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Fréchet functional equations with restricted spans: Solution on real numbers

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ABSTRACT: Our aim is to determine the general solution of the Fréchet functional equation, $\Delta_{h_1,\ldots,h_{m+1}} f(x) = 0$, when the span h_i is restricted to an open interval (a_i, b_i) for each $i = 1, \ldots, m+1$.

KEYWORDS: finite difference functional equation, conditional functional equation, generalized polynomial

INTRODUCTION

The problem of functional equations with restricted domains is one of an interesting framework. It is challenging to determine their solutions when some conditions are imposed on the domains of the variables. Some authors[1–4] have described this phenomenon by the term *conditional functional equations*. A classical problem is the Cauchy functional equation,

$$f(x+y) = f(x) + f(y) \quad \text{for all} \quad (x,y) \in \Omega, \quad (1)$$

when the variables, x and y, satisfy some conditions or restrictions. We sometimes call the domain of variables Ω the *domain of validity*[1–3,5,6]. Readers who are interested in other functional equations with restricted domains should refer to e.g., Refs. 1–9.

In this paper, we will determine the solutions of the Fréchet functional equation, $\Delta_{h_1,\ldots,h_{m+1}} f(x) = 0$, when the spans h_i's are restricted to the open intervals (a_i, b_i) for each $i = 1, \ldots, m+1$.

PRELIMINARIES

For our purposes, in this section, the related theorems and definitions of the Fréchet functional equation will be introduced (for further details see Refs. 10, 11).

Let f be an arbitrary function on $\mathbb{R}$. We will first define the *difference operator* with a *span* $h \in \mathbb{R}$, denoted by Δ_h, as

$$\Delta_h f(x) = f(x+h) - f(x), \qquad \text{for all} \quad x \in \mathbb{R}.$$

Furthermore, for all $h_1, \ldots, h_m \in \mathbb{R}$ where m is a positive integer, we denote the composition of the difference operator with difference spans $h_1, \ldots, h_m$ by

$$\Delta_{h_1,\ldots,h_m} f(x) = \Delta_{h_1} \ldots \Delta_{h_m} f(x).$$

It is possible to expand the iterative difference operators by the formula

$$\Delta_{h_1,\ldots,h_m} f(x) = \sum_{\varepsilon_1,\ldots,\varepsilon_m \in \{0,1\}} (-1)^{m-(\varepsilon_1+\cdots+\varepsilon_m)} f(x+\varepsilon_1 h_1+\cdots+\varepsilon_m h_m). \quad (2)$$

For instance, $\Delta_{h_1,h_2} f(x) = f(x+h_1+h_2) - f(x+h_1) - f(x+h_2) + f(x)$. From (2), it is easy to see that $\Delta_{h_1,h_2} f(x) = \Delta_{h_2,h_1} f(x)$ for all $h_1, h_2 \in \mathbb{R}$. That is, the difference operators are commutative. A *Fréchet functional equation* is the generalization of Cauchy's equation. This functional equation has the form

$$\Delta_{h_1,\ldots,h_m} f(x) = 0. \quad (3)$$

In particular, if $h_1 = \ldots = h_m = h$, $h \in \mathbb{R}$, then (3) can be rewritten succinctly as

$$\Delta_h^m f(x) = 0. \quad (4)$$

The general solution of $\Delta_h^{m+1} f(x) = 0$ where m is a nonnegative integer, will be called a *polynomial function of order* m and possesses some important properties as in the following two theorems[10]. Let X and Y be linear spaces over $\mathbb{R}$.

Theorem 1 (Czerwik[10]) *Let m be a nonnegative integer. A function $f : X \to Y$ is a polynomial function of order m, that is $\Delta_h^{m+1} f(x) = 0$, if and only if there exist k-additive symmetric functions $A_k : X^k \to Y$, $k = 0, 1, \ldots, m$ such that*

$$f(x) = A^0(x) + A^1(x) + A^2(x) + \ldots + A^m(x) \quad (5)$$

for all $x \in X$ *where* $A^k : X \to Y$, $k = 0, 1, \dots, m$ *is the diagonalization of* A_k *and is defined by*

$$A^k(x) = A_k(\underbrace{x, \dots, x}_{k}), \quad \text{for all} \quad x \in X.$$

The following theorem encompasses the solution to the equation $\Delta_{h_1,\dots,h_{m+1}} f(x) = 0$.

Theorem 2 (Czerwik[10]) *A function* $f : X \to Y$ *is a polynomial function of order* m *if and only if* $\Delta_{h_1,\dots,h_{m+1}} f(x) = 0$ *for every* $x, h_1, \dots, h_m \in X$.

That is, the general solution of the Fréchet functional equation, $\Delta_{h_1,\dots,h_{m+1}} f(x) = 0$, is also a *polynomial function of order* m and f takes the form as in (5).

MAIN RESULTS

In this section, we will prove that the domain of validity of a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the Fréchet functional equation can be extended to $\mathbb{R}$. Before going to the main results, some notations are defined for use in this section. Let m be a nonnegative integer and let ε_i, $\delta_i \in \{0, 1\}$ for every $i = 1, \dots, m+1$. For our convenience, we will denote sets $E_{m+1} = \{\varepsilon_1, \dots, \varepsilon_{m+1}\}$, $D_{m+1} = \{\delta_1, \dots, \delta_{m+1}\}$ and the sum

$$\sum_{E_{m+1}} g(\varepsilon_1, \dots, \varepsilon_{m+1}) = \sum_{\{\varepsilon_1,\dots,\varepsilon_{m+1}\} \in \{0,1\}} g(\varepsilon_1, \dots, \varepsilon_{m+1}),$$

where g is a function. Define a function S by

$$S(A) = \prod_{\varepsilon \in A} (-1)^{1-\varepsilon},$$

where A is a finite set of integers. With this notation, (2) can be rewritten as

$$\Delta_{h_1,\dots,h_m} f(x) = \sum_{E_{m+1}} S(E_{m+1}) f(x + \varepsilon_1 h_1 + \dots + \varepsilon_m h_m). \quad (6)$$

Lemma 1 *Let* m *be a nonnegative integer and let* $a_i, b_i \in \mathbb{R}$ *with* $a_i < b_i$, $i = 1, \dots, m+1$. *If a function* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the Fréchet functional equation*

$$\Delta_{h_1,\dots,h_{m+1}} f(x) = 0 \quad (7)$$

for all $x \in \mathbb{R}$ *and all* $h_i \in (a_i, b_i), i = 1, \dots, m+1$, *then* f *also satisfies* (7) *for all* $x \in \mathbb{R}$ *and all* $h_i \in (n_i a_i, n_i b_i)$ *for all positive integers* $n_1, n_2, \dots, n_{m+1}$.

Proof: Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the hypothesis of the theorem. For any spans $y_i \in (a_i, b_i), i = 1, \dots, m+1$. We can see that $\Delta_{y_1,\dots,y_{m+1}} f(x + jy_1) = 0$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ and let n_1 be a positive integer. Then,

$$\sum_{j=0}^{n_1-1} \Delta_{y_1,\dots,y_{m+1}} f(x + jy_1) = 0. \quad (8)$$

Applying (6) to (8) yields

$$\sum_{j=0}^{n_1-1} \sum_{E_{m+1}} S(E_{m+1}) f(x + (\varepsilon_1 + j) y_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k) = 0.$$

Note that in the case $m = 0$, $\sum_{k=2}^{m+1} \varepsilon_k y_k$ is understood to be zero. If we evaluate the sum over ε_1 in E_{m+1}, then we get

$$\sum_{j=0}^{n_1-1} \sum_{E_{m+1} \setminus \{\varepsilon_1\}} S(E_{m+1} \setminus \{\varepsilon_1\}) \left(f\left(x + (1+j) y_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right) - f\left(x + jy_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right)\right) = 0. \quad (9)$$

Swapping the order of two summations, it can be observed that

$$\sum_{j=0}^{n_1-1} \left(f\left(x + (1+j) y_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right) - f\left(x + jy_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right)\right) = f\left(x + n_1 y_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right) - f\left(x + \sum_{k=2}^{m+1} \varepsilon_k y_k\right).$$

Then (9) is reduced to

$$\sum_{E_{m+1} \setminus \{\varepsilon_1\}} S(E_{m+1} \setminus \{\varepsilon_1\}) \left(f\left(x + n_1 y_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right) - f\left(x + \sum_{k=2}^{m+1} \varepsilon_k y_k\right)\right) = 0$$

which can be rewritten as

$$\sum_{E_{m+1} \setminus \{\varepsilon_1\}} S(E_{m+1} \setminus \{\varepsilon_1\}) \left(\sum_{\varepsilon_1 \in \{0,1\}} (-1)^{1-\varepsilon_1} f\left(x + \varepsilon_1 n_1 y_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right)\right) = 0.$$

That is,

$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + \varepsilon_1 n_1 y_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right) = 0.$$

Using (6) again, it becomes

$$\Delta_{n_1 y_1, y_2, \ldots, y_{m+1}} f(x) = 0.$$

Therefore, we can see that the domain of validity of y_1 can be extended from the open interval (a_1, b_1) to $(n_1 a_1, n_1 b_1)$. Since the order of the difference operators can be permuted; that is, $\Delta_{y_1, y_2, \ldots, y_{m+1}} f(x) = \Delta_{y_2, y_1, y_3, \ldots, y_{m+1}} f(x)$ for all $i = 1, \ldots, m+1$, the domain of validity of each variable y_i can be extended from (a_i, b_i) to $(n_i a_i, n_i b_i)$ for each i and every positive integers n_i. Thus we obtain

$$\Delta_{n_1 y_1, \ldots, n_{m+1} y_{m+1}} f(x) = 0.$$

Therefore, if we let $h_i = n_i y_i, i = 1, \ldots, m+1$, then we can conclude that

$$\Delta_{h_1, \ldots, h_{m+1}} f(x) = 0$$

for all $x \in \mathbb{R}$ and all $h_i \in (n_i a_i, n_i b_i)$ for every positive integers $n_1, n_2, \ldots, n_{m+1}$. □

Next, we will consider (7) when the domains of validity are half-open infinite intervals (p_i, ∞) or $(-\infty, p_i)$ where $p_i \in \mathbb{R}$, $i = 1, \ldots, m+1$; that is, f satisfies (7) for all $x \in \mathbb{R}$ and for all $h_i \in (p_i, \infty)$, when $i = 1, \ldots, n$ and for all $h_i \in (-\infty, p_i)$, when $i = n+1, \ldots, m+1$. We note that when $n = 0$, $h_i \in (-\infty, p_i)$ for all $i = 1, \ldots, m+1$ and when $n = m+1$, $h_i \in (p_i, \infty)$ for all $i = 1, \ldots, m+1$. With this restriction of domains of the spans h_i's, we will prove that (7) still holds for all real numbers h_i's as in the following theorem.

Lemma 2 *Let m and n be nonnegative integers with $n \leqslant m+1$ and for each $i = 1, \ldots, m+1$, let $p_i \in \mathbb{R}$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies (7) for all $x \in \mathbb{R}$ and for all $h_i \in (p_i, \infty)$ when $i = 1, \ldots, n$ and for all $h_i \in (-\infty, p_i)$ when $i = n+1, \ldots, m+1$, then (7) also holds for all $x, h_i \in \mathbb{R}, i = 1, \ldots, m+1$.*

Proof: Let a function $f : \mathbb{R} \to \mathbb{R}$ satisfy equation

$$\Delta_{h_1, \ldots, h_{m+1}} f(x) = 0 \tag{10}$$

for all $x \in \mathbb{R}$ and for all $h_i \in (p_i, \infty)$ when $i = 1, \ldots, n$ and for all $h_i \in (-\infty, p_i)$ when $i = n+1, \ldots, m+1$. Let $x, h_i \in \mathbb{R}, i = 1, \ldots, m+1$. Choose

$$t_i = \begin{cases} \max\{p_i, p_i + h_i\} + 1 \text{ if } i = 1, \ldots, n, \\ \min\{p_i, p_i + h_i\} - 1 \text{ if } i = n+1, \ldots, m+1. \end{cases}$$

Then for all $\varepsilon_i \in \{0, 1\}$, $i = 1, \ldots, m+1$, we obtain that for all $i = 1, \ldots, n$, $t_i - \varepsilon_i h_i > p_i$ which means $t_i - \varepsilon_i h_i \in (p_i, \infty)$. Similarly, for all $i = n+1, \ldots, m+1$, $t_i - \varepsilon_i h_i < p_i$ which means $t_i - \varepsilon_i h_i \in (-\infty, p_i)$. Consider (10) at point $x + \sum_{i=1}^{m+1} \varepsilon_i h_i$ with spans $t_i - \varepsilon_i h_i$'s. It follows that

$$\Delta_{(t_1 - \varepsilon_1 h_1), \ldots, (t_{m+1} - \varepsilon_{m+1} h_{m+1})} f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i\right) = 0,$$

and then for $E_{m+1} = \{\varepsilon_1, \ldots, \varepsilon_{m+1}\}$,

$$\sum_{E_{m+1}} S(E_{m+1}) \Delta_{(t_1 - \varepsilon_1 h_1), \ldots, (t_{m+1} - \varepsilon_{m+1} h_{m+1})} f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i\right) = 0 \tag{11}$$

where $S(E_{m+1}) = \prod_{\varepsilon_i \in E_{m+1}} (-1)^{1-\varepsilon_i}, i = 1, \ldots, m+1$. From (6), we can rewrite (11) as

$$\sum_{E_{m+1}} S(E_{m+1}) \sum_{D_{m+1}} S(D_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right) = 0$$

where $\delta_i \in \{0, 1\}, i = 1, \ldots, m+1$ and $D_{m+1} = \{\delta_1, \ldots, \delta_{m+1}\}$. Swapping two summations, it yields

$$\sum_{D_{m+1}} S(D_{m+1}) \sum_{E_{m+1}} S(E_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right) = 0. \tag{12}$$

We consider the term

$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right)$$

in the cases that there exist $\delta_j = 1$ for some $j \in \{1, \ldots, m+1\}$. If we choose the smallest number $j \in \{1, \ldots, m+1\}$ such that $\delta_j = 1$, then we obtain

$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right) = \sum_{E_{m+1}} S(E_{m+1}) f\left(x + t_j + \sum_{\substack{i=1 \\ i \neq j}}^{m+1} ((1 - \delta_i)\varepsilon_i h_i + \delta_i t_i)\right).$$

We then separate the sum over ε_j to yield;

$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + t_j + \sum_{\substack{i=1\\ i\neq j}}^{m+1} ((1-\delta_i)\varepsilon_i h_i + \delta_i t_i)\right)$$
$$= \sum_{E_{m+1}\setminus\{\varepsilon_j\}} S(E_{m+1}\setminus\{\varepsilon_j\}) \sum_{\varepsilon_j\in\{0,1\}} (-1)^{1-\varepsilon_j} f\left(x + t_j + \sum_{\substack{i=1\\ i\neq j}}^{m+1} ((1-\delta_i)\varepsilon_i h_i + \delta_i t_i)\right).$$

Since the term

$$f\left(x + t_j + \sum_{\substack{i=1\\ i\neq j}}^{m+1} ((1-\delta_i)\varepsilon_i h_i + \delta_i t_i)\right)$$

does not depend on ε_j,

$$\sum_{\varepsilon_j\in\{0,1\}} (-1)^{1-\varepsilon_j} f\left(x + t_j + \sum_{\substack{i=1\\ i\neq j}}^{m+1} ((1-\delta_i)\varepsilon_i h_i + \delta_i t_i)\right) = 0.$$

Therefore, we can conclude that

$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right) = 0 \quad (13)$$

for every case which there exists $\delta_i = 1$ for some $i \in \{1,\dots,m+1\}$. We evaluate the sum over D_{m+1} in (12) and apply (13). Then we arrive at

$$\sum_{E_{m+1}} (-1)^{(\varepsilon_1+\cdots+\varepsilon_{m+1})} f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i\right) = 0,$$

that is, $\Delta_{h_1,\dots,h_{m+1}} f(x) = 0$. □

Theorem 3 *Let m be a nonnegative integer and let $a_i, b_i \in \mathbb{R}$ with $a_i < b_i, i = 1,\dots,m+1$. If $f : \mathbb{R} \to \mathbb{R}$ is a function fulfilling (7) for all $x \in \mathbb{R}$ and all $h_i \in (a_i, b_i)$, $i = 1,\dots,m+1$, then (7) also holds for all $x, h_i \in \mathbb{R}$, $i = 1,\dots,m+1$.*

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying (7) for all $x \in \mathbb{R}$ and all $h_i \in (a_i, b_i)$, $i = 1,\dots,m+1$. We apply Lemma 1 to get that (7) holds for all $x \in \mathbb{R}$ and for all $h_i \in (n_i a_i, n_i b_i), i = 1,\dots,m+1$ for every positive integers $n_1, n_2, \dots, n_{m+1}$. Defining $H_i = \bigcup_{k=1}^{\infty} (ka_i, kb_i), i = 1,\dots,m+1$, then f also fulfils

$$\Delta_{h_1,\dots,h_{m+1}} f(x) = 0 \quad (14)$$

for all $x \in \mathbb{R}$ and for all $h_i \in H_i, i = 1,\dots,m+1$. For each $i = 1,\dots,m+1$, the interval (a_i, b_i) will satisfy one of the following three cases.
Case 1: $a_i < 0 < b_i$. Then $H_i = (-\infty,\infty)$. We note that if $a_i < 0 < b_i$ for all $i = 1,\dots,m+1$, then f satisfies (7) for all $x, h_i \in \mathbb{R}, i = 1,\dots,m+1$.
Case 2: $0 \leqslant a_i < b_i$. Choose an integer $M_i > \frac{a_i}{b_i - a_i}$. Then $jb_i - (j+1)a_i = j(b_i - a_i) - a_i > 0$ for all $j > M_i$. It follows that $(M_i a_i, \infty) \subset H_i$.
Case 3: $a_i < b_i \leqslant 0$. Choose an integer $m_i > \frac{-b_i}{b_i - a_i}$. Thus we get that $(j+1)b_i - ja_i = j(b_i - a_i) + b_i > 0$ for all $j > m_i$; consequently, $(-\infty, m_i b_i) \subset H_i$.
For each $i = 1,\dots,m+1$, let

$$p_i = \begin{cases} 0 & \text{if} \quad a_i < 0 < b_i, \\ M_i a_i & \text{if} \quad 0 \leqslant a_i < b_i, \\ m_i b_i & \text{if} \quad a_i < b_i \leqslant 0. \end{cases}$$

Thus it follows that, from (14) and the cases, (7) holds for all $x \in \mathbb{R}$ and for each $i = 1,\dots,m+1$, for all $h_i \in (p_i, \infty)$ or for all $h_i \in (-\infty, p_i)$. Since the order of the difference operators can be permuted, we can rearrange the order of the difference operators to

$$\Delta_{h_{j_1},\dots,h_{j_{m+1}}} f(x) = 0,$$

where $h_{j_i} \in (p_{j_i}, \infty)$ if $i = 1,\dots,n$ and $h_{j_i} \in (-\infty, p_{j_i})$ if $i = n+1,\dots,m+1$ for some non-negative integer n with $n \leqslant m+1$. Therefore, we can apply Lemma 2, which then yields that f also fulfills (7) for all $x, h_i \in \mathbb{R}, i = 1,\dots,m+1$ as desired. □

The following corollary will prove that the general solution of the Fréchet functional equation with restricted spans to the open interval (a_i, b_i), $i = 1,\dots,m+1$, is also a polynomial function of order m.

Corollary 1 *Let m be a nonnegative integer and let $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$, $i = 1,\dots,m+1$. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies (7) for all $x \in \mathbb{R}$ and all $h_i \in (a_i, b_i)$, $i = 1,\dots,m+1$ if and only if the function f is given by (5).*

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ satisfy an (7) for all $x \in \mathbb{R}$ and all $h_i \in (a_i, b_i)$, $i = 1,\dots,m+1$. Applying

Theorem 3, we obtain that (7) also holds for all $x, h_i \in \mathbb{R}$, $i = 1, \dots, m+1$. Thus it follows from Theorem 2 that f takes the same form as (5). Conversely, from Theorems 1 and 2, it is obvious that f satisfies (7) for all $x \in \mathbb{R}$ and all $h_i \in (a_i, b_i)$, $i = 1, \dots, m+1$. □

REFERENCES

1. Brzdek J, Sikorska J (2010) A conditional exponential functional equation and its stability. *Nonlin Anal* **72**, 2923–34.
2. Fochi M (2005) An alternative function equation on restricted domain. *Aequations Math* **70**, 201–12.
3. Paganoni L, Rätz J (1995) Conditional functional equations and orthogonal additivity. *Aequations Math* **50**, 135–42.
4. Sikorska J (2009) On two conditional Pexider functional equations and their stabilities. *Nonlin Anal* **70**, 2673–84.
5. Fochi M (2009) An alternative-conditional function equation and the orthogonally additive functionals. *Aequations Math* **78**, 309–20.
6. Shalit OM (2007) The overdeterminedness of a class of functional equations. *Aequations Math* **74**, 242–8.
7. Kuczma M (1978) Function equations on restricted domains. *Aequations Math* **18**, 1–34.
8. Sikorska J (2010) Exponential function equation on spheres. *Appl Math Lett* **23**, 156–60.
9. Skof F (2007) The general solution of the exponential Cauchy equation on a bounded restricted domain. *Aequations Math* **73**, 144–55.
10. Czerwik S (2002) *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore.
11. Kuczma M (1985) *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Scientific Publishers and Silesian Univ Press, Warszawa-Kraków-Katowice.