant \liminf_{k\to\infty}\langle u-x^*, J(x_{n_k+1}-z_t)\rangle + \varepsilon \leqslant \varepsilon.$$

Since ε is arbitrary, we obtain that

$$\liminf_{k\to\infty}\langle u-x^*, J(x_{n_k+1}-x^*)\rangle \leqslant 0.$$

It follows from (12) that

$$c = \liminf_{k\to\infty}\langle u-x^*, J(x_{n_k+1}-x^*)\rangle \leqslant 0.$$

Therefore, (11) is proved.

We next show $x_n\to x^*$. In fact,

$$\begin{aligned}\|T_nx_n-x^*\| &\leqslant \frac{1}{n+1}\sum_{j=0}^{n}\|T^jx_n-x^*\| \\ &\leqslant \frac{1}{n+1}\sum_{j=0}^{n}k_j\|x_n-x^*\| \\ &= (b_n+1)\|x_n-x^*\|.\end{aligned}$$

It follows from the equality (9) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \alpha_n\langle u-x^*, J(x_{n+1}-x^*)\rangle \\ &\quad + (1-\alpha_n)\langle T_n x_n - x^*, J(x_{n+1}-x^*)\rangle \\ &\leqslant \alpha_n\langle u-x^*, J(x_{n+1}-x^*)\rangle \\ &\quad + (1-\alpha_n)\|T_n x_n - x^*\|\|x_{n+1}-x^*\| \\ &\leqslant \alpha_n\langle u-x^*, J(x_{n+1}-x^*)\rangle \\ &\quad + (1-\alpha_n)(b_n+1)\|x_n-x^*\|\|x_{n+1}-x^*\| \\ &\leqslant \alpha_n\langle u-x^*, J(x_{n+1}-x^*)\rangle \\ &\quad + (1-\alpha_n)\frac{(b_n+1)^2\|x_n-x^*\|^2 + \|x_{n+1}-x^*\|^2}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leqslant (1-\alpha_n)\|x_n-x^*\|^2 \\ &\quad + (1-\alpha_n)[(b_n+1)^2-1]\|x_n-x^*\|^2 \\ &\quad + 2\alpha_n\langle u-x^*, J(x_{n+1}-x^*)\rangle \\ &\leqslant (1-\alpha_n)\|x_n-x^*\|^2 + b_n(b_n+2)\|x_n-x^*\|^2 \\ &\quad + 2\alpha_n\langle u-x^*, J(x_{n+1}-x^*)\rangle,\end{aligned}$$

that is,

$$\|x_{n+1}-x^*\|^2 \leqslant (1-\alpha_n)\|x_n-x^*\|^2 + \gamma_n\alpha_n, \quad (15)$$

where $\gamma_n = \frac{b_n}{\alpha_n}(b_n+2)\|x_n-p\|^2 + 2\langle u - x^*, J(x_{n+1}-x^*)\rangle$.

It follows from the condition $\lim_{n\to\infty} b_n/\alpha_n = 0$ and the boundedness of $\{x_n\}$ along with the inequality (11) that

$$\limsup_{n\to\infty} \gamma_n \leqslant 0.$$

Applying Lemma 2 to the inequality (15), we conclude that $x_n \to x^*$. This completes the proof. □

Corollary 1 *Let K be a nonempty closed convex subset of a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Suppose that $T : K \to K$ is a non-expansive mapping. Let $\{x_n\}$ be defined by (9). Assume that $\alpha_n \in (0,1)$ satisfies (i) $\lim_{n\to\infty}\alpha_n = 0$, (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then as $n \to \infty$, $\{x_n\}$ converges strongly to some fixed point x^* of T.*

Remark 1 Our results are new even in a Hilbert space and their proofs are independent of not only $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, but also the convergence of the Browder type iteration path $z_t = tu + (1-t)Tz_t$, (see Ref. 28).

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