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On Quasi-gamma-ideals in Gamma-semigroups

Ronnason Chinram*

Department of Mathematics, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, 90112, Thailand.

* Corresponding author, E-mail: ronnason.c@psu.ac.th

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ABSTRACT: The concept of quasi-ideals in semigroups was introduced in 1956 by O. Steinfeld. The class of quasi-ideals in semigroups is a generalization of one-sided ideals in semigroups. It is well-known that the intersection of a left ideal and a right ideal of a semigroup S is a quasi-ideal of S and every quasi-ideal of S can be obtain in this way. In 1981, M. K. Sen have introduced the concept of Γ-semigroups. One can see that Γ-semigroups are a generalization of semigroups. In this research, quasi-Γ-ideals in Γ-semigroups are introduced and some properties of quasi-Γ-ideals in Γ-semigroups are provided.

KEYWORDS: Γ-semigroups, quasi-Γ-ideals, minimal quasi-Γ-ideals, quasi-simple Γ-semigroups.

INTRODUCTION

Let S be a semigroup. A nonempty subset Q of S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. Let Q be a quasi-ideal of S. Then $Q^2 \subseteq SQ \cap QS \subseteq Q$. Hence Q is a subsemigroup of S. The concept of quasi-ideals in semigroups was introduced in 1956 by O. Steinfeld (see [1]). The author has studied some properties of quasi-ideals in semigroups (See [2] and [3]).

Example 1.1. Let $S = [0, 1]$. Then S is a semigroup under usual multiplication. Let $Q = [0, ½]$. Thus $SQ \cap QS = [0, ½] \subseteq Q$. Therefore, Q is a quasi-ideal of S.

A nonempty subset L of S is called a *left ideal* of S if $SL \subseteq L$ and a nonempty subset R of S is called a *right ideal* of S if $RS \subseteq R$. Clearly, every left ideal and every right ideal of a semigroup S is a subsemigroup of S. Next, let L and R be a left ideal and a right ideal of a semigroup S. By the definition of quasi-ideals of semigroups, it is easy to prove that $L \cap R$ is a quasi-ideal of S (See [4]). Let Q be a quasi-ideal of a semigroup. Then $Q = (Q \cup SQ) \cap (Q \cup QS)$. It is easy to show that $(Q \cup SQ)$ is a left ideal of S and $Q \cup QS$ is a right ideal of S. Then every quasi-ideal Q of S can be written as the intersection of a left ideal and a right ideal of S.

Example 1.2. Let $\mathbf{Z}$ be the set of all integers and $M_2(\mathbf{Z})$, the set of all 2×2 matrices over $\mathbf{Z}$. We have known that $M_2(\mathbf{Z})$ is a semigroup under the usual multiplication. Let

$$L = \{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \mid x, y \in \mathbf{Z} \}$$

and

$$R = \{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathbf{Z} \}.$$

Then L is a left ideal of $M_2(\mathbf{Z})$, R is a right ideal of $M_2(\mathbf{Z})$ and $L \cap R = \{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid x \in \mathbf{Z} \}$ is a quasi-ideal of $M_2(\mathbf{Z})$.

In 1981, the notion of Γ–semigroups was introduced by M. K. Sen (See [5], [6] and [7]). Let M and Γ be any two nonempty sets. If there exists a mapping $M \times \Gamma \times M \rightarrow M$, written (a, γ, b) by $a\gamma b$, M is called a Γ–*semigroup* if M satisfies the identities $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in M$ and $\gamma, \mu \in \Gamma$. Let K be a nonempty subset of M. Then K is called a *sub* Γ–*semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

Example 1.3. Let S be a semigroup and Γ be any nonempty set. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. Then S is a Γ–semigroup.

Example 1.4. Let $M = [0,1]$ and

$$\Gamma = \{ \frac{1}{n} \mid n \text{ is a positive integer} \}.$$

Then M is a Γ–semigroup under the usual multiplication. Next, let $K = [0, ½]$. We have that K is a nonempty subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a sub Γ–semigroup of M.

From example 1.3, we have that every semigroup is a Γ–semigroup. Therefore, Γ–semigroups are a generalization of semigroups.

In this research, we generalize some properties of quasi-ideals of semigroups to some properties of quasi–Γ–ideals in Γ–semigroups.

Main Results

Let M be a Γ–semigroup. A nonempty subset Q of M is called a *quasi–Γ–ideal* of M if $M\Gamma Q \cap Q\Gamma M \subseteq Q$. Let Q be a quasi–Γ–ideal of M. Then $Q\Gamma Q \subseteq M\Gamma Q \cap Q\Gamma M \subseteq Q$. This implies that Q is a subΓ–semigroup of M.

Example 2.1. Let S be a semigroup and Γ be any nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. From example 1.3, S is a Γ–semigroup. Let Q be a quasi-ideal of S. Thus $SQ \cap QS \subseteq Q$. We have that $S\Gamma Q \cap Q\Gamma S = SQ \cap QS \subseteq Q$. Hence, Q is a quasi–Γ–ideal of S.

Example 2.1 implies that the class of quasi–Γ–ideals in Γ–semigroups is a Generalization of quasi-ideals in semigroups.

Theorem 2.1. Let M be a Γ–semigroup and Q_i a quasi–Γ–ideal of M for each $i \in I$. If $\bigcap_{i\in I} Q_i$ is a nonempty set, then $\bigcap_{i\in I} Q_i$ is a quasi–Γ–ideal of M.

Proof. Let M be a Γ–semigroup and Q_i a quasi–Γ–ideal of M for each $i \in I$. Assume that $\bigcap_{i\in I} Q_i$ is a nonempty set. Take any $a, b \in \bigcap_{i\in I} Q_i$, $m_1, m_2 \in M$ and $\gamma, \mu \in \Gamma$ such that $m_1\mu b = a\gamma m_2$. Then $a, b \in Q_i$ for all $i \in I$. Since Q_i is a quasi–Γ–ideal of M for all $i \in I$, $m_1\mu b = a\gamma m_2 \in M\Gamma Q_i \cap Q_i\Gamma M \in Q_i$ for all $i \in I$. Therefore $m_1\mu b = a\gamma m_2 \in \bigcap_{i\in I} Q_i$. Thus $M\Gamma \bigcap_{i\in I} Q_i \cap \bigcap_{i\in I} Q_i \ \Gamma M \in \bigcap_{i\in I} Q_i$. Hence, $\bigcap_{i\in I} Q_i$ is a quasi–Γ–ideal of M.

In Theorem 2.1, the condition $\bigcap_{i\in I} Q_i$ is a nonempty set is necessary. For example, let $\mathbf{N}$ be the set of all positive integers and $\Gamma = \{1\}$. Then M is a Γ–semigroup. For $n \in \mathbf{N}$, let $Q_n = \{n+1, n+2, n+3, \ldots\}$. It is easy to show that each Q_n is a quasi-Γ-ideal of M for all $n \in \mathbf{N}$ but $\bigcap_{n\in N} Q_n$ is a empty set.

Let A be a nonempty subset of a Γ–semigroup M and $\mathfrak{I} = \{Q \mid Q$ is a quasi–Γ–ideal of M containing $A\}$. Then $\mathfrak{I}$ is a nonempty set because $M \in \mathfrak{I}$. Let $(A)_q = \bigcap_{Q\in\mathfrak{I}} Q$. It is clear to see that $A \subseteq (A)_q$. By Theorem 2.1, $(A)_q$ is a quasi–Γ–ideal of M. Moreover, $(A)_q$ is the smallest quasi–Γ–ideal of M containing A. $(A)_q$ is called *the quasi–Γ–ideal of M Generated by A*.

Theorem 2.2. Let A be a nonempty subset of a Γ–semigroup M. Then

$$(A)_q = A \cup (M\Gamma A \cap A\Gamma M).$$

Proof. Let A be a nonempty subset of a Γ–semigroup M. Let $Q = A \cup (M\Gamma A \cap A\Gamma M)$. It is easy to see that $A \subseteq Q$. We have that $M\Gamma Q \cap Q\Gamma M = M\Gamma\, [A \cup (M\Gamma A \cap A\Gamma M)] \cap [A \cup (M\Gamma A \cap A\Gamma M)]\, \Gamma M \subseteq M\Gamma\, (A \cup M\Gamma A) \cap [A \cup (A\Gamma M)]\, \Gamma M \subseteq M\Gamma A \cap A\Gamma M \subseteq Q$. Therefore, Q is a quasi–Γ–ideal of M.

Let C be any quasi–Γ–ideal of M containing A. Since C is a quasi–Γ–ideal of M and $A \subseteq C$, $M\Gamma A \cap A\Gamma M \subseteq C$. Therefore, $Q = A \cup (M\Gamma A \cap A\Gamma M) \subseteq C$.

Hence, Q is the smallest quasi–Γ–ideal of M containing A. Therefore,

$(A)_q = A \cup (M\Gamma A \cap A\Gamma M)$, as required.

Example 2.2. Let $\mathbf{N}$ be the set of natural integers and $\Gamma = \{5\}$. Then $\mathbf{N}$ is a Γ–semigroup under usual addition.

(i) Let $A = \{2\}$. We have that

$(A)_q = \{2\} \cup \{8, 9, 10, \ldots\}$.

(ii) Let $A = \{3, 4\}$. We have that

$(A)_q = \{3, 4\} \cup \{9, 10, 11, \ldots\}$.

Let M be a Γ–semigroup. A subΓ–semigroup L of M is called *a left Γ–ideal* of M if $M\Gamma L \subseteq L$ and a subΓ–semigroup R of M is called a *right Γ–ideal* of M if $R\Gamma M \subseteq R$. The following theorem is true.

Theorem 2.3. Let M be a Γ–semigroup. Let L and R be a left Γ–ideal and a right Γ–ideal of M, respectively. Then $L \cap R$ is a quasi–Γ–ideal of M.

Proof. Let L and R be any left Γ–ideal and any right Γ–ideal of a Γ–semigroup M, respectively. By properties of L and R, we have $R\Gamma L \subseteq L \cap R$. This implies that $L \cap R$ is a nonempty set. We have that

$M\Gamma\, (L\cap R) \cap (L\cap R)\, \Gamma M \subseteq M\Gamma L \cap R\Gamma M \subseteq L \cap R$.

Hence, $L \cap R$ is a quasi–Γ–ideal of M.

Theorem 2.4. Every quasi–Γ–ideal Q of a Γ-semigroup M is the intersection of a left Γ–ideal and a right Γ–ideal of M.

Proof. Let Q be any quasi–Γ–ideal of a Γ–semigroup M. Let $L = Q \cup M\Gamma Q$ and $R = Q \cup Q\Gamma M$.

Then $M\Gamma L = M\Gamma\, (Q \cup M\Gamma Q) = M\Gamma Q \cup M\Gamma M\Gamma Q \subseteq M\Gamma Q \subseteq L$ and $R\Gamma M = (Q \cup Q\Gamma M)\, \Gamma M = Q\Gamma M \cup Q\Gamma M\Gamma M \subseteq Q\Gamma M \subseteq R$. Then L and R is a left Γ–ideal and a right Γ–ideal of M, respectively.

Next, we claim that $Q = L \cap R$. It is easy to see that $Q \subseteq (Q \cup M\Gamma Q) \cap (Q \cup Q\Gamma M) \subseteq L \cap R$. Conversely, $L \cap R = Q \cup M\Gamma Q) \cap (Q \cup Q\Gamma M) \subseteq Q \cup (M\Gamma Q \cap Q\Gamma M) \subseteq Q$. Hence, $Q = L \cap R$.

Let M be a Γ–semigroup. M is called a *quasi-simple*

Γ–*semigroup* if M is a unique quasi–Γ–ideal of M. A quasi–Γ–ideal Q of M is called a *minimal quasi*–Γ–*ideal* of M if Q does not properly contain any quasi–Γ–ideals of M.

Example 2.3. Let G be a group and $\Gamma = \{e_G\}$. It is easy to see that Γ is a unique quasi–Γ–ideal of Γ under the usual binary operation. Then G is a quasi-simple Γ–semigroup.

Theorem 2.5. Let M be a Γ–semigroup. Then M is a quasi-simple Γ–semigroup if and only if $M\Gamma m \cap m\Gamma M = M$ for all $m \in M$.

Proof. Let M be a Γ–semigroup.

The proof of ($\rightarrow$) : Assume that M is a quasi-simple Γ–semigroup. Take any $m \in M$. First, we claim that $M\Gamma m \cap m\Gamma M$ is a quasi-ideal of M. We have that $m\Gamma m \in M\Gamma m \cap m\Gamma M$, this implies $M\Gamma m \cap m\Gamma M$ is a nonempty set. Moreover, $M\Gamma (M\Gamma m \cap m\Gamma M) \cap (M\Gamma m \cap m\Gamma M)\,\Gamma M \subseteq M\Gamma (M\Gamma m) \cap (m\Gamma M)\,\Gamma M = (M\Gamma M)\,\Gamma m \cap m\Gamma (M\Gamma M) \subseteq M\Gamma m \cap m\Gamma M$. Therefore, $M\Gamma m \cap m\Gamma M$ is a quasi–Γ–ideal of M. Since M is a quasi-simple Γ–semigroup, $M\Gamma m \cap m\Gamma M = M$.

The proof of ($\leftarrow$) : Assume that $M\Gamma m \cap m\Gamma M = M$ for all $m \in M$. Let Q be a quasi–Γ–ideal of M and $q \in Q$. By assumption, $M = M\Gamma q \cap q\Gamma M$. Since Q is a quasi–Γ–ideal of M, $M = M\Gamma q \cap q\Gamma M \subseteq M\Gamma Q \cap Q\Gamma M \subseteq Q$. Therefore $Q = M$. Hence, M is a quasi-simple Γ–semigroup.

Theorem 2.6. Let M be a Γ–semigroup and Q a quasi–Γ–ideal of M. If Q is a quasi-simple Γ–semigroup, then Q is a minimal quasi–Γ–ideal of M.

Proof. Suppose M be a Γ–semigroup and Q a quasi–Γ–ideal of M. Assume that Q is a quasi-simple Γ–semigroup. Let C be a quasi–Γ–ideal of M such that $C \subseteq Q$. Then $Q\Gamma C \cap C\Gamma Q \subseteq M\Gamma C \cap C\Gamma M \subseteq C$. Therefore, C be a quasi–Γ–ideal of Q. Since Q is a quasi-simple Γ–semigroup, $C = Q$. Then Q is a minimal quasi–Γ–ideal of M.

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