SHORT REPORT

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Regularity-preserving elements of regular rings

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ABSTRACT: Let R be a ring and $a \in R$. A variant of R with respect to a is a ring R under usual addition and multiplication $\circ$ defined by $x \circ y = xay$ for all $x, y \in R$. In this paper, we characterize the regularity-preserving elements of regular rings.

KEYWORDS: variants, rings of linear transformations, rings of integers modulo n

INTRODUCTION

Variants of semigroups were first studied by Hickey[1], although variants of concrete semigroups of relations had earlier been considered by Magil[2,3]. We can see some properties of variants of semigroups in Refs. 1, 4, 5.

In this paper, we give the definition of variants of rings by using the concept of variants of semigroups and we characterize the regularity-preserving elements of regular rings.

REGULARITY-PRESERVING ELEMENTS OF REGULAR RINGS

Let R be a ring and $a \in R$. A new product $\circ$ is defined on R by $x \circ y = xay$ for all $x, y \in R$. Then $(R, +, \circ)$ is a ring. We usually write $(R, +, a)$ rather that $(R, +, \circ)$ to make the element a explicit. The ring $(R, +, a)$ is called a *variant* of R with respect to a.

An element a of a ring R is said to be *regular* if there exists $x \in R$ such that $a = axa$. A ring R is called a *regular ring* if every element of R is regular.

Let R be a ring. An element $a \in R$ is said to be a *regularity-preserving element* of R if the ring $(R, +, a)$ is regular. Denote the set of all regularity-preserving elements of R by $RP(R)$.

Theorem 1 *If R is not regular, then $RP(R)$ is an empty set.*

Proof: Assume that $RP(R)$ is a nonempty set. Then there exists $a \in R$ such that $(R, +, a)$ is regular. Thus, for each $x \in R$, there exists $y_x \in R$ such that $x = x \circ y_x \circ x$. Therefore, for all $x \in R$, $x = x \circ y_x \circ x = xay_xax = x(ay_xa)x$. So x is regular in R. This implies that R is regular, a contradiction. □

Question Let R be a regular ring. Is $RP(R)$ a nonempty set?

The author has not been able to answer this question yet. However, the following theorem is true.

Theorem 2 *If $RP(R)$ is a nonempty set, then $RP(R)$ is a subsemigroup of $(R, \cdot)$.*

Proof: Let $a, b \in RP(R)$ and $x \in R$. Then there exist $y, z, s, t \in R$ such that $x = xayax$, $x = xbzbx$, $a = absba$, and $b = batab$. Thus

$$\begin{aligned} x = xayax &= x(absba)ya(xbzbx) \\ &= x(ab)(sbayaxbz)bx \\ &= x(ab)(sbayaxbz)(batab)x \\ &= x(ab)(sbayaxbzbat)(ab)x. \end{aligned}$$

Therefore x is regular in $(R, +, ab)$. Then $ab \in RP(R)$. Hence $RP(R)$ is a subsemigroup of $(R, \cdot)$. □

Now the author studies regularity-preserving elements of regular rings having an identity.

Let R be a ring with identity 1. An element $a \in R$ is called a *unit* of R if there exist $x, y \in R$ such that $ax = 1 = ya$ (see Ref. 6). It is easy to prove that $x = y$. The following theorem holds.

Theorem 3 *Let R be a regular ring and $a \in R$. If R has an identity 1, then a is a regularity-preserving element of R if and only if a is a unit of R.*

Proof: Assume a is regularity-preserving. Then 1 is a regular element in $(R, +, a)$, so there exists $x \in R$ such that $1 = 1 \circ x \circ 1$. Therefore $1 = 1 \circ x \circ 1 = 1axa1 = axa$. Thus a is a unit of R.

Conversely, suppose that a is a unit of R. Let $b \in R$. Since R is regular, $b = bxb$ for some $x \in R$, and so $b = bxb = b1x1b = b(aa^{-1})x(a^{-1}a)b = ba(a^{-1}xa^{-1})ab$. Therefore b is a regular element in $(R, +, a)$. Hence a is a regularity-preserving element of R. □

The following corollary is obtained directly from Theorem 2 and Theorem 3.

Corollary 1 *Let R be a regular ring. If R has an identity, then $RP(R)$ is a subgroup of $(R,\cdot)$.*

Proof: It follows from Theorem 3 and the fact that the set of all units of R is a group under usual multiplication of R. □

Theorem 4 *Let R be a regular ring. If a is regularity-preserving, then $RbR \subseteq RaR$ for every $b \in R$.*

Proof: Let a be a regularity-preserving element of R. Let $b \in R$. Then there exists $x \in R$ such that $b = b \circ x \circ b = baxab$. Then $b \in RaR$. Therefore, $RbR \subseteq RaR$. □

The following two corollaries can be obtained directly from Theorem 4.

Corollary 2 *Let R be a regular ring. Then $RaR = RbR$ for all $a, b \in RP(R)$.*

Corollary 3 *Let R be a regular ring. If R has an identity, then $RaR = R$ for all $a \in RP(R)$.*

REGULARITY-PRESERVING ELEMENTS OF RINGS OF LINEAR TRANSFORMATIONS

Let V be a vector space over a field F and $L(V)$ be the set of all linear transformations on V. We know that $(L(V),+,\circ)$ is a ring where $\circ$ is a composition of functions[6]. We have that the identity map on V is an identity of a ring $L(V)$. Moreover, $L(V)$ is a regular ring[7]. The following proposition is well-known.

Proposition 1 *$\alpha \in L(V)$ is a unit of $L(V)$ if and only if α is an isomorphism.*

By Theorem 3 and Proposition 1, the following corollary holds.

Corollary 4 $RP(L(V)) = \{\alpha \in L(V) \mid \alpha$ *is an isomorphism*$\}$.

Let F be a field and $M_n(F)$ denote the set of all $n \times n$ matrices on F. It is easy to prove that $(M_n(F),+,\cdot)$ is a ring where $+$ and $\cdot$ is usual addition and usual multiplication of matrices, respectively. Moreover, the identity $n \times n$ matrix on F is an identity of a ring $M_n(F)$. Let V be a vector space over F. If $\dim V = n$, we know that a ring $(M_n(F),+,\cdot)$ is isomorphic to a ring $(L(V),+,\circ)$[6]. Therefore a ring $M_n(F)$ is a regular ring. The following corollary follows from Corollary 4.

Corollary 5 $RP(M_n(F)) = \{A \in M_n(F) \mid A$ *is an invertible matrix*$\}$.

REGULARITY-PRESERVING ELEMENTS OF RINGS $(\mathbb{Z}_n,+,\cdot)$

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all integers and the set of all positive integers, respectively. For $n \in \mathbb{N}$, let $(\mathbb{Z}_n,+,\cdot)$ denote the ring of integers modulo n. For $k \in \mathbb{Z}$, let $\bar{k}$ be the equivalence class of integers modulo n containing k. We have that $\bar{1}$ is an identity of a ring $\mathbb{Z}_n$. The following proposition is well-known[6].

Proposition 2 *Let $\bar{k} \in \mathbb{Z}_n$. Then $\bar{k}$ is a unit of a ring $\mathbb{Z}_n$ if and only if $\gcd(k,n) = 1$.*

Proposition 3 (Ehrlich[8]) *For any $n \in \mathbb{N}$, the ring $(\mathbb{Z}_n,+,\cdot)$ is regular if and only if n is square-free.*

Then the following corollary is true.

Corollary 6 *Let $n \in \mathbb{N}$. If n is not square-free, then $RP(\mathbb{Z}_n)$ is an empty set.*

Proof: It follows from Theorem 1 and Proposition 3. □

Next, let n be a square-free number. By Proposition 3, the ring $\mathbb{Z}_n$ is regular.

Corollary 7 *Let $n \in \mathbb{N}$. If n is square-free, then* $RP(\mathbb{Z}_n) = \{\bar{k} \in \mathbb{Z}_n \mid \gcd(k,n) = 1\}$.

Proof: It follows from Theorem 3, Proposition 2, and Proposition 3. □

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