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Inequalities for Kronecker products and Hadamard products of positive definite matrices

Pattrawut Chansangiam*, **Patcharin Hemchote**, **Praiboon Pantaragphong**

Department of Mathematics and Computer Science, Faculty of Science,
King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

* Corresponding author, e-mail: kcpattra@kmitl.ac.th

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ABSTRACT: The purpose of this paper is to develop inequalities for Kronecker products and Hadamard products of positive definite matrices. A number of inequalities involving powers, Kronecker powers, and Hadamard powers of linear combination of matrices are presented. In particular, Hölder inequalities and arithmetic mean-geometric mean inequalities for Kronecker products and Hadamard products are obtained as special cases.

KEYWORDS: Löwner partial order, jointly concave map

INTRODUCTION

Inequalities have proved to be powerful tools in mathematics. Matrix inequalities arise in various branches of mathematics and science such as system and control theory[1], optimization[2] and semidefinite programming[3]. Matrix inequalities are also important tools in quantum statistical inference and quantum information theory[4]. This paper is concerned with inequalities for Kronecker products and Hadamard products of special kinds of matrices. Let us begin with some terminology and notation.

As usual, $\mathbb{R}$ stands for the set of real numbers. Denote the set of all $m \times n$ complex matrices by $\mathbb{M}_{m,n}$ and abbreviate $\mathbb{M}_{n,n}$ to $\mathbb{M}_n$ for convenience. The identity matrix in $\mathbb{M}_n$ is denoted by I_n or I if the size of matrix is clear. A matrix $A \in \mathbb{M}_n$ is *Hermitian* if $A^* = A$, where * denotes the conjugate transpose. $A \in \mathbb{M}_n$ is *positive semidefinite* if $x^*Ax \geqslant 0$ for all vectors $x \in \mathbb{C}^n$ ($\mathbb{C}^n$ is the set of n-tuples of complex numbers). If A is positive semidefinite and invertible, then A is *positive definite*. An equivalent condition for $A \in \mathbb{M}_n$ to be positive definite is that A is Hermitian and all eigenvalues of A are positive real numbers. A Hermitian matrix with the positivity property (a positive definite matrix), which plays a similar role to a positive real number, is an important kind of matrix. The set of all $n \times n$ positive definite matrices is represented by $\mathbb{P}_n$.

By inequalities for matrices we mean inequalities in the Löwner sense. Given Hermitian matrices A and B of the same size, $A \leqslant B$ means that $B - A$ is positive semidefinite. In particular, $A \geqslant \mathbf{0}$ indicates that A is positive semidefinite. The relation "$\leqslant$" is reflexive, antisymmetric and transitive, i.e., "$\leqslant$" forms a partial order on the space of Hermitian matrices. This is known as the *Löwner partial order*[5]. In fact, this relation does not form a total order on the space of Hermitian matrices; there are Hermitian matrices A, B of same size which are not comparable, i.e., neither $A \leqslant B$ nor $A \geqslant B$ holds.

Now we introduce matrix products that differ from the ordinary matrix multiplication. A notion which is useful in the study of matrix equations and other applications is the Kronecker product of matrices. The *Kronecker product* of $A = [a_{ij}] \in \mathbb{M}_{m,n}$ and $B = [b_{ij}] \in \mathbb{M}_{p,q}$ is defined to be the block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{M}_{mp,nq}.$$

This matrix product has very nice properties. The most important is the mixed-product property:

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

for matrices A, B, C, D with appropriate sizes. The set of positive semidefinite matrices is closed under the Kronecker product, i.e.

$$A \geqslant \mathbf{0}, B \geqslant \mathbf{0} \quad \Longrightarrow \quad A \otimes B \geqslant \mathbf{0}.$$

The *Hadamard product* of $A, B \in \mathbb{M}_{m,n}$ is the entrywise product

$$A \circ B := [a_{ij}b_{ij}] \in \mathbb{M}_{m,n}.$$

The Hadamard product differs from the ordinary product in many ways. The most important is that Hadamard multiplication is commutative. The most basic properties of the product is the closure of the cone of positive semidefinite matrices under the Hadamard product (Schur product theorem[6]), i.e.

$$A \geqslant \mathbf{0}, B \geqslant \mathbf{0} \quad \Longrightarrow \quad A \circ B \geqslant \mathbf{0}.$$

Inequalities for the Kronecker product and the Hadamard product of matrices have a long history[6–10]. There are many results associated with them. In this paper we study inequalities for Kronecker products and Hadamard products involving powers, Kronecker powers and Hadamard powers of linear combinations of matrices. We derive some inequalities and then investigate results under special cases. Among these we obtain Hölder and arithmetic mean-geometric mean (AM-GM) inequalities.

PRELIMINARIES

We recall the concept of maps on matrix spaces. Let f be a continuous real-valued function on a real interval Ω. Let $A \in \mathbb{M}_n$ be Hermitian with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ contained in Ω. Since A is Hermitian, A can be decomposed as

$$A = U \operatorname{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]\, U^*$$

where U is unitary (i.e. $U^*U = I$) and $\operatorname{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$ is the diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. The *functional calculus* for A is defined as

$$f(A) = U \operatorname{diag}[f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)]\, U^*.$$

For example, if $A \in \mathbb{P}_n$ and $r \in \mathbb{R}$, then

$$A^r = U \operatorname{diag}[\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r]\, U^*.$$

The following property involving Kronecker products of positive definite matrices can be derived from the mixed-product property.

Lemma 1 (Horn and Johnson[6]) *For any $A, B \in \mathbb{P}_n$ and $r \in \mathbb{R}$, we have $(A \otimes B)^r = A^r \otimes B^r$.*

A map $\Phi : \mathbb{M}_n \to \mathbb{M}_m$ is *unital* if Φ maps unit element to unit element, i.e. $\Phi(I_n) = I_m$. Φ is *positive* if Φ maps positive element to positive element, i.e. $A \geqslant \mathbf{0} \Rightarrow \Phi(A) \geqslant \mathbf{0}$. Φ is *linear* if Φ preserves addition and scalar multiplication, i.e.,

$$\Phi(\alpha A + \beta B) = \alpha\Phi(A) + \beta\Phi(B)$$

where α, β are scalars. A map $\psi : \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_m$ is *jointly concave* if for any $A, B, C, D \in \mathbb{P}_n$ and any $0 < \epsilon < 1$,

$$\begin{aligned}\psi\big(\epsilon A + (1-\epsilon)B, \epsilon C + (1-\epsilon)D\big)\\ \geqslant \epsilon\, \psi(A, C) + (1-\epsilon)\psi(B, D).\end{aligned}$$

The following fact is well-known.

Lemma 2 (Ando[7]) *A map Φ defined by $\Phi(A, B) = (A^{-1} + B^{-1})^{-1}$ for $A, B \in \mathbb{P}_n$ is jointly concave.*

The following identity is used in the next section.

Lemma 3 *The following identity holds for any $A, B \in \mathbb{P}_n$ and $s > 0$:*

$$\begin{aligned}&\big((s^{-1}A \otimes I)^{-1} + (I \otimes B)^{-1}\big)^{-1}\\ &= (A \otimes B^{-1})\Big((A \otimes B^{-1}) + (sI \otimes I)\Big)^{-1}(I \otimes B).\end{aligned} \tag{1}$$

Proof: Since A and B are positive definite and s is positive, all terms in the equality (1) are existent, positive definite and nonsingular. For convenience, write $X = s^{-1}A \otimes I$, $Y = I \otimes B$, $Z = A \otimes B^{-1}$ and $P = X + Y$. It follows from the mixed-product property of the Kronecker product that

$$\begin{aligned}&\big(Z + (sI \otimes I)\big)\Big((s^{-1}I \otimes I) - (s^{-1}Y)P^{-1}(s^{-1}Z)\Big)\\ &\quad= Z(s^{-1}I \otimes I) + (sI \otimes I)(s^{-1}I \otimes I)\\ &\qquad - Z(s^{-1}Y)(X+Y)^{-1}(s^{-1}Z)\\ &\qquad - (sI \otimes I)(s^{-1}Y)(X+Y)^{-1}(s^{-1}Z)\\ &\quad= (s^{-1}Z) + (I \otimes I) - X(X+Y)^{-1}(s^{-1}Z)\\ &\qquad - Y(X+Y)^{-1}(s^{-1}Z)\\ &\quad= (s^{-1}Z) + I_{n^2} - (X+Y)(X+Y)^{-1}(s^{-1}Z)\\ &\quad= I_{n^2}.\end{aligned}$$

That is

$$\begin{aligned}(s^{-1}I \otimes I) - (s^{-1}Y)(X+Y)^{-1}(s^{-1}Z)\\ = \big(Z + (sI \otimes I)\big)^{-1}.\end{aligned}$$

Again, the mixed-product property yields

$$\begin{aligned}&Z^{-1}\big(X^{-1} + Y^{-1}\big)^{-1}Y^{-1}\\ &\quad= Z^{-1}\big(X - X(X+Y)^{-1}X\big)Y^{-1}\\ &\quad= (A^{-1} \otimes B)X(Y^{-1})\\ &\qquad - (A^{-1} \otimes B)X(X+Y)^{-1}XY^{-1}\\ &\quad= (s^{-1}I \otimes I) - (s^{-1}Y)(X+Y)^{-1}(s^{-1}Z)\\ &\quad= \big(Z + (sI \otimes I)\big)^{-1}.\end{aligned}$$

Thus, $(X^{-1}+Y^{-1})^{-1} = Z\big(Z+(sI\otimes I)\big)^{-1}Y$ which is (1). □

INEQUALITIES FOR KRONECKER PRODUCTS

In this section we derive inequalities for the Kronecker product of positive definite matrices in the form $(\alpha A+\beta B)^r \otimes (\alpha C+\beta D)^s$ and $\alpha(A^r \otimes C^s)+\beta(B^r \otimes D^s)$ where A, B, C, D are positive definite matrices and α, β, r, s are positive real numbers such that $r+s=1$.

Theorem 1 *For* $A, B, C, D \in \mathbb{P}_n$ *and* $\alpha, \beta, r, s > 0$ *such that* $r+s=1$*,*

$$(\alpha A+\beta B)^r \otimes (\alpha C+\beta D)^s \geqslant \alpha(A^r \otimes C^s)+\beta(B^r \otimes D^s). \quad (2)$$

Proof: Let f be a real-valued function defined by $f(t)=t^r$ for $t>0$ and $0<r<1$. Clearly, f is continuous. Recall an integral representation of f (see Ref. 8):

$$t^r = \frac{\sin r\pi}{\pi}\int_0^\infty \frac{s^{r-1}t}{s+t}\,\mathrm{d}s.$$

For convenience, write $Y=I\otimes B$ and $Z=A\otimes B^{-1}$. Hence, the functional calculus for $A\otimes B^{-1}$, namely, $f(A\otimes B^{-1})=(A\otimes B^{-1})^r$ can be written as

$$\frac{\sin r\pi}{\pi}\int_0^\infty (sI\otimes I)^{r-1}Z\big(Z+(sI\otimes I)\big)^{-1}\,\mathrm{d}s.$$

It follows from Lemma 1 that

$$\begin{aligned} A^r\otimes B^{1-r} &= (A^r I)\otimes(B^{-r}B)\\ &= (A^r\otimes B^{-r})(I\otimes B)\\ &= (A\otimes B^{-1})^r(I\otimes B). \end{aligned}$$

Hence, by Lemma 3 we obtain

$$\begin{aligned} &A^r\otimes B^{1-r}\\ &= \frac{\sin r\pi}{\pi}\int_0^\infty (sI\otimes I)^{r-1}Z\big(Z+(sI\otimes I)\big)^{-1}\,\mathrm{d}s\;Y\\ &= \frac{\sin r\pi}{\pi}\int_0^\infty s^{r-1}Z\big(Z+(sI\otimes I)\big)^{-1}\,\mathrm{d}s\;Y\\ &= \frac{\sin r\pi}{\pi}\int_0^\infty s^{r-1}Z\big(Z+(sI\otimes I)\big)^{-1}Y\,\mathrm{d}s\\ &= \frac{\sin r\pi}{\pi}\int_0^\infty s^{r-1}\big((s^{-1}A\otimes I)^{-1}+Y^{-1}\big)^{-1}\,\mathrm{d}s. \end{aligned}$$

Since $s^{-1}A\otimes I$ and $I\otimes B$ are positive definite, by Lemma 2 we have that the map $\Phi:\mathbb{P}_{n^2}\times\mathbb{P}_{n^2}\to\mathbb{P}_{n^2}$ defined by

$$\Phi(s^{-1}A\otimes I, I\otimes B)=\big((s^{-1}A\otimes I)^{-1}+(I\otimes B)^{-1}\big)^{-1}$$

is jointly concave. It is well-known that the positive linear combination of the jointly concave maps is jointly concave. Hence, from the viewpoint of the Riemann integral, the integrand is also jointly concave, and so is $A^r\otimes B^{1-r}$. This means that for any $A, B, C, D\in\mathbb{P}_n$ and scalar $0<\epsilon<1$,

$$(\epsilon A+(1-\epsilon)B)^r\otimes(\epsilon C+(1-\epsilon)D)^s \geqslant \epsilon(A^r\otimes C^s)+(1-\epsilon)(B^r\otimes D^s)$$

for $s>0$ and $r+s=1$. Since $0<\alpha/(\alpha+\beta)<1$, by setting $\epsilon=\alpha/(\alpha+\beta)$, we get (2). □

From this theorem, we obtain the Hölder inequality for positive definite matrices as a special case. Recall that real numbers p, q are *conjugate exponents* if p, q are positive and $1/p+1/q=1$.

Corollary 1 *For* $A, B, C, D \in \mathbb{P}_n$ *and conjugate exponents* p, q*, we have*

$$(A\otimes B)+(C\otimes D)\leqslant (A^p+C^p)^{\frac{1}{p}}\otimes(B^q+D^q)^{\frac{1}{q}}. \quad (3)$$

Proof: First, set $\alpha=\beta=1$, $r=1/p$ and $s=1/q$ in Theorem 1. Then swap B with C. Finally, replace A, B, C, D with A^p, B^q, C^p, D^q, respectively. □

The case $p=2$ (hence $q=2$) in (3) is the Cauchy-Schwarz inequality. See Ref. 7 for generalizations of the Hölder inequality for positive definite matrices. Recall that the *Kronecker sum* of $A\in\mathbb{M}_n$ and $B\in\mathbb{M}_m$ is defined as

$$A\oplus B := (A\otimes I_m)+(I_n\otimes B)\ \in\mathbb{M}_{mn}.$$

With this notation, we obtain the following from the case $B=C=I$ in Corollary 1.

Corollary 2 *For* $A, B\in\mathbb{P}_n$ *and conjugate exponents* p, q*, we have*

$$A\oplus B\leqslant (A^p+I)^{\frac{1}{p}}\otimes(B^q+I)^{\frac{1}{q}}. \quad (4)$$

The equality in (2) occurs if $A=B$, $C=D$. Now we investigate results under other special cases of Theorem 1. We consider the cases (i) $A=C$, $B=D$, (ii) $A=D$, $B=C$, (iii) $r=s$, (iv) $\alpha=\beta$.

Many inequalities can be obtained via combining these cases. For $A, B, C, D\in\mathbb{P}_n$ and $\alpha, \beta, r, s>0$ such that $r+s=1$, we have the following results:

$$(\alpha A+\beta B)^r\otimes(\alpha A+\beta B)^s \geqslant \alpha(A^r\otimes A^s)+\beta(B^r\otimes B^s)\,, \quad (5)$$

$$(\alpha A+\beta B)^r\otimes(\beta A+\alpha B)^s \geqslant \alpha(A^r\otimes B^s)+\beta(B^r\otimes A^s)\,, \quad (6)$$

$$\big((\alpha A+\beta B)\otimes(\alpha C+\beta D)\big)^{\frac{1}{2}} \geqslant \alpha(A\otimes C)^{\frac{1}{2}}+\beta(B\otimes D)^{\frac{1}{2}}, \tag{7}$$

$$(A+B)^r\otimes(C+D)^s \geqslant (A^r\otimes C^s)+(B^r\otimes D^s). \tag{8}$$

From inequalities (5)-(8), it follows that

$$\big((A+B)\otimes(C+D)\big)^{\frac{1}{2}} \geqslant (A\otimes C)^{\frac{1}{2}}+(B\otimes D)^{\frac{1}{2}}, \tag{9}$$

$$(A+B)^r\otimes(A+B)^s \geqslant (A^r\otimes A^s)+(B^r\otimes B^s), \tag{10}$$

$$(A+B)^r\otimes(A+B)^s \geqslant (A^r\otimes B^s)+(B^r\otimes A^s), \tag{11}$$

$$\big((\alpha A+\beta B)\otimes(\beta A+\alpha B)\big)^{\frac{1}{2}} \geqslant \alpha(A\otimes B)^{\frac{1}{2}}+\beta(B\otimes A)^{\frac{1}{2}}. \tag{12}$$

Recall that the kth *Kronecker power* of $A\in\mathbb{M}_n$ is defined inductively for any positive integer k by $A^{\otimes 1}:=A$ and $A^{\otimes k}:=A\otimes A^{\otimes(k-1)}$, for $k>1$. It is easy to see that for any $A\in\mathbb{P}_n$, positive integer k, and real number r,

$$(A^{\otimes k})^r=(A^r)^{\otimes k}. \tag{13}$$

With this notation and property (13), we get the following results:

$$\big((\alpha A+\beta B)^{\frac{1}{2}}\big)^{\otimes 2} \geqslant \alpha(A^{\frac{1}{2}})^{\otimes 2}+\beta(B^{\frac{1}{2}})^{\otimes 2}, \tag{14}$$

$$\big((A+B)^{\frac{1}{2}}\big)^{\otimes 2} \geqslant (A^{\frac{1}{2}})^{\otimes 2}+(B^{\frac{1}{2}})^{\otimes 2}, \tag{15}$$

$$\big((A+B)^{\frac{1}{2}}\big)^{\otimes 2} \geqslant (A\otimes B)^{\frac{1}{2}}+(B\otimes A)^{\frac{1}{2}}. \tag{16}$$

The next result is the AM-GM inequality for matrices involving the Kronecker product.

Corollary 3 *If $A,B\in\mathbb{P}_n$ commute under the Kronecker product, then*

$$(A\otimes B)^{\frac{1}{2}} \leqslant \frac{1}{2}\big((A+B)^{\frac{1}{2}}\big)^{\otimes 2} \tag{17}$$

with equality if and only if $A=B$.

Proof: The inequality (16) becomes (17) if $A\otimes B=B\otimes A$. The equality part follows from the fact that $A\otimes B=B\otimes A$ if and only if there is a scalar c such that $A=cB$ or $B=cA$. □

Remark 1 (i) If A and B are comparable, then (15) is sharper than (16). The reason is that the closure of $\mathbb{P}_n$ under the Kronecker product yields

$$(A^{\frac{1}{2}}-B^{\frac{1}{2}})\otimes(A^{\frac{1}{2}}-B^{\frac{1}{2}})\geqslant\mathbf{0}$$

and hence

$$(A^{\frac{1}{2}})^{\otimes 2}+(B^{\frac{1}{2}})^{\otimes 2}\geqslant(A\otimes B)^{\frac{1}{2}}+(B\otimes A)^{\frac{1}{2}}.$$

Similarly, if A and B are comparable, then (10) is sharper than (11). Note that giving positive definite matrices A,B does not always imply $A\geqslant B$ or $A\leqslant B$. (ii) Without the commutativity of A and B under the Kronecker product, the inequality (16) becomes the AM-GM inequality for real numbers when $n=1$.

INEQUALITIES FOR HADAMARD PRODUCTS

In this section we derive inequalities for Hadamard products of positive definite matrices.

Theorem 2 *For $A,B,C,D\in\mathbb{P}_n$ and $\alpha,\beta,r,s>0$ such that $r+s=1$,*

$$(\alpha A+\beta B)^r\circ(\alpha C+\beta D)^s \geqslant \alpha(A^r\circ C^s)+\beta(B^r\circ D^s). \tag{18}$$

Proof: Let us define $\Phi:\mathbb{P}_n\times\mathbb{P}_n\to\mathbb{P}_{n^2}$ by $\Phi(A,B)=A^r\otimes B^s$. Recall that the Hadamard product of matrices is the principal submatrix of the Kronecker product of matrices. Consequently, there exists a unital positive linear map $\varphi:\mathbb{P}_{n^2}\to\mathbb{P}_n$ such that $\varphi(A\otimes B)=A\circ B$. Hence,

$$(\varphi\circ\Phi)(A,B)=\varphi(\Phi(A,B)) =\varphi(A^r\otimes B^s)=A^r\circ B^s.$$

Since Φ is jointly concave (by Theorem 1) and φ is positive and linear, the composition $\varphi\circ\Phi$ is also jointly concave. This means that for any $A,B,C,D\in\mathbb{P}_n$ and any scalar $0<\epsilon<1$,

$$(\epsilon A+(1-\epsilon)B)^r\circ(\epsilon C+(1-\epsilon)D)^s \geqslant \epsilon(A^r\circ C^s)+(1-\epsilon)(B^r\circ D^s).$$

Since $0<\alpha/(\alpha+\beta)<1$, by replacing ϵ with $\alpha/(\alpha+\beta)$, we get (18). □

We obtain the Hölder inequality for positive definite matrices as a special case of Theorem 2.

Corollary 4 *For $A,B,C,D\in\mathbb{P}_n$ and conjugate exponents p,q, we have*

$$(A\circ B)+(C\circ D)\leqslant(A^p+C^p)^{\frac{1}{p}}\circ(B^q+D^q)^{\frac{1}{q}}. \tag{19}$$

Proof: First set $\alpha=\beta=1$, $r=1/p$ and $s=1/q$ in (18). Then swap B with C. Finally, replace A,B,C,D with A^p,B^q,C^p,D^q, respectively. □

The case $p = 2$ (hence $q = 2$) in (19) is the Cauchy-Schwarz inequality. The Hadamard sum of matrices is defined and studied in Ref. 11. The *Hadamard sum* of $A, B \in \mathbb{M}_n$ is

$$A \bullet B := A \circ I + I \circ B.$$

In particular, from Corollary 4, we get

Corollary 5 *For $A, B \in \mathbb{P}_n$ and conjugate exponents p, q, we have*

$$A \bullet B \leqslant (A^p + I)^{\frac{1}{p}} \circ (B^q + I)^{\frac{1}{q}}. \tag{20}$$

The equality in (18) occurs if $A = B$, $C = D$. Now we investigate results under other special cases of Theorem 2. We consider the cases (i) $A = C$, $B = D$, (ii) $A = D$, $B = C$, (iii) $r = s$, (iv) $\alpha = \beta$.

Many inequalities can be derived from these cases. The following inequalities hold for any $A, B, C, D \in \mathbb{P}_n$ and $\alpha, \beta, r, s > 0$ with $r + s = 1$:

$$(\alpha A + \beta B)^r \circ (\alpha A + \beta B)^s \geqslant \alpha(A^r \circ A^s) + \beta(B^r \circ B^s), \tag{21}$$

$$(\alpha A + \beta B)^r \circ (\beta A + \alpha B)^s \geqslant \alpha(A^r \circ B^s) + \beta(A^s \circ B^r), \tag{22}$$

$$(\alpha A + \beta B)^{\frac{1}{2}} \circ (\alpha C + \beta D)^{\frac{1}{2}} \geqslant \alpha(A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + \beta(B^{\frac{1}{2}} \circ D^{\frac{1}{2}}), \tag{23}$$

$$(A + B)^r \circ (C + D)^s \geqslant (A^r \circ C^s) + (B^r \circ D^s). \tag{24}$$

From inequalities (21)-(24), it follows that

$$(A + B)^{\frac{1}{2}} \circ (C + D)^{\frac{1}{2}} \geqslant (A^{\frac{1}{2}} \circ C^{\frac{1}{2}}) + (B^{\frac{1}{2}} \circ D^{\frac{1}{2}}), \tag{25}$$

$$(A + B)^r \circ (A + B)^s \geqslant (A^r \circ A^s) + (B^r \circ B^s), \tag{26}$$

$$(A + B)^r \circ (A + B)^s \geqslant (A^r \circ B^s) + (A^s \circ B^r), \tag{27}$$

$$(\alpha A + \beta B)^{\frac{1}{2}} \circ (\beta A + \alpha B)^{\frac{1}{2}} \geqslant (\alpha + \beta)(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}). \tag{28}$$

Recall that the kth *Hadamard power* of $A \in \mathbb{M}_n$ is defined by $A^{(k)} := [a_{ij}^k]$, k being a positive integer. It is easy to see that $(\alpha A)^{(k)} = \alpha^k A^{(k)}$ and

$$A^{(k)} = A \circ A^{(k-1)}, \quad k = 2, 3, \ldots .$$

With this notation, we have the following results:

$$\big((\alpha A + \beta B)^{\frac{1}{2}}\big)^{(2)} \geqslant \alpha(A^{\frac{1}{2}})^{(2)} + \beta(B^{\frac{1}{2}})^{(2)}, \tag{29}$$

$$\big((A + B)^{\frac{1}{2}}\big)^{(2)} \geqslant (A^{\frac{1}{2}})^{(2)} + (B^{\frac{1}{2}})^{(2)}, \tag{30}$$

$$\big((A + B)^{\frac{1}{2}}\big)^{(2)} \geqslant 2(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}). \tag{31}$$

Remark 2 If A and B are comparable, then (30) is sharper than (31). Indeed, the closure of $\mathbb{P}_n$ under the Hadamard product implies

$$(A^{\frac{1}{2}} - B^{\frac{1}{2}}) \circ (A^{\frac{1}{2}} - B^{\frac{1}{2}}) \geqslant \mathbf{0}$$

and hence

$$(A^{\frac{1}{2}})^{(2)} + (B^{\frac{1}{2}})^{(2)} \geqslant 2(A^{\frac{1}{2}} \circ B^{\frac{1}{2}}).$$

Similarly, if A and B are comparable, then (26) is sharper than (27).

Now we obtain, from (31), the AM-GM inequality for matrices involving the Hadamard product.

Corollary 6 *For $A, B \in \mathbb{P}_n$, we have the following inequality:*

$$A^{\frac{1}{2}} \circ B^{\frac{1}{2}} \leqslant \frac{1}{2}\big((A + B)^{\frac{1}{2}}\big)^{(2)}. \tag{32}$$

CONCLUSIONS

We have obtained many matrix inequalities involving Kronecker products and Hadamard products of positive definite matrices using the concept of maps on matrix spaces. We believe that one can get other inequalities by appropriate use of this concept. Our results should be applicable in fields related to matrix theory.

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