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Some results concerning invertible matrices over semirings

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ABSTRACT: It is well-known that a square matrix A over a commutative ring R with identity is invertible over R if and only if $\det A$ is a multiplicatively invertible element of R. Additively inverse commutative semirings with zero 0 and identity 1 are a generalization of commutative rings with identity. In this paper, we generalize the above known result as follows. An $n \times n$ matrix A over an additively inverse commutative semiring $S = (S, +, \cdot)$ with 0, 1 is invertible over S if and only if $\det^+ A + (\det^- A)'$ is multiplicatively invertible in S and $A_{ij}A_{ik}$ $[A_{ji}A_{ki}]$ is additively invertible in S for all $i, j, k \in \{1, \dots, n\}$ with $j \neq k$ where $\det^+ A$ and $\det^- A$ are the positive determinant and the negative determinant of A, respectively, and $(\det^- A)'$ is the unique inverse of $\det^- A$ in the inverse semigroup $(S, +)$.

KEYWORDS: additively inverse semiring

INTRODUCTION

A *semiring* is a triple $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups and $\cdot$ is distributive over $+$. A semiring $(S, +, \cdot)$ is called *commutative* if it is both additively and multiplicatively commutative. An element $0 \in S$ is called a *zero* of $(S, +, \cdot)$ if $x + 0 = 0 + x = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$. By an *identity* of $(S, +, \cdot)$ we mean an element 1 of S such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in S$. A semiring $(S, +, \cdot)$ is called *additively inverse* if $(S, +)$ is an inverse semigroup, i.e., for each $x \in S$, there is a unique $x' \in S$ such that $x = x + x' + x$ and $x' = x' + x + x'$. Additively inverse semirings were studied by Karvellas[1]. We see that an additively inverse commutative semiring with zero and identity is a generalization of a commutative ring with identity.

Matrices over semirings have been widely studied. One of the most interesting topics in this area is determining the invertible matrices over a specific semiring. Invertible matrices over semirings of various types have been studied. Luce[2] characterized the invertible matrices over a Boolean algebra of at least 2 elements. He showed that they must be orthogonal matrices. Rutherford[3] showed that a square matrix over a Boolean algebra of 2 elements is invertible if and only if it is a permutation matrix. The invertible matrices over a special commutative antiring were characterized by Tan[4]. Dolzand and Oblak[5] generalized Tan's result to an arbitrary commutative antiring. Semifields are a generalization of fields. The invertible matrices over a semifield S which is not a field were investigated by Saranrakskul et al[6]. It was shown that a square matrix A over S is invertible if and only if every row and every column of A contains exactly one non-zero element. It is well-known that for a square matrix A over a field F, A is invertible over F if and only if $\det A \neq 0$. The following theorem is a generalization of this fact.

Theorem 1 (Hoffman and Kunze[7]) *Let R be a commutative ring with identity. A square matrix A over R is invertible over R if and only if $\det A$ is a multiplicatively invertible element of R.*

In this paper, we propose a theorem which generalizes Theorem 1 to some semirings. Our purpose is to characterize the invertible matrices over an additively inverse commutative semiring with zero and identity in terms of the positive and negative determinants.

PRELIMINARIES

An element x of a semiring $(S, +, \cdot)$ with zero 0 [identity 1] is said to be *additively* [*multiplicatively*] *invertible* if $x + y = y + x = 0$ $[x \cdot y = y \cdot x = 1]$ for some $y \in S$. Such an element y is obviously unique.

The following properties of an additively inverse semiring given by Karvellas[1] will be used.

Proposition 1 (Karvellas[1]) *If $(S, +, \cdot)$ is an additively inverse semiring, then for all $x, y \in S$,*

(i) $(x')' = x$,

(ii) $(x+y)' = y' + x'$,

(iii) $(xy)' = x'y = xy'$,

(iv) $x'y' = xy$.

Also, the following fact is clearly seen.

Proposition 2 *If $(S, +, \cdot)$ is an additively inverse semiring with zero 0 and $x, y \in S$ are such that $x + y = 0$, then $y = x'$.*

We give some examples of additively inverse commutative semirings with zero and identity which are not commutative rings with identity as follows.

Example 1 Define $x \oplus y = \max\{x, y\}$ and $x \odot y = \min\{x, y\}$ for all $x, y \in [0,1]$. Then $([0,1], \oplus, \odot)$ and $([0,1], \oplus, \cdot)$ are clearly additively inverse commutative semirings with zero 0 and identity 1 which are not rings where $\cdot$ is the usual multiplication. These semirings may be written as $([0,1], \max, \min)$ and $([0,1], \max, \cdot)$. Notice that for $x \in [0,1]$, $x' = x$ in these semirings.

If R is a ring, then $([0,1], \max, \min) \times R$ and $([0,1], \max, \cdot) \times R$ under coordinatewise operations are both additively inverse semirings with zero $(0,0)$ for which $(x,y)' = (x, -y)$ for all $x \in [0,1]$ and $y \in R$. Also, these semirings are additively commutative. Moreover, if R is a commutative ring with identity 1, then these two direct products of semirings are additively inverse commutative semirings having $(0,0)$ and $(1,1)$ as its zero and identity, respectively.

In fact, a commutative inverse semigroup is a union of disjoint abelian groups[8]. Therefore if $(S, +, \cdot)$ is an additively inverse commutative semiring, then $(S, +)$ is a union of disjoint abelian groups. In Example 1, we have that $[0,1] \times R = \dot{\cup}\{\{a\} \times R \mid a \in [0,1]\}$ and $\{a\} \times R$ is an abelian group under the addition of the direct products $([0,1], \max, \min) \times R$ and $([0,1], \max, \cdot) \times R$. Here $\dot{\cup}$ means a disjoint union.

Let S be a commutative semiring with zero 0 and identity $1 \neq 0$, n a positive integer and $M_n(S)$ the set of all $n \times n$ matrices over S. Then under usual matrix addition and matrix multiplication, $M_n(S)$ is an additively commutative semiring. The zero matrix of order n and the identity matrix of order n over S are the zero and the identity of $M_n(S)$, respectively. If $n > 1$, then $M_n(S)$ is not multiplicatively commutative. For $A \in M_n(S)$ and $i, j \in \{1, \dots, n\}$, let A_{ij} be the entry of A in the ith row and jth column. The transpose of A will be denoted by A^{T}. Then for all $A, B \in M_n(S)$, $(A^{\mathrm{T}})^{\mathrm{T}} = A$, $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$, and $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. A matrix $A \in M_n(S)$ is called *invertible* over S if $AB = BA = I_n$ for some $B \in M_n(S)$ where I_n is the identity matrix of order n over S. Notice that B is also unique. It is clear that for $A \in M_n(S)$, A is invertible over S if and only if A^{T} is invertible over S.

Let $\mathcal{S}_n$ be the symmetric group of degree $n \geqslant 2$, $\mathcal{A}_n$ the alternating group of degree n, and $\mathcal{B}_n = \mathcal{S}_n \smallsetminus \mathcal{A}_n$, that is,

$$\mathcal{A}_n = \{\, \sigma \in \mathcal{S}_n \mid \sigma \text{ is an even permutation} \,\},$$
$$\mathcal{B}_n = \{\, \sigma \in \mathcal{S}_n \mid \sigma \text{ is an odd permutation} \,\}.$$

For $A \in M_n(S)$, the *positive determinant* and the *negative determinant* of A are defined, respectively, as follows:

$$\det{}^{+} A = \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right),$$
$$\det{}^{-} A = \sum_{\sigma \in \mathcal{B}_n} \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right)$$

(see Ref. 9). We can see that

$$\mathcal{A}_n = \{\, \sigma^{-1} \mid \sigma \in \mathcal{A}_n \} \text{ and } \mathcal{B}_n = \{\, \sigma^{-1} \mid \sigma \in \mathcal{B}_n \},$$

$\det^{+} I_n = 1$ and $\det^{-} I_n = 0$ and for $A \in M_n(S)$,

$$\begin{aligned}
\det{}^{+}(A^{\mathrm{T}}) &= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A^{\mathrm{T}}_{i\sigma(i)} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{\sigma(i),i} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{\sigma^{-1}(i),i} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{\sigma^{-1}(i),\sigma(\sigma^{-1}(i))} \right) \\
&= \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} A_{i\sigma(i)} \right) = \det{}^{+} A.
\end{aligned}$$

It can be shown similarly that $\det^{-}(A^{\mathrm{T}}) = \det^{-} A$. Notice that if R is a commutative ring with identity and $A \in M_n(R)$, then $\det A = \det^{+} A - \det^{-} A$.

Reutenauer and Straubing[9] gave the following significant results.

Theorem 2 (Reutenauer and Straubing[9]) *Let S be a commutative semiring with zero and identity and n a* *positive integer $\geqslant 2$. If A, $B \in M_n(S)$, then there is an element $r \in S$ such that*

$$\det^+(AB) = (\det^+ A)(\det^+ B) + (\det^- A)(\det^- B) + r,$$
$$\det^-(AB) = (\det^+ A)(\det^- B) + (\det^- A)(\det^+ B) + r.$$

Theorem 3 (Reutenauer and Straubing[9]) *Let S be a commutative semiring with zero and identity and n a positive integer. For A, $B \in M_n(S)$, if $AB = I_n$, then $BA = I_n$.*

INVERTIBLE MATRICES OVER ADDITIVELY INVERSE COMMUTATIVE SEMIRINGS WITH 0, 1

Throughout, let n be a positive integer greater than 1. To characterize invertible matrices over an additively inverse commutative semiring with 0, 1, the following lemmas are needed. Recall that $|\mathcal{S}_n| = n!$, $|\mathcal{A}_n| = n!/2 = |\mathcal{B}_n|$, and $\sigma\mathcal{A}_n = \mathcal{B}_n$ whenever $\sigma \in \mathcal{B}_n$ where $|X|$ stands for the cardinality of a set X.

Lemma 1 *For distinct i, $j \in \{1, 2, \dots, n\}$, $\sigma \mapsto (\sigma(i), \sigma(j))\sigma$ is a bijection from $\mathcal{A}_n$ onto $\mathcal{B}_n$.*

Proof: Let i, $j \in \{1, \dots, n\}$ be distinct. If $\sigma \in \mathcal{A}_n$, then $(\sigma(i), \sigma(j))\sigma \in \mathcal{B}_n$, so $\{(\sigma(i), \sigma(j))\,\sigma \mid \sigma \in \mathcal{A}_n\} \subseteq \mathcal{B}_n$. Assume that σ_1, $\sigma_2 \in \mathcal{A}_n$ such that $\sigma_1 \neq \sigma_2$.
Case 1: $(\sigma_1(i), \sigma_1(j)) = (\sigma_2(i), \sigma_2(j))$. By the cancellation property of $\mathcal{S}_n$, we have $(\sigma_1(i), \sigma_1(j))\sigma_1 \neq (\sigma_2(i), \sigma_2(j))\sigma_2$.
Case 2: $(\sigma_1(i), \sigma_1(j)) \neq (\sigma_2(i), \sigma_2(j))$. Then $\{\sigma_1(i), \sigma_1(j)\} \neq \{\sigma_2(i), \sigma_2(j)\}$. We may assume without loss of generality that $\sigma_1(i) \notin \{\sigma_2(i), \sigma_2(j)\}$. Then $\sigma_1(i) \neq \sigma_2(i)$, so

$$\begin{aligned}(\sigma_1(i), \sigma_1(j))\sigma_1(j) &= \sigma_1(i)\\ &\neq \sigma_2(i)\\ &= (\sigma_2(i), \sigma_2(j))\sigma_2(j).\end{aligned}$$

This implies that $(\sigma_1(i), \sigma_1(j))\sigma_1 \neq (\sigma_2(i), \sigma_2(j))\sigma_2$. This shows that $|\{(\sigma(i), \sigma(j))\,\sigma \mid \sigma \in \mathcal{A}_n\}| = |\mathcal{A}_n|$. But since $|\mathcal{A}_n| = |\mathcal{B}_n|$ and $\{(\sigma(i), \sigma(j))\,\sigma \mid \sigma \in \mathcal{A}_n\} \subseteq \mathcal{B}_n$, the desired result follows. □

Lemma 2 *Let S be a commutative semiring with zero 0 and identity 1 and $A \in M_n(S)$. If A is invertible over S, then $A_{ij}A_{ik}$ is additively invertible in S for all i, j, $k \in \{1, \dots, n\}$ with $j \neq k$.*

Proof: First, we note that if $a_1, \dots, a_t \in S$ are additively invertible in S, then so is $c_1a_1 + c_2a_2 + \dots + c_ta_t$ for all $c_1, \dots, c_t \in S$. Let $B \in M_n(S)$ be such that $AB = BA = I_n$. Then for distinct p, $q \in \{1, \dots, n\}$,

$$0 = (BA)_{pq} = \sum_{l=1}^{n} B_{pl}A_{lq}.$$

This shows that $B_{pl}A_{lq}$ are additively invertible in S for all l, p, $q \in \{1, \dots, n\}$ with $p \neq q$. Let i, j, $k \in \{1, \dots, n\}$ be such that $j \neq k$. Then

$$\begin{aligned}A_{ij}A_{ik} &= (A_{ij}A_{ik})(AB)_{ii}\\ &= A_{ij}A_{ik}\Big(\sum_{l=1}^{n} A_{il}B_{li}\Big)\\ &= A_{ik}^2(B_{ki}A_{ij}) + \sum_{\substack{l=1\\ l\neq k}}^{n} A_{ij}A_{il}(B_{li}A_{ik}),\end{aligned}$$

so by the above results, $A_{ij}A_{ik}$ is additively invertible in S. □

Theorem 4 *Let S be an additively inverse commutative semiring with zero 0 and identity 1 and $A \in M_n(S)$. Then A is invertible over S if and only if*

(i) *$\det^+ A + (\det^- A)'$ is multiplicatively invertible in S and*

(ii) *$A_{ij}A_{ik}$ is additively invertible in S for all i, j, $k \in \{1, \dots, n\}$ with $j \neq k$.*

Proof: Assume that there exists a matrix $B \in M_n(S)$ such that $AB = BA = I_n$. By Theorem 2, there exists an element $r \in S$ such that

$$\det^+(AB) = (\det^+ A)(\det^+ B) + (\det^- A)(\det^- B) + r,$$
$$\det^-(AB) = (\det^+ A)(\det^- B) + (\det^- A)(\det^+ B) + r.$$

But since $\det^+(AB) = \det^+ I_n = 1$ and $\det^-(AB) = \det^- I_n = 0$, it follows that

$$1 = (\det^+ A)(\det^+ B) + (\det^- A)(\det^- B) + r, \tag{1}$$
$$0 = (\det^+ A)(\det^- B) + (\det^- A)(\det^+ B) + r. \tag{2}$$

From (2) and Proposition 2, we have that

$$r = \big((\det^+ A)(\det^- B) + (\det^- A)(\det^+ B)\big)'.$$

Proposition 1(ii) and (iii) yield the following result:

$$r = (\det^+ A)(\det^- B)' + (\det^- A)'(\det^+ B). \tag{3}$$

From (1) and (3) and Proposition 1(iv), we have

$$\begin{aligned}1 &= (\det{}^+A)(\det{}^+B) + (\det{}^-A)(\det{}^-B) + \\ &\quad (\det{}^+A)(\det{}^-B)' + (\det{}^-A)'(\det{}^+B) \\ &= (\det{}^+A)(\det{}^+B) + (\det{}^-A)'(\det{}^-B)' + \\ &\quad (\det{}^+A)(\det{}^-B)' + (\det{}^-A)'(\det{}^+B) \\ &= (\det{}^+A + (\det{}^-A)')(\det{}^+B + (\det{}^-B)').\end{aligned}$$

Hence $\det^+ A + (\det^- A)'$ is multiplicatively invertible, so (i) holds. Condition (ii) follows from Lemma 2.

Conversely, assume that (i) and (ii) hold. By (i), $x(\det^+ A + (\det^- A)') = 1$ for some $x \in S$. Also, by (ii) and Proposition 2, $A_{ij}A_{ik} + (A_{ij}A_{ik})' = 0$ for all $i, j, k \in \{1, \dots, n\}$ with $j \neq k$. Define $B \in M_n(S)$ by

$$B_{ij} = x\left(\sum_{\substack{\sigma\in\mathcal{A}_n\\ \sigma(j)=i}}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big) + \sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=i}}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'\right)$$

for all $i, j \in \{1, \dots, n\}$. Here the empty sum is 0. To show that $AB = I_n$, let $i, j \in \{1, \dots, n\}$. Then

$$\begin{aligned}(AB)_{ij} &= \sum_{t=1}^{n} A_{it}B_{tj} \\ &= \sum_{t=1}^{n} A_{it}x\left(\sum_{\substack{\sigma\in\mathcal{A}_n\\ \sigma(j)=t}}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big) + \sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=t}}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'\right) \\ &= x\left[\sum_{t=1}^{n}\Big(\sum_{\substack{\sigma\in\mathcal{A}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\big)\Big) + \sum_{t=1}^{n}\Big(\sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\big)'\Big)\right]. \quad (4)\end{aligned}$$

But since

$$\mathcal{A}_n = \dot{\bigcup_{t\in\{1,\dots,n\}}}\{\sigma \in \mathcal{A}_n \mid \sigma(j) = t\}$$

and

$$\mathcal{B}_n = \dot{\bigcup_{t\in\{1,\dots,n\}}}\{\sigma \in \mathcal{B}_n \mid \sigma(j) = t\},$$

we deduce that

$$\sum_{t=1}^{n}\left(\sum_{\substack{\sigma\in\mathcal{A}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)\right) = \sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big),$$

$$\sum_{t=1}^{n}\left(\sum_{\substack{\sigma\in\mathcal{B}_n\\ \sigma(j)=t}} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'\right) = \sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'.$$

These facts and (4) yield

$$(AB)_{ij} = x\left(\sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big) + \sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big)'\right). \quad (5)$$

Case 1: $i = j$. Then

$$\sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big) = \sum_{\sigma\in\mathcal{A}_n}\Big(\prod_{k=1}^{n} A_{k\sigma(k)}\Big) = \det{}^+A,$$

$$\sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\ k\neq j}}^{n} A_{k\sigma(k)}\Big) = \sum_{\sigma\in\mathcal{B}_n}\Big(\prod_{k=1}^{n} A_{k\sigma(k)}\Big) = \det{}^-A.$$

Since $x(\det^+ A + (\det^- A)') = 1$, by these results and (5), we have $(AB)_{ii} = 1$.

Case 2: $i \neq j$. If $n = 2$, then from (5), (ii) and Proposition 2, we have

$$\begin{aligned}(AB)_{ij} &= x\big(A_{ij}A_{ii} + A_{ii}A'_{ij}\big) \\ &= x\big(A_{ij}A_{ii} + (A_{ii}A_{ij})'\big) \\ &= 0\end{aligned}$$

Next, assume that $n > 2$. Then

$$\sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{n} A_{k\sigma(k)}\Big) = \sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big), \quad (6)$$

$$\sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{n} A_{k\sigma(k)}\Big)' = \sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big)' = \sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}A'_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big) \quad (7)$$

Note that (7) is obtained from Proposition 1(iii). It follows from (5), (6), and (7) that

$$(AB)_{ij} = x\left(\sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big) + \sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}A'_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big)\right).$$

For each $\sigma \in \mathcal{A}_n$, let $\overline{\sigma} = (\sigma(i), \sigma(j))\sigma \in \mathcal{B}_n$. It then follows from Lemma 1 that

$$(AB)_{ij} = x\left(\sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big) + \sum_{\sigma\in\mathcal{A}_n} (A_{i\overline{\sigma}(j)}A_{i\overline{\sigma}(i)})'\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\overline{\sigma}(k)}\Big)\right). \quad (8)$$

But since for $\sigma \in \mathcal{A}_n$, $\overline{\sigma}(i) = \sigma(j)$, $\overline{\sigma}(j) = \sigma(i)$, and $\overline{\sigma}(k) = \sigma(k)$ for all $k \in \{1, \dots, n\} \smallsetminus \{i, j\}$, we deduce from (8) that

$$(AB)_{ij} = x\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big)\Big[\sum_{\sigma\in\mathcal{A}_n}\Big(A_{i\sigma(j)}A_{i\sigma(i)} + (A_{i\sigma(i)}A_{i\sigma(j)})'\Big)\Big]$$

Since $\sigma(i) \neq \sigma(j)$ for all $\sigma \in \mathcal{A}_n$, by (ii) and Proposition 2, $A_{i\sigma(j)}A_{i\sigma(i)} + (A_{i\sigma(i)}A_{i\sigma(j)})' = 0$. Hence $(AB)_{ij} = 0$. This proves that $AB = I_n$. By Theorem 3, $BA = I_n$, so A is invertible over S. □

The following corollary is a direct consequence of Theorem 4 and the facts that $\det^+ A = \det^+(A^{\mathrm{T}})$, $\det^- A = \det^-(A^{\mathrm{T}})$, and A is invertible over S if and only if A^{T} is invertible over S.

Corollary 1 *Let S be an additively inverse commutative semiring with 0, 1 and $A \in M_n(S)$. Then A is invertible over S if and only if*

(i) $\det^+ A + (\det^- A)'$ *is multiplicatively invertible in S and*

(ii) $A_{ji}A_{ki}$ *is additively invertible in S for all i, j, $k \in \{1, \dots, n\}$ with $j \neq k$.*

We can see that Theorem 1 is a direct consequence of Theorem 4. Let us discuss the invertible matrices in $M_n(S)$ where S is either the semiring $([0, 1], \max, \min)$ or the semiring $([0, 1], \max, \cdot)$ in Example 1 by making use of Theorem 4 and Corollary 1. Assume that $A \in M_n(S)$ is an invertible matrix. We can see that 1 is the only multiplicatively invertible element and 0 is the only additively invertible element in S. In addition, for $x, y \in S$, $xy = 0$ if and only if $x = 0$ or $y = 0$, and $x' = x$ for all $x \in S$. We also have that for $x, y \in S$, $xy = 1$ if and only if $x = y = 1$. Since A is invertible, A cannot have a zero row or a zero column. It follows from Theorem 4 and Corollary 1 that $\det^+ A + (\det^- A)' = 1$ and $A_{ij}A_{ik} = 0 = A_{ji}A_{ki}$ for all $i, j, k \in \{1, \dots, n\}$ with $j \neq k$. From the above facts and the last equalities, we deduce that each row and each column of A contains exactly one non-zero element. Then for each $i \in \{1, \dots, n\}$, there exists a unique $\sigma(i) \in \{1, \dots, n\}$ such that $A_{i\sigma(i)} \neq 0$, and so $\sigma(i) \neq \sigma(j)$ if $i \neq j$. Then $\sigma \in S_n$. It is clearly seen that if $\sigma \in \mathcal{A}_n$, then $\det^+ A = A_{1\sigma(1)}A_{2\sigma(2)} \cdots A_{n\sigma(n)}$ and $\det^- A = 0$, and if $\sigma \in \mathcal{B}_n$, then $\det^+ A = 0$ and $\det^- A = A_{1\sigma(1)}A_{2\sigma(2)} \cdots A_{n\sigma(n)}$. It follows that $A_{1\sigma(1)}A_{2\sigma(2)} \cdots A_{n\sigma(n)} = 1$, and hence $A_{i\sigma(i)} = 1$ for all $i \in \{1, \dots, n\}$. We then deduce that every entry of A is either 0 or 1, and each row and each column of A contains exactly one 1, i.e., A is a permutation matrix.

If A is a permutation matrix in $M_n(S)$ then $AA^{\mathrm{T}} = A^{\mathrm{T}}A = I_n$. We conclude that for $A \in M_n(S)$, A is invertible over S if and only if A is a permutation matrix.

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