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On the strong law of large numbers for pairwise negative quadrant dependent identically distributed random variables with infinite means

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ABSTRACT: Kruglov has recently given a strong law of large numbers for identically distributed random variables with infinite means. He improved the work of Feller by assuming only pairwise independence of random variables. In this note we relax the condition from pairwise independence to pairwise negative quadrant dependence.

KEYWORDS: pairwise independence

INTRODUCTION

Let (X_n) be a sequence of identically distributed random variables. We shall say that (X_n) obeys the strong law of large numbers (SLLN) with respect to a sequence of positive numbers (a_n) if

$$\frac{S_n}{a_n} \xrightarrow{\text{a.s.}} \mu \text{ as } n \to \infty,$$

i.e.,

$$P\left(\lim_{n\to\infty} \frac{S_n}{a_n} = \mu\right) = 1$$

where $S_n = X_1 + X_2 + \cdots + X_n$ and $E[X_1] = E[X_2] = \cdots = \mu < \infty$. Note that a.s. stands for 'almost surely'. This concept is analogous to the concept of 'almost everywhere' in measure theory.

In the case that the X_n are independent identically distributed random variables with $E[|X_i|] < \infty$, many authors have given the conditions that make the (X_n) satisfy the SLLN.

If $\mu = 0$, the law becomes

$$\frac{S_n}{a_n} \xrightarrow{\text{a.s.}} 0 \text{ as } n \to \infty.$$

Chung (see p. 73 of Ref. 1) shows that

$$\frac{S_n}{a_n} \xrightarrow{\text{a.s.}} 0 \text{ if and only if } P\left(|\frac{S_n}{a_n}| \geqslant \epsilon \text{ i.o.}\right) = 0$$

for all $\epsilon > 0$, where $A = A_n$ i.o. (i.o. = infinitely often), means that the event A_n happens for infinitely many values of n. Formally,

$$A = \limsup_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.$$

Hence, one can say that (X_n) satisfies the SLLN with respect to (a_n) if

$$\frac{S_n}{a_n} \xrightarrow{\text{a.s.}} 0 \text{ as } n \to \infty \text{ or } P\left(\left|\frac{S_n}{a_n}\right| \geqslant \epsilon \text{ i.o.}\right) = 0$$

for all $\epsilon > 0$. This definition is better than the previous one because we do not need the existence of the means of X_i. In this note, we will consider the case of infinite means.

Theorem 1 (Feller[2]) *Assume that*

(a) $a_n > 0$ *and* (a_n/n) *is an increasing sequence,*

(b-1) X_n *are independent identically distributed random variables and* $E[|X_1|] = \infty$.

Then

(i) $P(|S_n| > a_n \text{ i.o.}) = 0$ *if and only if* $\sum_{n=1}^{\infty} P(|X_n| > a_n)$ *converges,*

(ii) $P(|S_n| > a_n \text{ i.o.}) = 1$ *if and only if* $\sum_{n=1}^{\infty} P(|X_n| > a_n)$ *diverges.*

Kruglov[3] improved the work of Feller by assuming only the pairwise independence of the X_n.

Theorem 2 (Kruglov[3]) *Assume that* (a) *holds and that* (b-2) X_n *are pairwise independent identically distributed random variables and* $E[X_1^-] < \infty$, $E[X_1^+] = \infty$.

Then

(i) $(|X_n|)$ *obeys the SLLN w.r.t.* (a_n) *if and only if* $\sum_{n=1}^{\infty} P(X_n > a_n) < \infty$,

(ii) $P(S_n > a_n \text{ i.o.}) = 1$ *if and only if* $\sum_{n=1}^{\infty} P(X_n > a_n) = \infty$.

In this note, we relax the condition from pairwise independence to pairwise negative quadrant dependence (NQD). A sequence of random variables (X_n) is said to be *pairwise negative quadrant dependent* if

$$P(X_i \leqslant x_i, X_j \leqslant x_j) \leqslant P(X_i \leqslant x_i)P(X_j \leqslant x_j)$$

for all $x_i, y_j \in \mathbb{R}$ and for all $i, j \geqslant 1$ and $i \neq j$. This concept of dependence was introduced in Refs. 4–6.

Remark 1 Many authors[7–9] have given a sequence of pairwise independent random variables which are not mutually independent. Examples of pairwise NQD random variables which are not pairwise independent can be found in Ref. 4.

Theorem 3 and Theorem 4 are our results.

Theorem 3 *Assume that* $\lim_{n\to\infty} a_n/n = \infty$ *and*

(b-3) X_n *are pairwise NQD identically distributed random variables and* $E[X_1^-] < \infty$, $E[X_1^+] = \infty$.

Then (X_n) *obeys the SLLN w.r.t.* (a_n) *if and only if* $(|X_n|)$ *obeys the SLLN w.r.t.* (a_n).

Theorem 4 *Assume that* (a) *and* (b-3) *hold. Then*

(i) $(|X_n|)$ *obeys the SLLN w.r.t.* (a_n) *if and only if* $\sum_{n=1}^{\infty} P(X_n > a_n) < \infty$

(ii) $P(S_n > a_n \text{i.o.}) = 1$ *if and only if* $\sum_{n=1}^{\infty} P(X_n > a_n) = \infty$.

SLLN OF NQD-RANDOM VARIABLES WITH FINITE MEANS

To prove our results, we apply the result of the SLLN of NQD-random variables in the case of finite means together with the argument of Kruglov[3].

Proposition 1 (Ebrahimi and Ghosh[10]) *Let* (X_n) *be an NQD sequence. Then the following results are true.*

(i) $(f_n(X_n))$ *is an NQD sequence for any sequence of monotonically increasing functions* (f_n).

(ii) $(f_n(X_n))$ *is an NQD sequence for any sequence of monotonically decreasing functions* (f_n).

(iii) (X_n^+) *and* (X_n^-) *are NQD sequences.*

(iv) $\operatorname{Cov}(X_i, X_j) \leqslant 0$ *for all* $i \neq j$.

Theorem 5 (Matula[11]) *Let* (Ω, F, P) *be a probability space and* (A_n) *a sequence of events.*

(i) *If* $\sum_{n=1}^{\infty} P(A_n) < \infty$, *then* $P(A_n \text{ i.o.}) = 0$.

(ii) *If* $\sum_{n=1}^{\infty} P(A_n) = \infty$ *and* $P(A_k \cap A_m) \leqslant P(A_k)P(A_m)$ *for* $k \neq m$, *then* $P(A_n \text{ i.o.}) = 1$.

To prove Theorem 4, we need the SLLN for NQD random variables in the case of finite variances.

Theorem 6 *Let* (X_n) *be a sequence of NQD and not necessary identically distributed random variables and* $E[X_n^2] < \infty$ *for all* $n \in \mathbb{N}$. *If*

(i) $\sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n} E\left[\left|X_k - E[X_k]\right|\right] < \infty$,

(ii) $\sum_{n=1}^{\infty} \frac{\operatorname{Var}[X_n]}{n^2} < \infty$,

then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(X_k - E[X_k]) = 0 \text{ a.s.}$$

Proof: To prove the theorem, we follow the proof of Theorem 1 of Ref. 12. They used the pairwise independence only to show that

$$\operatorname{Var}[\sum_{k=1}^{n} X_k] = \sum_{k=1}^{n} \operatorname{Var}[X_k].$$

In fact, in their proof, they need only the fact that

$$\operatorname{Var}[\sum_{k=1}^{n} X_k] \leqslant \sum_{k=1}^{n} \operatorname{Var}[X_k]. \quad (1)$$

Similarly, (1) holds by using the NQD-property which follows from Proposition 1 (iv). □

Corollary 1 *Let* (X_n) *be an NQD sequence and not necessary identically distributed random variables. If* $E[X_n^2] < \infty$ *for all* $n \in \mathbb{N}$ *and*

(i) $\sup_{n\in\mathbb{N}} E[|X_n|] < \infty$ *and*

(ii) $\sum_{n=1}^{\infty} \frac{\operatorname{Var}[X_n]}{n^2} < \infty$, *then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(X_k - E[X_k]) = 0 \text{ a.s.}$$

Proof: It follows from Theorem 6 and the fact that

$$\sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{k=1}^{n}E\Big[\Big|X_k - [EX_k]\Big|\Big] \leqslant 2\sup_{n\in\mathbb{N}}E[|X_n|].$$

□

Corollary 2 *Let* (X_n) *be a sequence of NQD identically distributed random variables with* $E[|X_1|] < \infty$. *Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(X_k - E[X_k]) = 0 \text{ a.s.}$$

Proof: Use Corollary 1 and the fact that

$$\sum_{n=1}^{\infty}\frac{\mathrm{Var}[X_n]}{n^2} \leqslant \sum_{n=1}^{\infty}\frac{E[X_1^2]}{n^2} < \infty.$$

□

Observe that Corollary 1 and Corollary 2 are the main results of Azarnoosh[13].

PROOF OF MAIN RESULTS

Proof of Theorem 3: It is obvious that if $(|X_n|)$ obeys the SLLN w.r.t. (a_n), then (X_n) obeys the SLLN w.r.t. (a_n) too. Then we assume that (X_n) obeys the SLLN w.r.t. (a_n). First we will show that

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}X_k^- = 0 \text{ a.s.} \tag{2}$$

for all (a_n) such that

$$\lim_{n\to\infty}\frac{a_n}{n} = \infty.$$

We note that if (X_n) is a sequence of identically distributed random variables, then (X_n^+) and (X_n^-) are also. Since the X_i^- are pairwise NQD and $E[X_1^-] = E[X_2^-] = \cdots < \infty$, by Corollary 2,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}X_k^- = E[X_1^-] \text{ a.s.}$$

Hence,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}X_k^- &= \lim_{n\to\infty}\left(\frac{n}{a_n}\cdot\frac{1}{n}\sum_{k=1}^{n}X_k^-\right)\\ &= \Big(\lim_{n\to\infty}\frac{n}{a_n}\Big)\Big(\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}X_k^-\Big)\\ &= 0 \text{ a.s.}\end{aligned}$$

We have that

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}X_k = 0 \quad \text{a.s.}$$

and

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}X_k^- = 0 \quad \text{a.s.}$$

implies that

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}X_k^+ = 0 \quad \text{a.s.}$$

Hence

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}|X_k| = \lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}(X_k^- + X_k^+) = 0 \text{ a.s.}$$

This completes the proof. □

Proof of Theorem 4: (i) Assume that $(|X_n|)$ obeys the SLLN w.r.t. (a_n). Since

$$P\left(\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}|X_k| = 0\right) = 1,$$

by the result on p. 73 of Ref. 1, $P(|X_1|+\cdots+|X_n| > a_n \text{ i.o.}) = 0$. This implies that $P(|X_n| > a_n \text{ i.o.}) = 0$. Suppose that

$$\sum_{n=1}^{\infty}P(X_n > a_n) = \infty$$

and let $A_n = \{X_n > a_n\}$. Then by Theorem 5 (ii), $P(X_n > a_n \text{ i.o.}) = 1$ which implies that $P(|X_n| > a_n \text{ i.o.}) = 1$. This is a contradiction. Hence, $\sum_{n=1}^{\infty}P(X_n > a_n) < \infty$. On the other hand, we assume that $\sum_{n=1}^{\infty}P(X_n > a_n) < \infty$. This implies

$$\lim_{n\to\infty}\frac{a_n}{n} = \infty$$

by the result on p. 891 of Ref. 3. This with (a) and $E[X_1^-] < \infty$ implies

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}X_k^- = 0 \text{ a.s.}$$

In order to prove that $(|X_n|)$ obeys the SLLN w.r.t. (a_n), it suffices to show that

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}X_k^+ = 0 \text{ a.s.} \tag{3}$$

To do this, let

$$Y_n = \frac{n}{a_n} X_n^+ \mathbb{I}(X_n^+ \leqslant 2a_n), \quad Z_n = 2n\mathbb{I}(X_n^+ > 2a_n)$$

and $W_n = Y_n + Z_n$, where $\mathbb{I}$ is the indicator function. Kruglov[3] showed that

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}[Y_n]}{n^2} < \infty, \tag{4}$$

$$\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} E\Big[\Big|Y_k - E[Y_k]\Big|\Big] < \infty, \tag{5}$$

and that if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (Y_k - E[Y_k]) = 0 \text{ a.s.}, \tag{6}$$

then (3) holds. Let

$$f_n(t) = \frac{n}{a_n}\Big(t\mathbb{I}(t \leqslant 2a_n) + 2a_n\mathbb{I}(t > 2a_n)\Big) \text{ and}$$
$$g_n(t) = 2n\mathbb{I}(t > 2a_n).$$

Observe that f_n and g_n are increasing functions, $W_n = f_n(X_n^+)$ and $Z_n = g_n(X_n^+)$. Hence, by Proposition 1 (i), we have that (W_n) and (Z_n) are sequences of pairwise NQD random variables. Next we will show that

$$\frac{1}{n}\sum_{k=1}^{n}(Y_k - E[Y_k]) + \frac{1}{n}\sum_{k=1}^{n}(Z_k - E[Z_k]) = \frac{1}{n}\sum_{k=1}^{n}(W_k - E[W_k]) \xrightarrow{a.s.} 0. \tag{7}$$

By Theorem 6, we need to prove

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} E\Big[\Big|W_k - E[W_k]\Big|\Big] \\
&\leqslant \sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} E\Big[\Big|Y_k - E[Y_k]\Big|\Big] \\
&\quad + \sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} E\Big[\Big|Z_k - E[Z_k]\Big|\Big] \\
&< \infty,
\end{aligned} \tag{8}$$

and

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}[W_n]}{n^2} < \infty. \tag{9}$$

To prove (8), we note that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} E\Big[\Big|Z_k - E[Z_k]\Big|\Big] \\
&\leqslant 2\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} E[Z_k] \\
&= 4\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} kP(X_k^+ > 2a_k) \\
&\leqslant 4\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} P(X_k^+ > a_k) \\
&= 4\sum_{k=1}^{\infty} P(X_k^+ > a_k) \\
&= 4\sum_{k=1}^{\infty} P(X_k > a_k) \\
&< \infty.
\end{aligned} \tag{10}$$

From (5) and (10), we have (8).

To prove (9), we note from the fact $Y_nZ_n = 0$ for all $n \in \mathbb{N}$ that

$$\begin{aligned}
\mathrm{Var}[W_n] &= \mathrm{Var}[Y_n + Z_n] \\
&= \mathrm{Var}[Y_n] + \mathrm{Var}[Z_n] + 2\,\mathrm{Cov}(Y_n, Z_n) \\
&= \mathrm{Var}[Y_n] + \mathrm{Var}[Z_n] - 2E[Y_n]E[Z_n] \\
&\leqslant \mathrm{Var}[Y_n] + \mathrm{Var}[Z_n].
\end{aligned}$$

Hence, by (4) and the fact that

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}[Z_n]}{n^2} \leqslant \sum_{n=1}^{\infty} \frac{E[Z_n^2]}{n^2} = 4\sum_{n=1}^{\infty} P(X_n^+ > 2a_n) < \infty, \tag{11}$$

we have

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}[W_n]}{n^2} \leqslant \sum_{n=1}^{\infty} \frac{\mathrm{Var}[Y_n]}{n^2} + \sum_{n=1}^{\infty} \frac{\mathrm{Var}[Z_n]}{n^2} < \infty.$$

Hence, (9) holds.

From (10) and (11), by Theorem 6,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (Z_k - E[Z_k]) = 0 \text{ a.s.} \tag{12}$$

From (7) and (12),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (Y_k - E[Y_k]) = 0 \text{ a.s.}$$

Hence, by Kruglov's result that if (6) then (3) holds,

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} X_k^+ = 0 \text{ a.s.}$$

(ii) The "pairwise independence" condition was needed in two places in the proof of Kruglov[3]. These were in the proof that $P(A_n \text{ i.o.}) = 1$, and when applying Etemadi theorem[14]. In our proof, which closely follows that of Kruglov, we need to avoid using this condition. In the first place, we instead apply Theorem 5(ii) by setting $A_n = \{w | X_n(w) > a_n\}$. Since X_n's are NQD, $P(A_k \cap A_m) \leqslant P(A_k)P(A_m)$ for all $k \neq m$. Then $P(A_n \text{ i.o.}) = 1$. In the second place, we avoid applying the Etemadi theorem by using Corollary 2. Then we follow the proof of Theorem 2 (ii) and the theorem is proved. □

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