SHORT REPORT

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Some results on semigroups admitting ring structure

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ABSTRACT: Lawson has given a sufficient condition for a semigroup S which guarantees that S does not admit a ring structure. From Lawson's theorem, we have that the multiplicative interval semigroup $[0,1]$ does not admit a ring structure. In this paper we give an elementary proof of this fact. We then show that the multiplicative interval semigroup $[a,1]$ with $-1 \leqslant a < 0 < a^2 \leqslant 1$ does not admit the structure of a ring, a fact which cannot be derived from Lawson's theorem. These facts are then applied to show that every nontrivial multiplicative bounded interval semigroup on $\mathbb{R}$ does not admit a ring structure.

KEYWORDS: multiplicative interval semigroup

INTRODUCTION

The multiplicative structure of any ring is by definition a semigroup with zero. Then it is valid to ask whether S^0 is isomorphic to the multiplicative structure of some ring for a given semigroup S where $S^0 = S$ if S has a zero and S contains more than one element, and if S has no zero or S contains only one element, then S^0 is the semigroup with zero 0 adjoined. We say that a semigroup S *admits a ring structure* if S^0 is isomorphic to the multiplicative structure of some ring $(R,+,\cdot)$. If ϕ is an isomorphism from the semigroup S^0 onto the semigroup $(R,\cdot)$ and we define an operation $\oplus$ on S^0 by

$$x \oplus y = \phi^{-1}(\phi(x)+\phi(y)) \text{ for all } x,y \in S^0,$$

then $(S^0,\oplus,\cdot)$ is a ring isomorphic to $(R,+,\cdot)$ through ϕ. It follows that S admits a ring structure if and only if there exists an operation $+$ on S^0 such that $(S^0,+,\cdot)$ is a ring. Semigroups admitting ring structure have long been studied. Peinado[1] gave a brief survey of some results on this topic. For some further results, see Refs. 2–6.

Lawson[7] has proved that if S is a semigroup of order greater than two satisfying the conditions (i) for $x,y \in S$, $S^1x = S^1y \Rightarrow x = y$ (ii) for all $x,y \in S$, either $S^1x \subseteq S^1y$ or $S^1y \subseteq S^1x$, then S does not admit a ring structure where $S^1 = S$ if S has an identity, and if S has no identity, then S^1 is the semigroup S with identity 1 adjoined. We can see from Lawson's theorem that the multiplicative interval semigroup $[0,1]$ does not admit a ring structure[1]. Consider the multiplicative interval semigroup $[a,1]$ where $-1 \leqslant a < 0 < a^2 \leqslant 1$. If $a=-1$, then $1[a,1] = [-1,1] = -1[a,1]$. Also, if $a > -1$, then $a[a,1] = [a,a^2]$, $-a[a,1] = [-a^2,-a]$, $a < -a^2$, and $a^2 < -a$. These show that Lawson's theorem cannot be applied to determine whether the multiplicative interval semigroup $[a,1]$ admits a ring structure. In this paper we provide elementary proofs to show that the multiplicative interval semigroups $[0,1]$ and $[a,1]$ with $-1 \leqslant a < 0 < a^2 \leqslant 1$ do not admit a ring structure. As a consequence, we have that every nontrivial multiplicative bounded interval semigroup on $\mathbb{R}$ does not admit a ring structure. We note that for a nontrivial multiplicative bounded interval semigroup S on $\mathbb{R}$, S^0 is one of the following types:

$[0,a]$ or $[0,a)$ where $0 < a \leqslant 1$,
$[a,b]$, $(a,b]$, (a,b) where $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$,
$[a,b)$ where $-1 < a < 0 < a^2 < b \leqslant 1$

(see Ref. 8).

It is implicit throughout that the multiplication $\cdot$ on a semigroup of real numbers is the usual multiplication between numbers.

MAIN RESULTS

The following two lemmas are needed to show that the multiplicative interval semigroups $[0,1]$ and $[a,1]$ with $-1 \leqslant a < 0 < a^2 \leqslant 1$ do not admit a ring structure.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$.

Lemma 1 *Assume that S is a subsemigroup of the semigroup $(\mathbb{R}_0^+,\cdot)$. If $\oplus$ is an operation on S^0 such that $(S^0,\oplus,\cdot)$ is a ring, then $x \oplus x = 0$ for all $x \in S^0$.*

Proof: Let $x \in S^0$. Then $x \oplus y = 0$ for some $y \in S^0$, so $x^2 \oplus xy = x(x \oplus y) = x0 = 0$ and $xy \oplus y^2 = (x \oplus y)y = 0y = 0$. Therefore x^2 and y^2 are additive inverses of xy in the ring $(S^0, \oplus, \cdot)$. This implies that $x^2 = y^2$. But $x, y \geqslant 0$, so $x = y$. Hence $x \oplus x = 0$. □

Lemma 2 *Assume that S is a subsemigroup of the semigroup $(\mathbb{R}, \cdot)$. If $\oplus$ is an operation on S^0 such that $(S^0, \oplus, \cdot)$ is a ring, then $x \oplus (-x) = 0$ for every $x \in S^0$ for which $-x \in S^0$.*

Proof: Let $x \in S^0 \setminus \{0\}$ be such that $-x \in S^0$. Then $x \oplus (-x) = c$ for some $c \in S^0$ and thus $xc = x(x \oplus (-x)) = x^2 \oplus (-x^2) = (-x)((-x) \oplus x) = -xc$. Since $x \neq 0$, we have that $c = -c$ which implies that $c = 0$. Hence the desired result follows. □

Theorem 1 *The multiplicative interval semigroup $[0, 1]$ does not admit a ring structure.*

Proof: Suppose that the semigroup $([0, 1], \cdot)$ admits a ring structure. Then there is an operation $\oplus$ on $[0, 1]$ such that $([0, 1], \oplus, \cdot)$ is a ring. By Lemma 1, $x \oplus x = 0$ for all $x \in [0, 1]$. Let $c \in (0, 1)$. Then $1 \oplus c \neq 0$ and $1 \oplus c \neq 1$. We also have that $c[0, 1] = [0, c]$ is an ideal of the ring $([0, 1], \oplus, \cdot)$. Since $0 < 1 \oplus c < 1$, there is a positive integer n such that $(1 \oplus c)^n \in [0, c]$. But

$$(1 \oplus c)^n = 1 \oplus \binom{n}{1}^{*} c \oplus \binom{n}{2}^{*} c^2 \oplus \cdots \oplus \binom{n}{n-1}^{*} c^{n-1} \oplus c^n$$

where $k^* c^i$ means $c^i \oplus c^i \oplus \cdots \oplus c^i$ (k times). Hence we have $(1 \oplus c)^n = 1 \oplus cy$ for some $y \in [0, 1]$. Since $(1 \oplus c)^n, cy \in [0, c]$ and $[0, c]$ is an ideal of $([0, 1], \oplus, \cdot)$, it follows that $cy \oplus (1 \oplus c)^n \in [0, c]$. Hence

$$1 = 1 \oplus 0 = 1 \oplus cy \oplus cy = (1 \oplus c)^n \oplus cy \in [0, c]$$

which is a contradiction. This proves that the semigroup $([0, 1], \cdot)$ does not admit a ring structure, as desired. □

Corollary 1 *For $0 < a \leqslant 1$, the multiplicative interval semigroup $[0, a]$ does not admit a ring structure.*

Proof: Let $0 < a \leqslant 1$ and assume that there is an operation $\oplus$ on $[0, a]$ such that $([0, a], \oplus, \cdot)$ is a ring. Define an operation $\oplus'$ on $[0, 1]$ by

$$x \oplus' y = \frac{ax \oplus ay}{a} \quad \text{for all } x, y \in [0, 1].$$

It is evident that $x \oplus' y = y \oplus' x$ for all $x, y \in [0, 1]$. Also, for $x, y, z \in [0, 1]$,

$$\begin{aligned}(x \oplus' y) \oplus' z &= \left(\frac{ax \oplus ay}{a}\right) \oplus' z \\ &= \frac{a\left(\frac{ax \oplus ay}{a}\right) \oplus az}{a} \\ &= \frac{(ax \oplus ay) \oplus az}{a} \\ &= \frac{ax \oplus (ay \oplus az)}{a} \\ &= \frac{ax \oplus a\left(\frac{ay \oplus az}{a}\right)}{a} \\ &= x \oplus' \left(\frac{ay \oplus az}{a}\right) \\ &= x \oplus' (y \oplus' z)\end{aligned}$$

and

$$\begin{aligned}x(y \oplus' z) &= x\left(\frac{ay \oplus az}{a}\right) \\ &= \frac{ax(ay \oplus az)}{a^2} \\ &= \frac{(ax)(ay) \oplus (ax)(az)}{a^2} \\ &= \frac{a(axy) \oplus a(axz)}{a^2} \\ &= \frac{a(axy \oplus axz)}{a^2} \\ &= \frac{axy \oplus axz}{a} \\ &= xy \oplus' xz.\end{aligned}$$

Let $x \in [0, 1]$. Then $x \oplus' 0 = (ax \oplus 0)/a = x$. Since $ax \in [0, a]$, $ax \oplus y = 0$ for some $y \in [0, a]$. It follows that $y/a \in [0, 1]$ and

$$x \oplus' \frac{y}{a} = \frac{ax \oplus a\,(y/a)}{a} = \frac{ax \oplus y}{a} = \frac{0}{a} = 0.$$

This shows that $([0, 1], \oplus', \cdot)$ is a ring which is contrary to Theorem 1. □

Corollary 2 *For $0 < a \leqslant 1$, the multiplicative interval semigroup $[0, a)$ does not admit a ring structure.*

Proof: Let $0 < a \leqslant 1$ and assume that there is an operation $\oplus$ on $[0, a)$ such that $([0, a), \oplus, \cdot)$ is a ring. Then for every $c \in (0, a)$, $[0, ca) = c[0, a)$ is an ideal of the ring $([0, a), \oplus, \cdot)$. Let $d \in (0, a)$ and $0 < \epsilon < a(a - d)$. Then $d < d + \epsilon/a < a$, so $[0, (d + \epsilon/a)a) = [0, da + \epsilon)$ is an ideal of the ring $([0, a), \oplus, \cdot)$. It follows that

$$[0, da] = \bigcap_{0<\epsilon<a(a-d)} [0, da + \epsilon)$$

is an ideal of $([0,a),\oplus,\cdot)$. Thus $0 < da < 1$ and $([0,da],\oplus,\cdot)$ is a ring which is contrary to Corollary 1, so the desired result follows. □

Theorem 2 *The multiplicative interval semigroup* $[a,1]$ *with* $-1 \leqslant a < 0 < a^2 \leqslant 1$ *does not admit a ring structure.*

Proof: Suppose on the contrary that there is an operation $\oplus$ on $[a,1]$ such that $([a,1],\oplus,\cdot)$ is a ring. From Lemma 2, $x \oplus (-x) = 0$ for all $x \in [a,|a|]$. Since $([a,1],\oplus)$ is a group, it follows that $\{1 \oplus x \mid x \in (0,\frac{1}{2}a^2]\}$ is an infinite subset of $[a,1]$. Then there is an element $c \in (0,\frac{1}{2}a^2]$ such that $1 \oplus c \neq 1$ and $1 \oplus c \neq a$. Thus $-1 \leqslant a < 1 \oplus c < 1$. It follows that $(1 \oplus c)^n \in [-\frac{1}{2}a^2,\frac{1}{2}a^2]$ for some positive integer n. By the binomial expansion of $(1 \oplus c)^n$ in the ring $([a,1],\oplus,\cdot)$, we have that $(1 \oplus c)^n = 1 \oplus cy$ for some $y \in [a,1]$ (see the proof of Theorem 1). It is clear that $cy \in [-\frac{1}{2}a^2,\frac{1}{2}a^2]$. Now, as both $(1 \oplus c)^n$ and $-cy$ lie in $[-\frac{1}{2}a^2,\frac{1}{2}a^2]$, we have that $[-\frac{1}{2}a^2,\frac{1}{2}|a|] = \frac{1}{2}|a|[a,1]$ is an ideal of the ring $([a,1],\oplus,\cdot)$ containing $[-\frac{1}{2}a^2,\frac{1}{2}a^2]$. This fact yields $(1 \oplus c)^n \oplus (-cy) \in [-\frac{1}{2}a^2,\frac{1}{2}|a|]$. Thus

$$\begin{aligned} 1 = 1 \oplus 0 = 1 \oplus cy \oplus (-cy) &= (1 \oplus c)^n \oplus (-cy) \\ &\in [\frac{a^2}{2},\frac{|a|}{2}] \end{aligned}$$

which is a contradiction. □

Corollary 3 *The multiplicative interval semigroup* $[a,b]$ *with* $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$ *does not admit a ring structure.*

Proof: Assume that there is an operation $\oplus$ on $[a,b]$ such that $([a,b],\oplus,\cdot)$ is a ring.

Case 1 $|a| \leqslant b$. Then $-1 \leqslant a/b < 0 < (a/b)^2 \leqslant 1$. Define an operation $\oplus'$ on $[a/b,1]$ by

$$x \oplus' y = \frac{bx \oplus by}{b} \quad \text{for all } x,y \in [a/b,1].$$

It can be shown in a similar way to the proof of Corollary 1 that $([a/b,1],\oplus',\cdot)$ is a ring which is contrary to Theorem 2.

Case 2 $|a| > b$. Then $a[a,b] = [ab,a^2]$ is an ideal of the ring $([a,b],\oplus,\cdot)$ and $|ab| < a^2$. From Case 1, this is a contradiction. Hence the result follows. □

Corollary 4 *If* S *is a multiplicative interval semigroup on* $\mathbb{R}$ *of one of the types:*

$(a,b]$ *or* (a,b) *where* $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$
$[a,b)$ *where* $-1 < a < 0 < a^2 < b \leqslant 1$,

then S *does not admit a ring structure.*

Proof: Assume that there is an operation $\oplus$ on S such that $(S,\oplus,\cdot)$ is a ring. Let $d \in (0,b)$, $k = b^2 - bd$ and $m = \max\{|a|,b\}$. Then $k > 0$ and $m > 0$. Let ϵ be such that $0 < \epsilon < k$. Since

$$0 < d + \frac{\epsilon}{m} < d + \frac{b^2 - bd}{m} \leqslant d + \frac{b^2 - bd}{b} = b,$$

we have that $d + \epsilon/m \in S$. Hence $(d+\epsilon/m)S$ is an ideal of the ring $(S,\oplus,\cdot)$. Also, we have

$$\left(d + \frac{\epsilon}{m}\right) S = \begin{cases} \left((d+\frac{\epsilon}{m})a, (d+\frac{\epsilon}{m})b\right], & S = (a,b], \\ \left((d+\frac{\epsilon}{m})a, (d+\frac{\epsilon}{m})b\right), & S = (a,b), \\ \left[(d+\frac{\epsilon}{m})a, (d+\frac{\epsilon}{m})b\right), & S = [a,b), \end{cases}$$

and

$$\begin{aligned} da &> \left(d + \frac{\epsilon}{m}\right) a \geqslant (d + \frac{\epsilon}{|a|})a = da - \epsilon, \\ db &< \left(d + \frac{\epsilon}{m}\right) b \leqslant \left(d + \frac{\epsilon}{b}\right) b = db + \epsilon. \end{aligned}$$

These imply that $[da,db] \subseteq (d+\epsilon/m)S \subseteq [da-\epsilon,da+\epsilon]$ for all ϵ with $0 < \epsilon < k$. Hence

$$\begin{aligned} [da,db] &\subseteq \bigcap_{0<\epsilon<k} (d+\epsilon/m)S \\ &\subseteq \bigcap_{0<\epsilon<k} [da-\epsilon,da+\epsilon] \\ &= [da,db]. \end{aligned}$$

Consequently, $[da,db] = \bigcap_{0<\epsilon<k}(d+\epsilon/m)S$ is an ideal of the ring $(S,\oplus,\cdot)$. This is contrary to Corollary 3. □

Remark 1 Let $0 < a \leqslant 1$. We define a mapping φ by $\varphi(x) = \frac{1}{x}$ if $x \neq 0$ and $\varphi(0) = 0$. Then φ is clearly an isomorphism between the pairs of interval semigroups $(0,a]$, $[\frac{1}{a},\infty)$ and $(0,a)$, $(\frac{1}{a},\infty)$. Hence from Corollary 1 and Corollary 2 we have that the multiplicative unbounded interval semigroups $[b,\infty)$, (b,∞) where $b \geqslant 1$ do not admit a ring structure. In particular, the semigroup $([1,\infty),\cdot)$ does not admit a ring structure. Note that Chu and Shyr[4] showed that the semigroup $(\mathbb{N},\cdot)$ admits a ring structure where $\mathbb{N} = \{1,2,3,\dots\}$. The proof was given by showing that $(\mathbb{N}\cup\{0\},\cdot) \cong (\mathbb{Z}_2[x],\cdot)$. From this fact, one can see that the multiplicative semigroup $\mathbb{Q}^+$ of positive rational numbers admits a ring structure as follows. If $(\mathbb{N}\cup\{0\},\oplus,\cdot)$ is a ring, then $(\mathbb{Q}^+\cup\{0\},\oplus',\cdot)$ is a ring where

$$\frac{a}{b} \oplus' \frac{c}{d} = \frac{ad \oplus bc}{bd} \quad \text{for all } a,c \in \mathbb{N}\cup\{0\} \text{ and } b,d \in \mathbb{N}.$$

Remark 2 From the above proofs and some simple modifications, one can see that the following statement holds. If F is a subfield of the field $\mathbb{R}$ of real numbers and I is a nontrivial multiplicative bounded interval semigroup on $\mathbb{R}$ with $\inf I, \sup I \in F$, then the multiplicative semigroup $I \cap F$ does not admit a ring structure. In addition, if $b \in F$ and $b \geqslant 1$, then the multiplicative semigroups $[b, \infty) \cap F$ and $(b, \infty) \cap F$ do not admit a ring structure.

Remark 3 We note here that Pearson[8] classified continuous semirings on intervals of $\mathbb{R}$. Rings are particular cases of semirings. Then Pearson's classification shows that none of the multiplicative interval semigroups considered in this paper admits the structure of a continuous ring.

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REFERENCES

1. Peinado RE (1970) On semigroups admitting ring structure. *Semigroup Forum* **1**, 189–208.
2. Satayanarayana M (1971) On semigroups admitting ring structure. *Semigroup Forum* **3**, 43–50.
3. Satayanarayana M (1973) On semigroups admitting ring structure II. *Semigroup Forum* **6**, 187–9.
4. Chu DD, Shyr HJ (1980) Monoids of languages admitting ring structure. *Semigroup Forum* **19**, 127–32.
5. Srichaiyarat S (1981) Generalized transformation semigroups admitting ring structure. MSc thesis, Chulalongkorn Univ.
6. Kemprasit Y, Siripitukdet M (2002) Matrix semigroups admitting ring structure. *Bull Calcutta Math Soc* **94**, 409–12.
7. Lawson LJM (1969) The multiplicative semigroup of a ring. PhD thesis, Univ of Tennessee.
8. Pearson KR (1966) Interval semirings on R_1 with ordinary multiplication. *J Aust Math Soc* **6**, 273–88.