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Weak and Strong Convergence Theorems of New Iterations with Errors for Nonexpansive Nonself-Mappings

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ABSTRACT: In this paper, a new three-step iterative scheme for nonexpansive nonself- mappings in Banach spaces is defined, and weak and strong convergence theorems are established for the new iterative scheme in a uniformly convex Banach space.

KEYWORDS: Nonexpansive nonself-mapping, completely continuous, uniformly convex, three-step iteration.

INTRODUCTION

Fixed-point iteration processes for nonexpansive nonself-mapping in Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors[1-13]. In 2001, Noor[14] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. In 1998, Jung and Kim[15] proved the existence of a fixed point for a nonexpansive nonself-mapping in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive self-mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space[11]. Suantai[16] defined a new three-step iteration, which is an extension of Noor iterations and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in uniformly Banach spaces. Recently, Shahzad[17] extended Tan and Xu's results[11] (Theorem 1, p.305) to the case of nonexpansive nonself-mapping in a uniformly convex Banach space. Motivating these facts, a new class of three-step iterative scheme is introduced and studied in this paper. The scheme is defined as follows.

Let X be a normed space, C a nonempty convex subset of X, $P: X \to C$ a nonexpansive retraction of X onto C and $T: C \to X$ a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P\big((1-a_n-b_n)x_n + a_nTx_n + b_nu_n\big), \\ y_n &= P\big((1-c_n-d_n)z_n + c_nTx_n + d_nv_n\big), \\ x_{n+1} &= P\big((1-\alpha_n-\beta_n)y_n + \alpha_nTx_n + \beta_nw_n\big), \quad n \ge 1, \end{aligned} \tag{1.1}$$

where $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0,1]$.

If $T: C \to C$ and $a_n = b_n = c_n = d_n = \beta_n \equiv 0$, then the iterative scheme (1.1) reduces to the usual Mann iterative scheme

$$x_{n+1} = \alpha_nTx_n + (1-\alpha_n)x_n, \quad n \ge 1,$$

where $\{\alpha_n\}$ are appropriate sequences in $[0,1]$.

The purpose of this paper is to establish several strong convergence theorems for the three-step scheme (1.1) for completely continuous nonexpansive nonself-mappings in a uniformly convex Banach space, and weak convergence theorems for the scheme (1.1) for nonexpansive nonself-mappings in a uniformly convex Banach space with Opial's condition.

Now, we recall the well known concepts and results.

Let X be normed space and C a nonempty subset of X. A mapping $T: C \to C$ is said to be nonexpansive on C if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$.

Recall that a Banach space X is said to satisfy *Opial's condition*[18] if $x_n \to x$ weakly as $n \to \infty$ and $x \ne y$ implying that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \quad \forall n = 1,2,\ldots .$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ exists.

(2) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0.$

Proof. See[11] (Lemma 1).

Lemma 1.2 Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z\|^2 \le \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|),$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$.

Proof. See[19] (Lemma 1.4).

Lemma 1.3 Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demi-closed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.

Proof. See[20].

Lemma 1.4 Let X be a Banach space which satisfies *Opial's condition* and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist.

If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and respectively, then $u = v$.

Proof. See[16] (Theorem 2.3).

Results

In this section, we prove weak and strong convergence theorems for the three-step iterative scheme (1.1) for a nonexpansive nonself-mapping in a uniformly convex Banach space. In order to prove our main results, the following lemma is needed.

Lemma 2.1 Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a nonexpansive nonself-mapping with $F(T) \ne \varnothing$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ are real sequences in $[0,1]$ such that $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.1).

(i) If p is a fixed point of T, then $\lim_{n\to\infty}\|x_n - p\|$ exists.

(ii) If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then $\lim_{n\to\infty}\|Tx_n - y_n\| = 0.$

(iii) If $0 < \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$ and $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1$, then $\lim_{n\to\infty}\|Tx_n - z_n\| = 0.$

(iv) If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1$ and $\limsup_{n\to\infty} a_n < 1$, then $\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$

Proof. Let $p \in F(T)$, and

$$M_1 = \sup\{\|u_n - p\| : n \ge 1\},$$
$$M_2 = \sup\{\|v_n - p\| : n \ge 1\},$$
$$M_3 = \sup\{\|w_n - p\| : n \ge 1\},$$
$$M = \max\{M_i : i = 1,2,3\}.$$

Using (1.1), we have

$$\begin{aligned}\|z_n - p\| &= \|P((1 - a_n - b_n)x_n + a_nTx_n + b_nu_n) - P(p)\| \\ &\le \|((1 - a_n - b_n)x_n + a_nTx_n + b_nu_n) - p\| \\ &\le (1 - a_n - b_n)\|x_n - p\| + a_n\|Tx_n - Tp\| \\ &\quad + b_n\|u_n - p\| \\ &\le (1 - a_n - b_n)\|x_n - p\| + a_n\|x_n - p\| + b_n\|u_n - p\| \\ &\le \|x_n - p\| + Mb_n,\end{aligned}$$

$$\begin{aligned}\|y_n - p\| &= \|P((1 - c_n - d_n)z_n + c_nTx_n + d_nv_n) - P(p)\| \\ &\le (1 - c_n - d_n)\|z_n - p\| + c_n\|x_n - p\| + Md_n \\ &\le (1 - c_n - d_n)(\|x_n - p\| + Mb_n) + c_n\|x_n - p\| \\ &\quad + Md_n \\ &\le \|x_n - p\| + Mb_n + Md_n,\end{aligned}$$

and so

$$\begin{aligned}\|x_{n+1} - p\| &= \|P((1 - \alpha_n - \beta_n)y_n + \alpha_nTx_n + \beta_nw_n) - P(p)\| \\ &\le (1 - \alpha_n - \beta_n)\|y_n - p\| + \alpha_n\|x_n - p\| \\ &\quad + \beta_n\|w_n - p\| \\ &\le (1 - \alpha_n - \beta_n)(\|x_n - p\| + Mb_n + Md_n) \\ &\quad + \alpha_n\|x_n - p\| + M\beta_n \\ &\le \|x_n - p\| + M(b_n + d_n + \beta_n).\end{aligned}$$

Hence the assertion (i) follows from Lemma 1.1.

(ii) By (i), we know that $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F(T)$. It follows that $\{x_n - p\}, \{Tx_n - p\}$ and $\{y_n - p\}$ are bounded. Also, $\{u_n - p\}, \{v_n - p\}$ and $\{w_n - p\}$ are bounded by the assumption. Now we set

$$r_1 = \sup\{\|x_n - p\| : n \ge 1\},$$
$$r_2 = \sup\{\|Tx_n - p\| : n \ge 1\},$$
$$r_3 = \sup\{\|y_n - p\| : n \ge 1\},$$
$$r_4 = \sup\{\|z_n - p\| : n \ge 1\},$$
$$r_5 = \sup\{\|u_n - p\| : n \ge 1\},$$
$$r_6 = \sup\{\|v_n - p\| : n \ge 1\},$$

$$r_7 = \sup\{\|w_n - p\| : n \ge 1\},$$
$$r = \max\{r_i : i = 1,2,3,4,5,6,7\}. \qquad (2.1)$$

By using Lemma 1.2 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|P((1-a_n-b_n)x_n + a_nTx_n + b_nu_n) - P(p)\|^2 \\
&\le \|(1-a_n-b_n)(x_n-p) + a_n(Tx_n-p) + b_n(u_n-p)\|^2 \\
&\le (1-a_n-b_n)\|x_n-p\|^2 + a_n\|Tx_n-p\|^2 \\
&\quad + b_n\|u_n-p\|^2 - a_n(1-a_n-b_n)g(\|Tx_n-x_n\|) \\
&\le (1-a_n-b_n)\|x_n-p\|^2 + a_n\|x_n-p\|^2 + b_n\|u_n-p\|^2 \\
&\le \|x_n-p\|^2 + r^2b_n,
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\|^2 &= \|P((1-c_n-d_n)z_n + c_nTx_n + d_nv_n) - P(p)\|^2 \\
&\le \|(1-c_n-d_n)(z_n-p) + c_n(Tx_n-p) + d_n(v_n-p)\|^2 \\
&\le (1-c_n-d_n)\|z_n-p\|^2 + c_n\|Tx_n-p\|^2 \\
&\quad + d_n\|v_n-p\|^2 - c_n(1-c_n-d_n)g(\|Tx_n-z_n\|) \\
&\le (1-c_n-d_n)\|z_n-p\|^2 + c_n\|x_n-p\|^2 \\
&\quad + d_n\|v_n-p\|^2 - c_n(1-c_n-d_n)g(\|Tx_n-z_n\|) \\
&\le (1-c_n-d_n)(\|x_n-p\|^2 + r^2b_n) + c_n\|x_n-p\|^2 \\
&\quad + r^2d_n - c_n(1-c_n-d_n)g(\|Tx_n-z_n\|) \\
&\le (1-d_n)\|x_n-p\|^2 + r^2b_n + r^2d_n \\
&\quad - c_n(1-c_n-d_n)g(\|Tx_n-z_n\|) \\
&\le \|x_n-p\|^2 + r^2b_n + r^2d_n,
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|P((1-\alpha_n-\beta_n)y_n + \alpha_nTx_n + \beta_nw_n) - P(p)\|^2 \\
&\le \|(1-\alpha_n-\beta_n)y_n + \alpha_nTx_n + \beta_nw_n - p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(y_n-p) + \alpha_n(Tx_n-p) + \beta_n(w_n-p)\|^2 \\
&\le (1-\alpha_n-\beta_n)\|y_n-p\|^2 + \alpha_n\|Tx_n-p\|^2 \\
&\quad + \beta_n\|w_n-p\|^2 - \alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\le (1-\alpha_n-\beta_n)\|y_n-p\|^2 + \alpha_n\|x_n-p\|^2 \\
&\quad + \beta_n\|w_n-p\|^2 - \alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\le (1-\alpha_n-\beta_n)(\|x_n-p\|^2 + r^2b_n + r^2d_n) + \alpha_n\|x_n-p\|^2 \\
&\quad + r^2\beta_n - \alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\le (1-\beta_n)\|x_n-p\|^2 + r^2b_n + r^2d_n + r^2\beta_n \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\le \|x_n-p\|^2 + r^2b_n + r^2d_n + r^2\beta_n \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\le \|x_n-p\|^2 + r^2(b_n+d_n+\beta_n) \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|),
\end{aligned}$$

which leads to the following:

$$\begin{aligned}
&\alpha_n(1-\alpha_n-\beta_n)g(\|Tx_n-y_n\|) \\
&\qquad \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + r^2(b_n+d_n+\beta_n),
\end{aligned} \qquad (2.2)$$

If $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, then there exist a positive integer n_0 and $\eta,\ \eta' \in (0,1)$ such that $0 < \eta < \alpha_n$ and $\alpha_n + \beta_n < \eta' < 1$ for all $n \ge n_0$. It follows from (2.2) that

$$\begin{aligned}
&\eta(1-\eta')g(\|Tx_n-y_n\|) \\
&\qquad \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + r^2(b_n+d_n+\beta_n),
\end{aligned} \qquad (2.3)$$

for all $n \ge n_0$. Applying (2.3) for $m \ge n_0$, we have

$$\begin{aligned}
\sum_{n=n_0}^{m} g(\|Tx_n-y_n\|) &\le \frac{1}{\eta(1-\eta')}\begin{pmatrix} \sum_{n=n_0}^{m}(\|x_n-p\|^2 - \|x_{n+1}-p\|^2) \\ +r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n) \end{pmatrix} \\
&\le \frac{1}{\eta(1-\eta')}\begin{pmatrix} \|x_{n_0}-p\|^2 \\ +r^2\sum_{n=n_0}^{m}(b_n+d_n+\beta_n) \end{pmatrix}
\end{aligned} \qquad (2.4)$$

Letting $m \to \infty$ in the inequality (2.4), we get that $\sum_{n=n_0}^{\infty} g(\|Tx_n-y_n\|) < \infty$, and therefore $\lim_{n\to\infty}\|Tx_n-y_n\| = 0$. Since g is strictly increasing and continuous at 0 with $g(0)=0$, it follows that $\lim_{n\to\infty}\|Tx_n-y_n\| = 0$.

(iii) If $0 < \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$ and $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n+d_n) < 1$, then by the same argument as that given in (ii), it can be shown that $\lim_{n\to\infty}\|Tx_n-z_n\| = 0$.

(iv) If $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}(\alpha_n+\beta_n) < 1$, $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n+d_n) < 1$ and $\limsup_{n\to\infty} a_n < 1$, by (ii) and (iii) we have

$$\lim_{n\to\infty}\|Tx_n-y_n\| = 0 \text{ and } \lim_{n\to\infty}\|Tx_n-z_n\| = 0. \qquad (2.5)$$

From $y_n = P((1-c_n-d_n)z_n + c_nTx_n + d_nv_n)$, we have

$$\begin{aligned}
\|y_n - x_n\| &= \|P((1-c_n-d_n)z_n + c_nTx_n + d_nv_n) - P(x_n)\| \\
&\le \|(1-c_n-d_n)z_n + c_nTx_n + d_nv_n - x_n\| \\
&= \|(z_n-x_n) + c_n(Tx_n-z_n) + d_n(v_n-z_n)\| \\
&\le \|z_n-x_n\| + c_n\|Tx_n-z_n\| + d_n\|v_n-z_n\| \\
&= \|P((1-a_n-b_n)x_n + a_nTx_n + b_nu_n) - P(x_n)\| \\
&\quad + c_n\|Tx_n-z_n\| + d_n\|v_n-z_n\| \\
&\le \|(1-a_n-b_n)x_n + a_nTx_n + b_nu_n - x_n\| \\
&\quad + c_n\|Tx_n-z_n\| + d_n\|v_n-z_n\| \\
&= \|a_n(Tx_n-x_n) + b_n(u_n-x_n)\| \\
&\quad + c_n\|Tx_n-z_n\| + d_n\|v_n-z_n\| \\
&\le a_n\|Tx_n-x_n\| + b_n\|u_n-x_n\| \\
&\quad + c_n\|Tx_n-z_n\| + d_n\|v_n-z_n\| \\
&\le a_n\|Tx_n-x_n\| + c_n\|Tx_n-z_n\| + 2rb_n + 2rd_n,
\end{aligned}$$

where r is defined by (2.1). Thus

$$\|Tx_n - x_n\| \le \|Tx_n - y_n\| + \|y_n - x_n\|$$
$$\le \|Tx_n - y_n\| + a_n\|Tx_n - x_n\| + c_n\|Tx_n - z_n\| + 2rb_n + 2rd_n,$$

and so

$$(1-a_n)\|Tx_n - x_n\| \le \|Tx_n - y_n\| + c_n\|Tx_n - z_n\| + 2rb_n + 2rd_n$$

Since $\limsup_{n\to\infty} a_n < 1$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} d_n = 0$, it follows from (2.5) that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$

Theorem 2.2 Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T: C \to X$ a completely continuous nonexpansive nonself-mapping with $F(T) \ne \emptyset$. Suppose that $\{\alpha_n\},\{\beta_n\},\{a_n\},\{b_n\},\{c_n\},\{d_n\}$ are sequences of real numbers in $[0,1]$ with $c_n + d_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \ge 1$, and $\sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty, \sum_{n=1}^{\infty} \beta_n < \infty$, and

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, and

(ii) $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1$ and $\limsup_{n\to\infty} a_n < 1$.

For $\{x_n\},\{y_n\}$ and $\{z_n\}$ being the sequences defined by the three-step iterative scheme (1.1), we have $\{x_n\},\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. By Lemma 2.1(iv), we have

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{2.6}$$

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (2.6), $\{x_{n_k}\}$ converges. Let $q = \lim_{k\to\infty} x_{n_k}$. By the continuity of T and (2.6) we have that $Tq = q$, so q is a fixed point of T. By Lemma 2.1(i), $\lim_{n\to\infty}\|x_n - q\|$ exists. Then $\lim_{k\to\infty}\|x_{n_k} - q\| = 0.$

Thus $\lim_{n\to\infty}\|x_n - q\| = 0$. Since $\|y_n - x_n\| \to 0$ as $n \to \infty$, and

$$\|z_n - x_n\| = \|P((1-a_n-b_n)x_n + a_nTx_n + b_nu_n) - P(x_n)\|$$
$$\le \|(1-a_n-b_n)x_n + a_nTx_n + b_nu_n - x_n\|$$
$$\le a_n\|Tx_n - x_n\| + b_n\|u_n - x_n\| \to 0 \text{ as } n\to\infty,$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q.$

For $a_n = b_n \equiv 0$, then Theorem 2.2 can be reduced to the two-step iteration with errors.

Corollary 2.3 Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T: C \to X$ a completely continuous nonexpansive nonself-mapping with $F(T) \ne \emptyset$. Suppose that $\{c_n\},\{d_n\},\{\alpha_n\},\{\beta_n\}$ are real sequences in $[0,1]$ satisfying

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, and

(ii) $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1.$

For a given $x_1 \in C$, define

$$y_n = P((1-c_n-d_n)x_n + c_nTx_n + d_nv_n),$$
$$x_{n+1} = P((1-\alpha_n-\beta_n)y_n + \alpha_nTx_n + \beta_nw_n), \quad n \ge 1.$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

In the next result, we prove the weak convergence of the three-step iterative scheme (1.1) for nonexpansive nonself-mappings in a uniformly convex Banach space satisfying *Opial's condition*.

Theorem 2.4 Let X be a uniformly convex Banach space which satisfies *Opial's condition*, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T: C \to X$ a nonexpansive nonself-mapping with $F(T) \ne \emptyset$. Suppose that $\{\alpha_n\},\{\beta_n\},\{a_n\},\{b_n\},\{c_n\},\{d_n\}$ are sequences of real numbers in $[0,1]$ with $c_n + d_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \ge 1$, and $\sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} d_n < \infty, \sum_{n=1}^{\infty} \beta_n < \infty$, and

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, and

(ii) $0 < \liminf_{n\to\infty} c_n \le \limsup_{n\to\infty}(c_n + d_n) < 1$ and $\limsup_{n\to\infty} a_n < 1$.

Let $\{x_n\}$ be the sequence defined by three-step iterative scheme (1.1). Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. By using the same proof as in Theorem 2.2, it can be shown that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 1.3, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 1.3,

$u, v \in F(T)$. By Lemma 2.1(i), $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. It follows from Lemma 1.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to fixed point of T.

When $a_n = b_n \equiv 0$ in Theorem 2.4, we obtain the weak convergence theorem of the two-step iteration with errors as follows:

Corollary 2.5 Let X be a uniformly convex Banach space which satisfies *Opial's condition*, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T: C \to X$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers in [0,1] such that

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, and

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$.

For a given $x_1 \in C$, define

$$y_n = P\big((1 - c_n - d_n)x_n + c_nTx_n + d_nv_n\big),$$
$$x_{n+1} = P\big((1 - \alpha_n - \beta_n)y_n + \alpha_nTx_n + \beta_nw_n\big), \quad n \geq 1.$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

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