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H_∞ optimal control of linear time-varying systems with time-varying delay via a controllability approach

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ABSTRACT: This paper addresses the H_∞ optimal control problem for a class of uncertain linear time-varying delay systems. The interesting features here are that the system in consideration is non-autonomous, the state delay is time-varying, and the controllers to be designed satisfy some exponential stability constraints on the closed-loop poles. Based on the Lyapunov-Krasovskii functional method, we show that the H_∞ optimal control problem for the system has a solution if some appropriate linear control delay-like system is globally controllable.

KEYWORDS: robust control, exponential stability, time delay, Riccati equation, Lyapunov functional

INTRODUCTION

In recent decades, considerable attention has been devoted to the problem of state estimation. When a priori information on external noises is not precisely known, the celebrated Kalman filtering scheme is no longer applicable. In such cases, H_∞ filtering can be used[1–4]. With H_∞ filtering the input signal is assumed to be energy bounded and the main objective is to minimize the H_∞ norm of the filtering error system. Other norms introduced for systems with uncertainties are L_2 and L_1, which have different physical meanings when used as performance indexes.

Time delays are frequently encountered in various engineering systems such as aircraft, long transmission lines in pneumatic models, and chemical or process control systems because of the time taken for transmission of measurement information. As these delays may be the source of instability and serious deterioration in the performance of closed-loop systems, the H_∞ control problem of systems with time delays has received considerable attention from many researchers in the last decade.

A significant new development in H_∞ optimal control theory has been the introduction of state-space methods. This has led to a rather transparent solution to the standard problem of H_∞ control theory, which is to find a feedback controller stabilizing a given system that satisfies some normed suboptimal level on perturbations/uncertainties (see, e.g., Refs. 5, 6). In the H_∞ control for time-invariant delay systems, the corresponding methods make use of the Lyapunov-Krasovskii functional approach and the sufficient conditions are obtained via solving either linear matrix inequalities, or algebraic Riccati-type equations[7,8]. However, this approach may not be readily applied for systems with time-varying parameters, which are frequently encountered in process control, filtering, and mobile communication systems. The difficulty is that the solution of a Riccati-type differential equation is, in general, not uniformly positive definite as is required for use in a Lyapunov-Krasovskii functional candidate. Hence the stability analysis becomes more complicated, and in particular when the system delay and uncertainties are also time-varying. Some results on the stabilization of linear time-varying (LTV) systems have been tackled in Refs. 6, 9, but without considering time delays. To find a H_∞ controller for LTV systems, the state-space approach is used in Refs. 6, 10 to derive sufficient conditions for the H_∞ control problem in terms of the solution of some Riccati differential equations. Based on the assumption of uniform controllability of the nominal control systems, some sufficient conditions for H_∞ of LTV systems were obtained in Refs. 7, 11.

To the best of our knowledge, this paper is the first to present a unified approach that addresses the problem of the H_∞ control problem for a class of LTV systems subject to time-varying state delay, system uncertainties, and an external disturbance. We consider the time-varying case of time-varying delays and norm-bounded time-varying uncertainties in the state and input matrices. By using the Lyapunov-Krasovskii functional method, we show that the H_∞

control problem has a solution if the appropriate linear control delay-like system is globally null-controllable in finite time. The feedback stabilizing controller is designed via the solution of matrix Riccati-type equations.

PRELIMINARIES

The following notation will be used throughout this paper. R^+ denotes the set of all non-negative real numbers, R^n denotes a n-dimensional Euclidean space with the scalar product $\langle\cdot,\cdot\rangle$, $L_2([t,\infty),R^n)$ denotes the set of all strongly measurable L_2-integrable R^n-valued functions on $[t,\infty)$, and I denotes the identity matrix. A matrix $Q \in M^{n\times n}$ is called non-negative definite ($Q \geqslant 0$) if $\langle Qx,x\rangle \geqslant 0$, for all $x \in R^n$. If for some $c > 0$ we have $\langle Qx,x\rangle \geqslant c\|x\|^2$ for all $x \in R^n$, then Q is called positive definite ($Q > 0$), and $A \geqslant B$ means $A - B \geqslant 0$. A matrix function $Q(t)$ is uniformly positive definite ($Q(t) \gg 0$) if

$$\exists\, c > 0:\ \langle Q(t)x,x\rangle \geqslant c\|x\|^2,\ \forall\,(t,x) \in R^+ \times R^n.$$

It is well known that if the matrix A is symmetric positive definite, then there is a matrix B such that $A = B^2$ and the matrix B is usually defined by $B = A^{\frac{1}{2}}$. Let $BM^+(0,\infty)$ denote the set of all symmetric non-negative definite matrix functions which are continuous and bounded on R^+, let $BMU^+(0,\infty)$ denote the set of all symmetric uniformly positive definite matrix functions which are continuous and bounded on R^+, and let $C([a,b],R^n)$ denote the set of all R^n-valued continuous functions on $[a,b]$.

Consider the following uncertain LTV system with time-varying delay:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + A_1(t)x(t-h(t))\\ &\quad + B(t)u(t) + B_1(t)w(t),\ t \in R^+,\\ z(t) &= C(t)x(t) + D(t)u(t),\\ x(t) &= \phi(t), t \in [-h,0],\end{aligned} \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control, $w \in R^p$ is the uncertain input, $z \in R^q$ is the observation output, $A(t)$, $A_1(t)$, $B(t)$, $B_1(t)$, $C(t)$, and $D(t)$ are given matrix functions continuous and bounded on R^+, and $\phi(t) \in C[-h,0]$ with the norm $\|\phi\| = \sup_{t\in[-h,0]}\|\phi(t)\|$. The time-delay function $h(t) \in C[-h,0]$ satisfies the condition

$$0 \leqslant h(t) \leqslant h, \quad \dot{h}(t) \leqslant \delta < 1, \quad \forall\, t \in R^+.$$

We say that the control u is admissible if $u \in L_2([0,s],R^m)$ for every $s \geqslant 0$, and the uncertainty w is admissible if $w \in L_2([0,\infty),R^p)$. Let x_t be the segment of the trajectory of $x(t)$ with the norm $\|x_t\| = \sup_{s\in[-h,0]}\|\|x(t+s)\|$.

Definition 1 Linear control system (1), where $w(t) = 0$, is exponentially stabilizable if there exists a feedback control $u(t) = K(t)x(t)$, such that the zero solution of the closed-loop delay system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) + A_1(t)x(t-h(t)), \tag{2}$$

is exponentially stable in the Lyapunov sense, i.e.

$$\exists\, N > 0, \alpha > 0 : \|x(t,\phi)\| \leqslant N\|\phi\|\mathrm{e}^{-\alpha t},$$

for all $t \geqslant 0$.

In this paper, we consider the following H_∞ optimal control problem with nonzero initial condition[8,12].

Definition 2 Given $\gamma > 0$, the H_∞ optimal control problem for the system (1) has a solution if there is a feedback control $u(t)K(t)x(t)$ such that (i) the system (1), where $w(t) = 0$, is exponentially stabilizable, (ii) there is a number $c_0 > 0$ such that

$$\sup \frac{\int_0^\infty \|z(t)\|^2\,\mathrm{d}t}{c_0\|\phi\|^2 + \int_0^\infty \|w(t)\|^2\,\mathrm{d}t} \leqslant \gamma, \tag{3}$$

where the supremum is taken over all initial states ϕ and non-zero admissible uncertainties $w(t)$. In this case we say that the feedback control $u(t) = K(t)x(t)$ exponentially stabilizes the system (1).

We recall the concept of global controllability from Ref. 13 which is concerned with the possibility of steering any state to another state of the system in finite time. We will be considering the following linear time-varying control system, briefly denoted by $[A(t),B(t)]$,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in R^+. \tag{4}$$

Definition 3 System (1) is globally null-controllable in finite time if for every initial state x_0, there exist a time $T > 0$ and an admissible control $u(t)$ such that the solution $x(t)$ of the system satisfies $x(0) = x_0$, $x(T) = 0$.

Proposition 1 (Klamka[14]) *Assume that the matrix functions $A(t), B(t)$ are analytic on R^+. The system $[A(t),B(t)]$ is globally null controllable in finite time if*

$$\exists\, t_0 > 0 : \operatorname{rank}[M_1(t_0), M_2(t_0), \dots, M_n(t_0)] = n, \tag{5}$$

where $M_1(t) = B(t)$ and

$$M_k(t) = -A(t)M_{k-1}(t) + \frac{\mathrm{d}}{\mathrm{d}t}M_{k-1}(t),$$

for $k = 2,\dots,n-1$.

Associated with the control system (4), we consider the following matrix Riccati equation

$$\begin{aligned}\dot{P}(t)+&A^{\mathrm{T}}(t)P(t)+P(t)A(t)\\&-P(t)B(t)B^{\mathrm{T}}(t)P(t)+Q(t)=0.\end{aligned}\tag{6}$$

Proposition 2 (Kalman et al[13]) *Assume that the system $[A(t), B(t)]$ is globally null-controllable in finite time. Then for any matrix $Q \in BM^+(0,\infty)$, there is a solution $P \in BM^+(0,\infty)$ to (6).*

Proposition 3 (Cauchy matrix inequality) *Let Q, S be symmetric matrices of appropriate dimensions and $S > 0$. Then*

$$2\langle Qy, x\rangle - \langle Sy, y\rangle \leqslant \langle QS^{-1}Q^{\mathrm{T}}x, x\rangle, \quad \forall\,(x, y).$$

The proof of the above proposition is easily derived from completing the square.

Proposition 4 *For any symmetric matrix function $A(t)$ bounded on R^+, there exists $Q \in BM^+(0,\infty)$ such that $Q(t) - A(t) \geqslant 0$.*

Proof: The matrix $Q(t)$ may be chosen as

$$Q(t) = \mathrm{diag}\{q_1(t), q_2(t), \dots, q_n(t)\},$$

where $q_i(t) \geqslant \max\{|q_i^0(t)|, 0\}$ and

$$q_i^0(t) = a_{ii}(t) + \frac{1}{4}\sum_{j\neq i}^{n} a_{ij}^2(t) + n - 1,$$

for $i = 1, \dots, n$. From this it is straightforward to show that $Q(t) - A(t) \geqslant 0$. □

Proposition 5 (Lyapunov stability theorem[15]) *Consider the functional differential equation $\dot{x} = f(t, x_t)$, with $x(t) = \phi(t)$ when $t \in [-h, 0]$. If there is a function $V(t, x_t)$ and positive numbers λ_i, $i = 1, 2, 3$ such that the solution $x(t)$ obeys $\lambda_1\|x(t)\|^2 \leqslant V(t, x_t) \leqslant \lambda_2\|x_t\|^2$ for $t \in R^+$, and $\dot{V}(x(t)) \leqslant -\lambda_3\|x(t)\|^2$, then the zero solution is asymptotically stable.*

MAIN RESULT

In this section we will omit the variable t of matrix functions if it does not cause any confusion. Consider the linear control system (1) where, as in Refs. 8, 12, we assume that

$$D^{\mathrm{T}}(t)[D(t) \quad C(t)] = [I \quad 0], \quad t \geqslant 0. \tag{7}$$

Given $\gamma > 0$, $\mu = 1/(1-\delta)$, we set

$$\begin{aligned}A_\gamma(t) &= A(t) + B_1(t)B_1^{\mathrm{T}}(t)/\gamma\\&\quad + \mu A_1(t)A_1^{\mathrm{T}}(t) - B(t)B^{\mathrm{T}}(t),\\B_\gamma(t) &= [B(t)B^{\mathrm{T}}(t) - B_1(t)B_1^{\mathrm{T}}(t)/\gamma\\&\quad - \mu A_1(t)A_1^{\mathrm{T}}(t)]^{\frac{1}{2}}.\end{aligned}$$

The main result is stated in the following.

Theorem 1 *Assume that for $t \geqslant 0$,*

$$B(t)B^{\mathrm{T}}(t) - \frac{1}{\gamma}B_1(t)B_1^{\mathrm{T}}(t) - \mu A_1(t)A_1^{\mathrm{T}}(t) \geqslant 0,$$

and linear control system $[A_\gamma(t), B_\gamma(t)]$ is globally null-controllable in finite time. Then the H_∞ optimal control problem for the system (1) has a solution.

The following lemma is needed for the proof of Theorem 1.

Lemma 1 *The H_∞ optimal control problem for the system (1) has a solution if there exist matrix functions $X, R \in BMU^+(0,\infty)$ such that the following matrix inequality holds*

$$\begin{aligned}\dot{X}+&A^{\mathrm{T}}X + XA - X[BB^{\mathrm{T}} - 1/\gamma B_1B_1^{\mathrm{T}}\\&- \mu A_1A_1^{\mathrm{T}}]X + C^{\mathrm{T}}C + I + R \leqslant 0, \quad t \geqslant 0.\end{aligned}\tag{8}$$

The feedback control is

$$u(t) = -B^{\mathrm{T}}(t)X(t)x(t), \quad t \geqslant 0. \tag{9}$$

Proof: Using (9), we consider the following Lyapunov function for the closed-loop system (2), where $w(t) = 0$, $K(t)x(t) = -B^{\mathrm{T}}(t)X(t)x(t)$:

$$V(t, x_t) = \langle X(t)x(t), x(t)\rangle + \int_{t-h(t)}^{t} \|x(s)\|^2\,\mathrm{d}s.$$

Since $X \gg 0$, there is a positive number λ_1 such that

$$\lambda_1\|x(t)\|^2 \leqslant V(t, x_t), \quad t \geqslant 0.$$

On the other hand, since the matrix function $X(t)$ is bounded on R^+ and $\|x(t)\| \leqslant \|x_t\|$, we have

$$V(t, x_t) \leqslant (\sup_{t\in R^+}\|X(t)\| + h)\|x_t\|^2, \quad t \geqslant 0,$$

and hence

$$\lambda_1\|x(t)\|^2 \leqslant V(t, x) \leqslant \lambda_2\|x_t\|^2, \quad \forall\, t \geqslant 0, \tag{10}$$

where $\lambda_2 = \sup_{t\in R^+} \|X(t)\| + h$. Taking the derivative of $V(t, x_t)$ along the solution $x(t)$ of the closed-loop system, we have

$$\begin{aligned}\dot{V}(t,x_t) &= \langle \dot{X}(t)x(t), x(t)\rangle + 2\langle X(t)\dot{x}(t), x(t)\rangle \\ &\quad + \|x(t)\|^2 - (1-\dot{h}(t)\|x(t-h(t))\|^2 \\ &= \langle (\dot{X} + A^{\mathrm{T}}X + XA + I)x(t), x(t)\rangle \\ &\quad + 2\langle XBu(t), x(t)\rangle \\ &\quad + 2\langle XA_1x(t-h(t)), x(t)\rangle \\ &\quad - (1-\dot{h}(t)\|x(t-h(t))\|^2 \\ &\leqslant \langle (\dot{X} + A^{\mathrm{T}}X + XA + I)x(t), x(t)\rangle \\ &\quad - 2\langle XBB^{\mathrm{T}}Xx(t), x(t)\rangle \\ &\quad + 2\langle XA_1x(t-h(t)), x(t)\rangle \\ &\quad - (1-\delta)\|x(t-h(t))\|^2\end{aligned}$$

Using Proposition 1 we have

$$\begin{aligned}2\langle XA_1x(t-h(t)), x(t)\rangle - (1-\delta)\|x(t-h(t))\|^2 \\ \leqslant \mu\langle XA_1A_1^{\mathrm{T}}x(t), x(t)\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{V}(t,x_t) &\leqslant \langle (\dot{X} + A^{\mathrm{T}}X + XA + I)x(t), x(t)\rangle \\ &\quad - 2\langle XBB^{\mathrm{T}}Xx(t), x(t)\rangle \\ &\quad + \mu\langle XA_1A_1^{\mathrm{T}}x(t), x(t)\rangle.\end{aligned}$$

Taking the matrix inequality (8) into account, we have

$$\begin{aligned}\dot{V}(t,x_t) &\leqslant -\langle XB_1B_1^{\mathrm{T}}Xx(t), x(t)\rangle/\gamma \\ &\quad - \langle XBB^{\mathrm{T}}Xx(t), x(t)\rangle \\ &\quad - \langle C^{\mathrm{T}}Cx(t), x(t)\rangle - \langle Rx(t), x(t)\rangle.\end{aligned} \tag{11}$$

Since $\langle C^{\mathrm{T}}Cx(t), x(t)\rangle \geqslant 0$, $\langle XBB^{\mathrm{T}}Xx(t), x(t)\rangle \geqslant 0$, $\langle XB_1B_1^{\mathrm{T}}Xx(t), x(t)\rangle \geqslant 0$, and $R(t) \gg 0$, from (11) it follows that there is a number $\lambda_3 > 0$ such that

$$\dot{V}(t, x(t)) \leqslant -\lambda_3\|x(t)\|^2, \quad \forall\, t \in R^+.$$

Therefore, by Proposition 5, the system is asymptotically stable. To determine the exponential factors, from (10) we have

$$\dot{V}(t,x_t) \leqslant -(\lambda_3/\lambda_2)V(t,x_t), \quad \forall\, t \geqslant 0.$$

Therefore,

$$V(t,x_t) \leqslant V(0,x_0)\mathrm{e}^{-(\lambda_3/\lambda_2)t}, \quad t \geqslant 0.$$

Applying the inequality (10) again gives

$$\begin{aligned}\|x(t,\phi)\| &\leqslant \sqrt{V(0,x_0)/\lambda_1}\,\mathrm{e}^{-(\lambda_3/2\lambda_2)t} \\ &\leqslant N\|\phi\|\mathrm{e}^{-(\lambda_3/2\lambda_2)t}\end{aligned}$$

where $N = \sqrt{\lambda_{\max}(X(0)) + h)/\lambda_1}$. The last inequality implies that the closed-loop system is exponentially stable, i.e., the system is exponentially stabilizable. To complete the proof of the lemma, it remains to show the γ-suboptimal condition (3). For this we consider the relation

$$\begin{aligned}&\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]\,\mathrm{d}s \\ &\quad = \int_0^t \Big[\|z(s)\|^2 - \gamma\|w(s)\|^2 \\ &\qquad + \dot{V}(s,x_s)\Big]\,\mathrm{d}s - \int_0^t \dot{V}(s,x_s)\,\mathrm{d}s,\end{aligned}$$

where $\dot{V}(t,x_t)$ is estimated as

$$\begin{aligned}\dot{V}(t,x_t) &\leqslant -\lambda_3\|x(t)\|^2 - \langle C^{\mathrm{T}}Cx(t), x(t)\rangle \\ &\quad - \langle XBB^{\mathrm{T}}Xx(t), x(t)\rangle \\ &\quad - 1/\gamma\langle XB_1B_1^{\mathrm{T}}Xx(t), x(t)\rangle \\ &\quad + 2\langle XB_1w(t), x(t)\rangle.\end{aligned} \tag{12}$$

Since $V(t,x_t) \geqslant 0$, we have

$$\begin{aligned}\int_0^t \dot{V}(s,x_s)\,\mathrm{d}s &= V(t,x(t)) - V(0,x(0)) \\ &\geqslant -V(0,x(0)) := -\langle X(0)x_0, x_0\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]\,\mathrm{d}s \leqslant \int_0^t \Big[\|z(s)\|^2 \\ &\quad - \gamma\|w(s)\|^2 + \dot{V}(s,x_s)\Big]\,\mathrm{d}s + \langle X(0)x_0, x_0\rangle.\end{aligned} \tag{13}$$

Taking the estimation of $\dot{V}(s,x_s)$ from (12) and using (7) for

$$\|z(t)\|^2 = \langle [C^{\mathrm{T}}C + XBB^{\mathrm{T}}X]x, x\rangle,$$

we obtain

$$\begin{aligned}&\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]\,\mathrm{d}s \leqslant \int_0^t \Big[[-\lambda_3\|x(s)\|^2 \\ &-\langle XB_1B_1^{\mathrm{T}}Xx(s), x(s)\rangle/\gamma + 2\langle XB_1w(s), x(s)\rangle \\ &\qquad -\gamma\langle w(s), w(s)\rangle\Big]\,\mathrm{d}s + \langle X(0)x_0, x_0\rangle.\end{aligned}$$

Applying Proposition 3 gives

$$2\langle XB_1w, x\rangle - \gamma\langle w, w\rangle \leqslant \langle XB_1B_1^{\mathrm{T}}Xx, x\rangle/\gamma.$$

Then,

$$\int_0^t [\|z\|^2 - \gamma\|w\|^2]\,\mathrm{d}s \leqslant -\lambda_3 \int_0^t \|x(s)\|^2\,\mathrm{d}s$$
$$+\langle X(0)x_0, x_0\rangle \leqslant \langle X(0)x_0, x_0\rangle, \forall\, t \in R^+.$$

Letting $t \to \infty$ we finally obtain that

$$\int_0^\infty \|z(t)\|^2 - \gamma\|w(t)\|^2]\,\mathrm{d}t \leqslant \langle X(0)x_0, x_0\rangle,$$

and hence

$$\int_0^\infty [\|z(t)\|^2\,\mathrm{d}t \leqslant \gamma\left\{\int_0^\infty \|w(t)\|^2\,\mathrm{d}t + (\|X(0)\|/\gamma)\|\phi\|^2\right\}.$$

Setting $c_0 = \|X(0)\|/\gamma$, we note that $\|X(0)\| \neq 0$ because $X \gg 0$, and hence $c_0 > 0$. From the last inequality we have

$$\frac{\int_0^\infty \|z(t)\|^2\,\mathrm{d}t}{c_0\|\phi\|^2 + \int_0^\infty \|w(t)\|^2\,\mathrm{d}t} \leqslant \gamma,$$

for all ϕ and non-zero $w(t) \in L_2([0,\infty), W)$. This completes the proof of the lemma. □

We are now in position to prove the main result.

Proof of Theorem 1: Assume that the system $[A_\gamma(t), B_\gamma(t)]$ is globally null controllable in finite time. Using Proposition 4, we find a matrix function $Q \in BM^+(0,\infty)$ such that

$$Q(t) \geqslant A(t) + A^{\mathrm{T}}(t) + C^{\mathrm{T}}(t)C(t) + 2I. \quad (14)$$

By Proposition 2, the matrix Riccati equation

$$\dot{P} + A_\gamma^{\mathrm{T}}P + PA_\gamma - PB_\gamma B_\gamma^{\mathrm{T}}P + Q = 0, \quad (15)$$

has a solution $P \in BM^+(0,\infty)$. We can reformulate (15) as

$$\dot{P} + A^{\mathrm{T}}(P+I) + (P+I)A - (P+I)[BB^{\mathrm{T}} - B_1B_1^{\mathrm{T}}/\gamma - \mu A_1A_1^{\mathrm{T}}](P+I) + Q - (A + A^{\mathrm{T}}) + BB^{\mathrm{T}} - B_1B_1^{\mathrm{T}}/\gamma - \mu A_1A_1^{\mathrm{T}} = 0.$$

Therefore, by taking (14) into account, we obtain

$$\dot{P} + A^{\mathrm{T}}(P+I) + (P+I)A - (P+I)[BB^{\mathrm{T}} - B_1B_1^{\mathrm{T}}/\gamma - \mu A_1A_1^{\mathrm{T}}](P+I)C^{\mathrm{T}}C + [BB^{\mathrm{T}} - B_1B_1^{\mathrm{T}}/\gamma - \mu A_1A_1^{\mathrm{T}}] + 2I \leqslant 0.$$

Putting $X(t) = P(t) + I$ and

$$R(t) = [B(t)B^{\mathrm{T}}(t) - 1/\gamma B_1(t)B_1^{\mathrm{T}}(t) - \mu A_1A_1^{\mathrm{T}}] + I,$$

we see that the matrices $X(t) \gg 0$ and $R(t) \gg 0$ satisfy the matrix Riccati inequality (8) and hence the proof is completed by using Lemma 1. □

Remark 1 From Theorem 1, to verify that a solution of the H_∞ control problem for system (1) exists, it suffices to check the global null-controllability of the linear control delay-like system $[A_\gamma(t), B_\gamma(t)]$. The stabilizing feedback control is defined by

$$u(t) = -B^{\mathrm{T}}(t)[P(t) + I]x(t), \quad t \in R^+,$$

where $P(t)$ is a solution of the matrix Riccati equation (15). The problem of finding solutions of matrix Riccati equations is in general still complicated. However, some efficient approaches to solving this problem can be found, for instance, in Refs. 16, 17 and the references therein.

Example 1 Consider (1) with $h(t) = 0.25\sin^2 t$, and

$$A(t) = \begin{pmatrix} -1.5 - 0.5\mathrm{e}^{-2\sin t} & 1 \\ -1 & -7/4 - 0.5\mathrm{e}^{-2\sin t} \end{pmatrix},$$

$$A_1(t) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix},\ B(t) = B_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$C(t) = \begin{pmatrix} 0.5\mathrm{e}^{-\sin t} & -0.5\mathrm{e}^{-\sin t} \\ -0.5\mathrm{e}^{-\sin t} & 0.5\mathrm{e}^{-\sin t} \end{pmatrix},$$

$$D = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

The assumption (7) holds as $D^{\mathrm{T}}(t)D(t) = I$ and $D^{\mathrm{T}}(t)C(t) = 0$. Since $\delta = 0.5$, for $\gamma = 4$, we have

$$BB^{\mathrm{T}} - 0.25B_1B_1^{\mathrm{T}} - 2A_1A_1^{\mathrm{T}} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0 \end{pmatrix} \geqslant 0,$$

$$B_\gamma = [BB^{\mathrm{T}} - 0.25B_1B_1^{\mathrm{T}} - 2A_1A_1^{\mathrm{T}}]^{1/2} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_\gamma = A - BB^{\mathrm{T}} + 0.25B_1B_1^{\mathrm{T}} + 2A_1A_1^{\mathrm{T}} = \begin{pmatrix} -1 - 0.5\mathrm{e}^{-2\sin t} & 1 \\ -1 & -0.25 - 0.5\mathrm{e}^{-2\sin t} \end{pmatrix},$$

and it is clear that both matrix functions A_γ and B_γ are analytic. Moreover,

$$M_1(t) = B_\gamma(t),\ M_2(t) = -A_\gamma(t)B_\gamma(t)$$

and it is easy to see that there exists $t_0 > 0$ so that the condition (5), namely

$$\mathrm{rank}\,[M_1(t_0),\ M_2(t_0)] = 2$$

holds. Thus, by Proposition 1, the linear control system $[A_\gamma(t),\ B_\gamma(t)]$ is globally null-controllable in finite time and by Theorem 1 the H_∞ optimal control problem for the system has a solution. To find the feedback stabilizing control, we take

$$Q = \begin{pmatrix} 15/4 + \mathrm{e}^{-2\sin t} & -0.5 \\ -0.5 & 7/4 + \mathrm{e}^{-2\sin t} \end{pmatrix}.$$

It is straightforward to show that $Q \in BM^+(0,\infty)$ and

$$Q \geqslant A(t) + A^{\mathrm{T}}(t) + C^{\mathrm{T}}(t)C(t) + 2I.$$

We can find the solution $P(t)$ of Riccati equation (15) as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} > 0.$$

Therefore, the feedback stabilizing control is

$$\begin{aligned} u(t) &= -B^{\mathrm{T}}(t)[P(t) + I]x(t) \\ &= -\begin{pmatrix} 2 & 0 \end{pmatrix} x(t) = -2x_1(t) \end{aligned}$$

where $x(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^{\mathrm{T}}$.

CONCLUSIONS

In this paper, we have shown that the H_∞ optimal control problem for LTV systems with time-varying delay has a solution if the appropriate linear control delay-like system is globally null-controllable in finite time. The feedback stabilizing controller is designed via the solution of matrix Riccati equations.

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